The Firebreak Problem: Scientific Aspects

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Firefighter problem (Hartnell, 1995)

- A fire (or contagion) breaks out on a vertex v_f of a graph G. The following process repeats until no new vertices burn:
  - A firefighter protects k unburnt vertices.
  - The fire then spreads to any unprotected neighbour of a burning vertex.

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  - Stop the fire's spread as quickly as possible (i.e. minimize the number of time steps before the process terminates)
  - Save a particular set of vertices from burning.
  - Determine the minimum value of k that will save a particular number or fraction of vertices.

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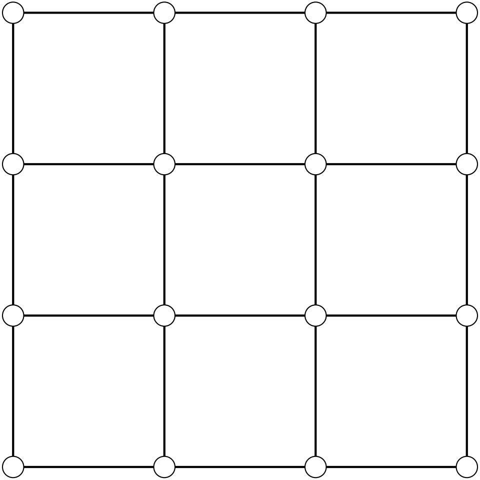

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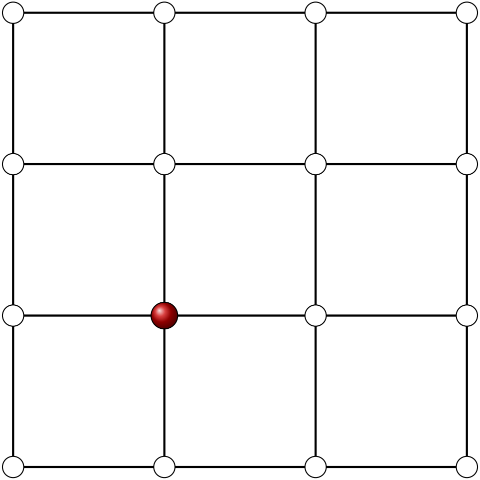

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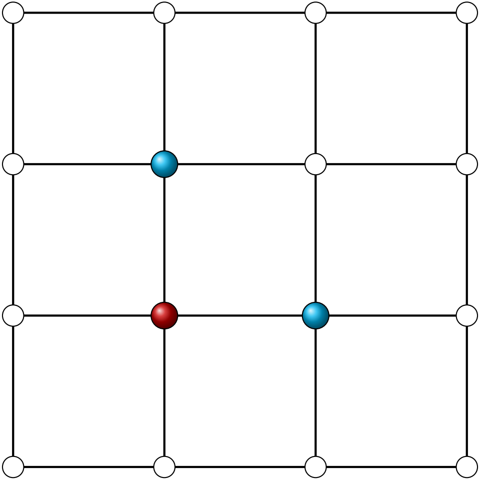

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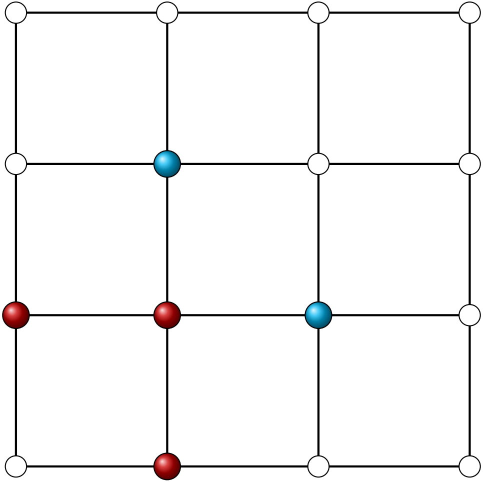

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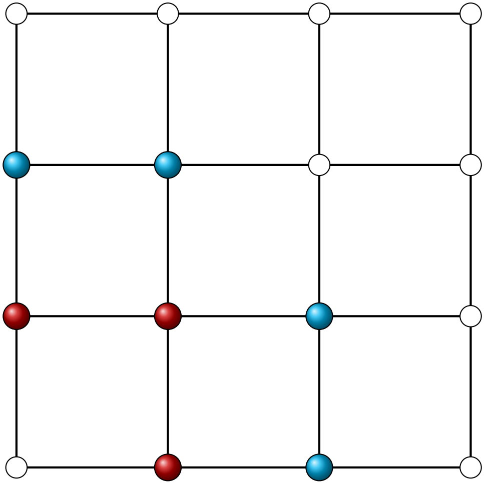

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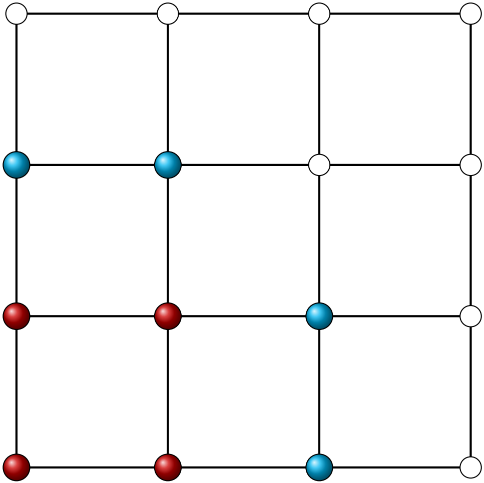

Background on FIREFIGHTER

- Consider the decision problem for the case $k = 1$:

FIREFIGHTER

Instance:	A graph G, an integer t, and a vertex $v_f \in V(G)$.
Question:	Can one firefighter save t vertices of G (in addition to those he protects)?

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- FIREFIGHTER is NP-complete for:
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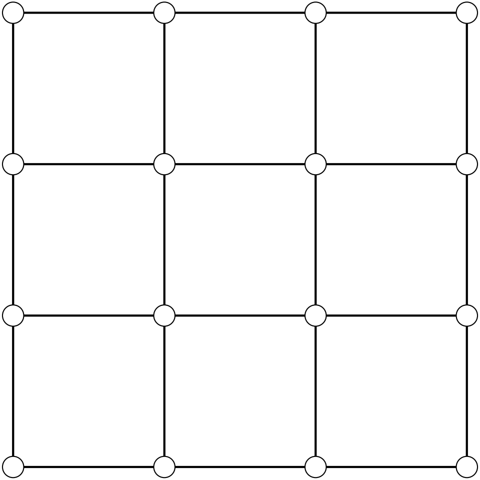
- See the survey by Finbow and MacGillivray (2009)

FIREBREAK Problem

- Suppose that the firefighter only has one move.
- When the fire breaks out, he may build a firebreak by protecting k unburnt vertices.
- Protected vertices will not burn, and the fire cannot spread through them.
- Thereafter, the fire spreads to any vertex reachable along paths through only unprotected vertices.
- How many vertices can we prevent from burning?

Example with $k = 2$ protected vertices



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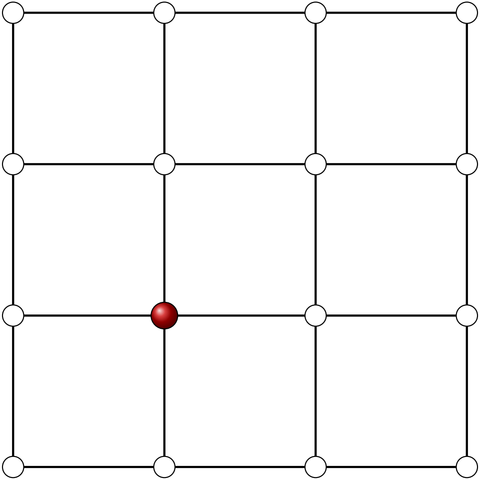

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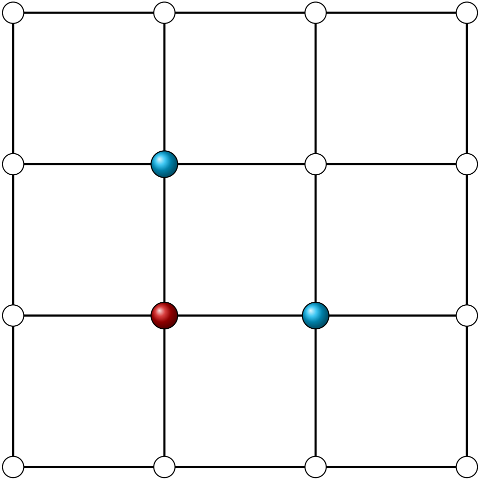

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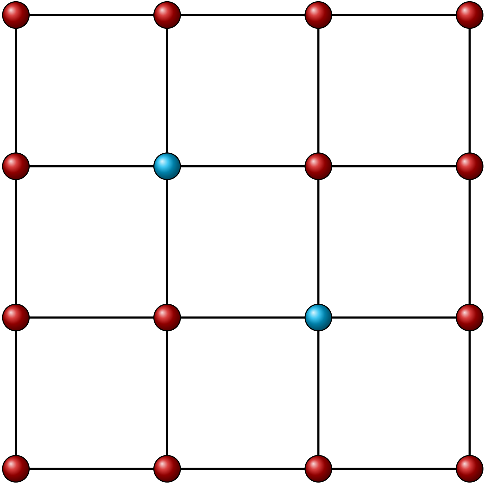

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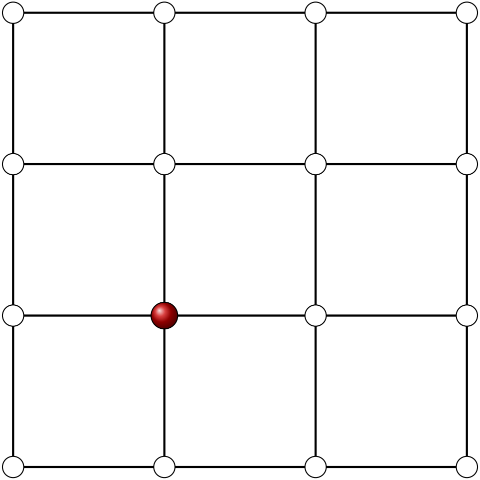

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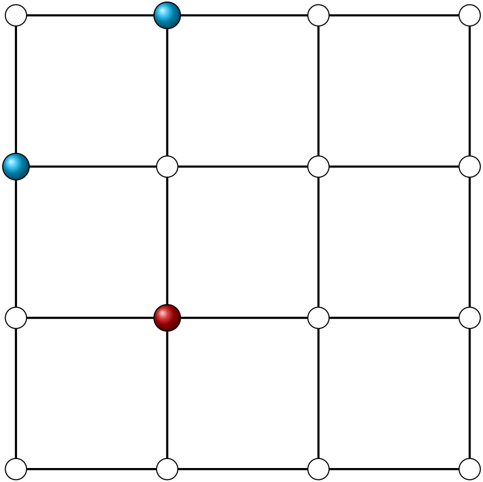

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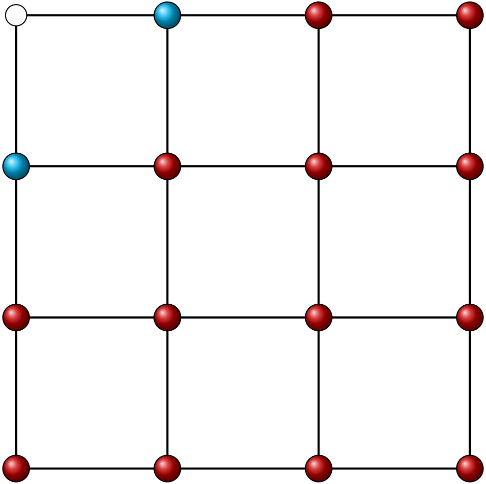


Firebreak problem

In order to save additional vertices, the set of protected vertices must form a vertex cut.

So we seek a vertex cut of size k (the protected vertices) that separates v_f from as many vertices as possible (the saved vertices).

Definition

$\mathcal{F}(G, k, v_f)$ is the maximum number of vertices outside the protected set that can be saved.

Can we compute $\mathcal{F}(G, k, v_f)$?

FIREBREAK decision problem

Instance: A graph G, integers k and t, and a vertex $v_f \in V(G)$.

Question: Does $V(G)$ contain a k-subset S such that $v_f \notin S$ and the number of vertices of $G - S$ that are separated from v_f is at least t?

What is the complexity of this problem?

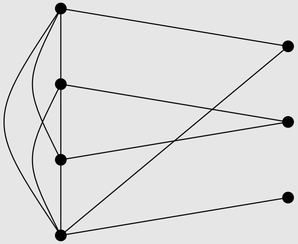
Split graphs

Definition

A split graph G has vertex set $V(G) = A \cup B$, where

- $G[A]$ is a maximum clique.
- $G[B]$ is an independent set.

Example



Key Player Problem

Consider the KEY PLAYER problem:

Instance: A graph G, integers k and t.

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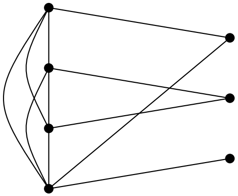
Theorem

FIREBREAK *and* KEY PLAYER *are computationally equivalent on split graphs.*

Computational equivalence

Using an oracle for FIREBREAK to solve KEY PLAYER:

- Let (G, k, t) be an instance of KEY PLAYER on a split graph



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- Let (G, k, t) be an instance of KEY PLAYER on a split graph
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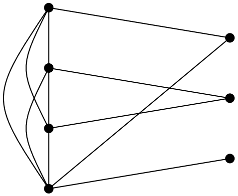

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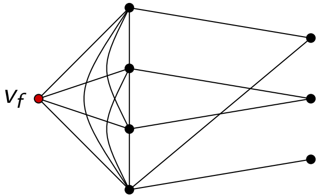
- Form G' by adding a new vertex v_f adjacent to each vertex of A.



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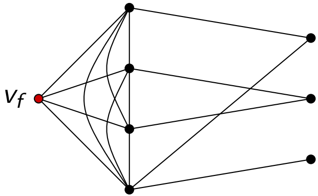
- Form G' by adding a new vertex v_f adjacent to each vertex of A.
- Answer to KEY PLAYER for (G, k, t) is affirmative iff answer to FIREBREAK for $(G', k, t-1, v_f)$ is affirmative.



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Using an oracle for KEY PLAYER to solve FIREBREAK

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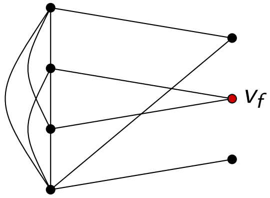
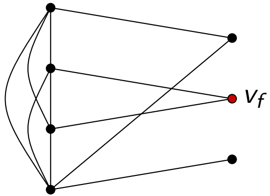


Computational equivalence

Using an oracle for Key Player to solve Firebreak

- Let (G, k, t, v_f) be an instance of Firebreak on a split graph G
- Assume $k < |N_G(v_f)|$ (otherwise, Firebreak can be solved in polynomial time)



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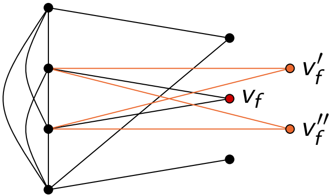
- Construct a graph G' by adding k "twins" of v_f



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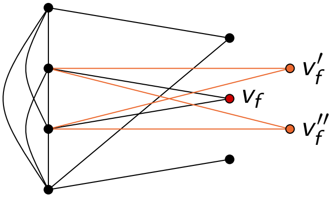
- Answer to Firebreak for (G, k, t, v_f) is affirmative iff answer to Key Player for $(G', k, t+1)$ is affirmative.



NP-complete on split graphs

Theorem

KEY PLAYER *is NP-complete on split graphs.*

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Proof.

Reduction from t-WAY VERTEX CUT, which is known to be NP-complete on split graphs (Berger, Grigoriev, van der Zwann, 2014):

t-WAY VERTEX CUT:

Instance:	Graph G, integer k, integer t
Question:	Does $V(G)$ contain a subset S such that $\lvert S\rvert \leq k$ and $G - S$ has at least t components?

□

NP-complete on split graphs

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Reduction from t-way vertex cut, which is known to be NP-complete on split graphs (Berger, Grigoriev, van der Zwann, 2014):

t-way vertex cut:

Instance:	Graph G, integer k, integer t
Question:	Does $V(G)$ contain a subset S such that $\|S\| \leq k$ and $G - S$ has at least t components?

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Corollary

Firebreak *is NP-complete on split graphs.*

Cubic graphs

Theorem

FIREBREAK *can be solved in polynomial time on cubic graphs.*

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Idea.

Given an instance (G, k, t, v_f) of FIREBREAK:

- If $k \geq 3$, let S consist of $N(v_f)$ along with any $k - 3$ other vertices.
- If $k < 3$, exhaustively check all k-subsets S of $V(G) \setminus \{v_f\}$.

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Theorem

KEY PLAYER *is NP-complete on cubic planar graphs.*

Idea.

Reduction from INDEPENDENT SET. □

Bipartite graphs

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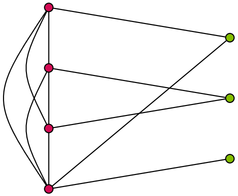
- An oracle for FIREBREAK on bipartite graphs can be used to solve FIREBREAK on split graphs.



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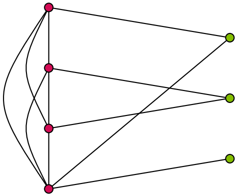
- Given a split graph G, consider an instance (G, k, t, v_f) of FIREBREAK. If $k \geq |N_G(v_f)|$, the problem is easy.



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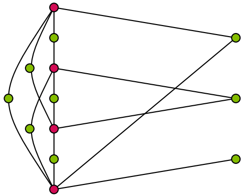

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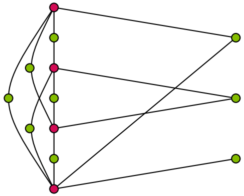
- Otherwise, subdivide edges in a max clique to form a bipartite graph G'.
- FIREBREAK has an affirmative answer for (G, k, t, v_f) iff it has an affirmative answer for $(G', k, t + \binom{k}{2}, v_f)$.



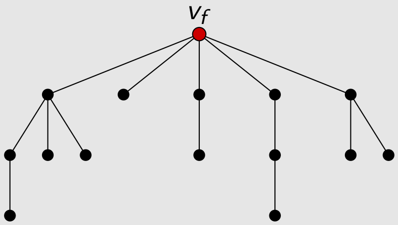
Trees

Theorem

FIREBREAK *can be solved in polynomial time on trees.*

Idea.

- Root the tree at v_f, and count vertices in the subtrees rooted at the neighbours of v_f.
- Protect the k neighbours of v_f with largest subtrees.



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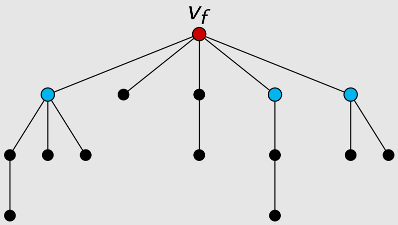

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Courcelle's Theorem

Theorem (Courcelle, 1990; Arnborg, Lagergren and Seese 1991)

- *Let G be a graph on n vertices and w a constant.*
- *Let $\mathcal{P}$ be a graph theoretical problem that can be expressed in the form of extended monadic second-order logic (EMSO).*
- *If $tw(G) \leq w$, then determining whether G has property $\mathcal{P}$ can be accomplished in time $\mathcal{O}(f(w) \cdot n)$.*

Theorem

FIREBREAK *can be solved in linear time for graphs with constant-bounded treewidth.*

Extended Monadic Second-Order logic

Monadic second-order logic (MSO) expressions for graphs are based on

- variables for vertices, edges, sets of vertices, sets of edges
- universal and existential quantifiers
- logical connectives of conjunction, disjunction and negation
- binary relations to assess set membership; adjacency of vertices; incidence of edges and vertices; equality for edges, vertices and sets

Extended monadic second-order logic (EMSO) also includes an evaluation relation that lets us consider set cardinalities.

See the survey by Langer, Reidl, Rossmanith and Sikdar (2014) for details.

Firebreak on graphs of bounded treewidth

Consider the MSO expression

$$\begin{aligned}\varphi \;=\; & (v_f \notin S) \wedge (v_f \notin X) \wedge (\forall y (y \in S) \Rightarrow (y \notin X)) \wedge \\ & (\forall x \forall y\, ((x \in X) \wedge (\mathrm{adj}(x, y) \wedge (y \notin S)) \Rightarrow (y \in X))\end{aligned}$$

The expression φ is true iff X is the set of vertices separated from v_f in $G - S$.

We need an evaluation relation ψ which is true iff $|S| = k$ and $|X| \geq t$.

Firebreak on graphs of bounded treewidth

The expression φ has two set variables (S and X).

Define a weight function $w_1 : V(G) \to \mathbb{R}$ by $w_1(v) = 1$ for all v and set

$$y_1 = \sum_{u \in S} w_1(u) \quad \text{and} \quad y_2 = \sum_{u \in X} w_1(u).$$

Note that $y_1 = |S|$ and $y_2 = |X|$.

We may also use numbers from the problem instance in the evaluation relation. Set $y_3 = k$ and $y_4 = t$.

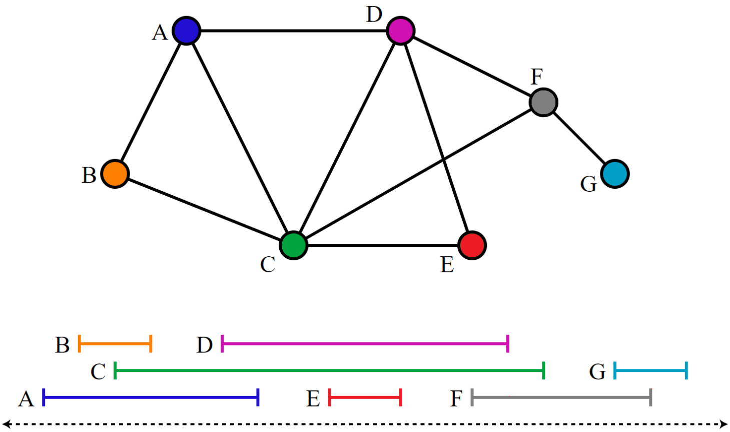
Define the evaluation relation ψ by

$$\psi : (y_1 = y_3) \wedge (y_2 \geq y_4).$$

Combining φ and ψ expresses FIREBREAK in EMSO.

Interval graphs

An interval graph is an intersection graph of a collection of intervals.



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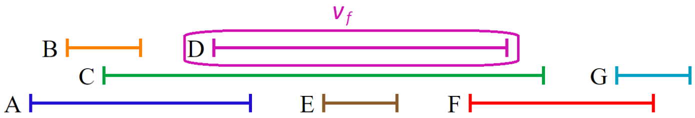

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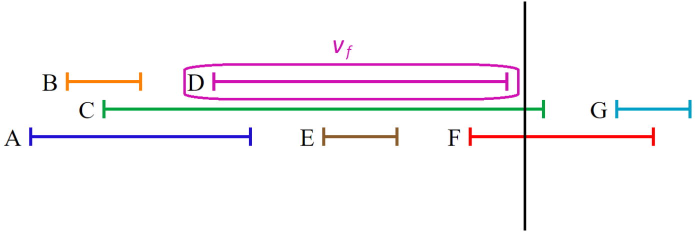
- Scan for minimal vertex separators ($\mathcal{O}(n^2)$)



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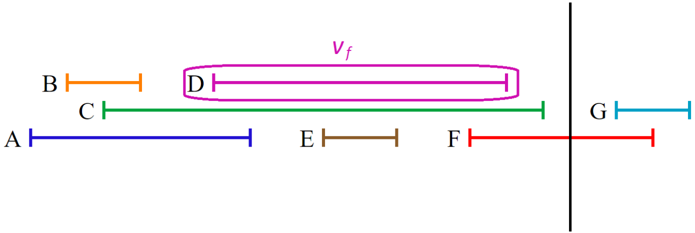

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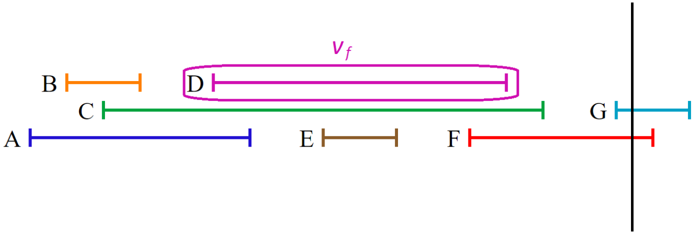

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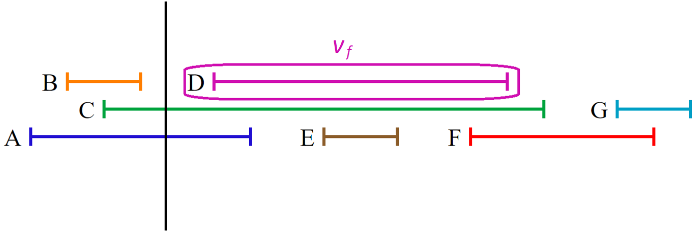

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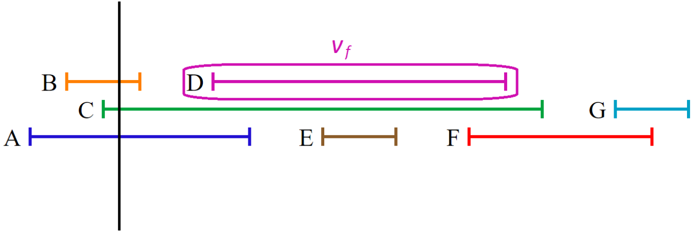

Solving FIREBREAK in polynomial time on interval graphs

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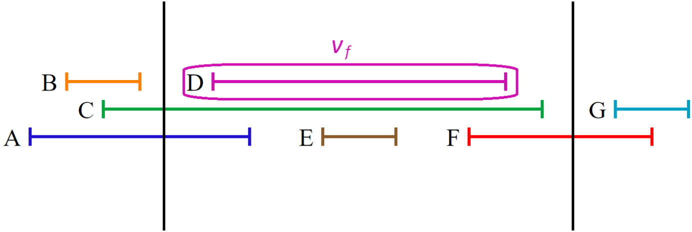

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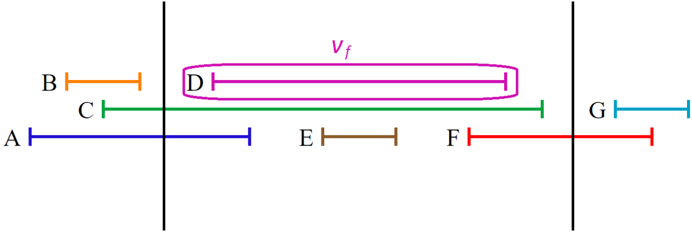

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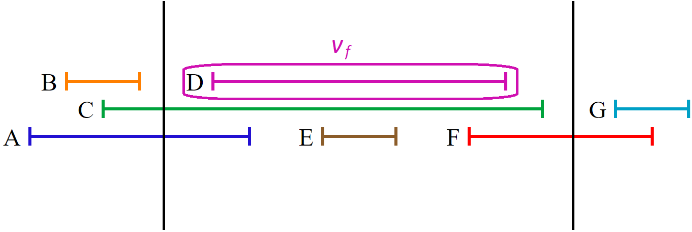
- So we need to check $\mathcal{O}(n^4)$ candidate solutions. Each can be checked in $\mathcal{O}(n)$ time.



Summary

- We have shown that FIREBREAK is:
  - NP-complete on:
    - split graphs
    - bipartite graphs
  - polynomial-time solvable on:
    - graphs of constant-bounded degree
    - graphs of constant-bounded treewidth
    - interval graphs
    - certain other classes of intersection graphs

Related questions

- In the firebreak problem, the firefighter has only one turn. In the firefighter problem, the number of firefighter turns is unrestricted. What if the number of firefighter turns is a fixed constant $m \geq 1$?

Related questions

- In the firebreak problem, the firefighter has only one turn. In the firefighter problem, the number of firefighter turns is unrestricted. What if the number of firefighter turns is a fixed constant $m \geq 1$?
- What if the fire can spread to all vertices within distance d at each timestep?
  - Firefighter: $d = 1$
  - Firebreak: $d = \infty$

Thanks!