This test paper has 5 questions and total of 40 marks.

It can not be taken from the examination room. You may use a non-programmable calculators.

SHOW ALL YOUR WORK. Duration: 50 minutes.

NAME:

STUDENT NO:

Marks

[8] **1)** Use Cramer's rule to solve the system

$$
\begin{aligned}
x + 2y &= 4 \\
-x + 2y &= 0
\end{aligned}
$$

Answer. We have

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix},\ A_1(b) = \begin{bmatrix} 4 & 2 \\ 0 & 2 \end{bmatrix},\ A_2(b) = \begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix}.$$

Thus,

$$\det A = 4, \qquad \det A_1(b) = 8, \qquad \det A_2(b) = 4.$$

Finally,

$$x = \frac{\det A_1(b)}{\det A} = 2 \qquad y = \frac{\det A_2(b)}{\det A} = 1.$$

[6] **2)** Let $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & -1 \\ -1 & 2 & -2 \end{bmatrix}$.

Find the cofactors C_{12}, C_{23}, C_{31} of A.

Answer.

$$C_{12} = -\begin{vmatrix} 2 & -1 \\ -1 & -2 \end{vmatrix} = 5, \quad C_{23} = -\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = -5, \quad C_{31} = \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix} = -7.$$

[10] **3)** Let $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & -1 & 3 & 2 \\ 0 & 2 & 6 & 6 \end{bmatrix}$.

(a) Find a basis for Col(A).
(b) Find a basis for Nul(A).
(c) Find rank(A) and dim(Nul(A)).

Answer: (a)

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & -1 & 3 & 2 \\ 0 & 2 & 6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 4 & 4 \\ 0 & 2 & 6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & -2 & -2 \end{bmatrix}.$$

Thus,

$$\text{Col}(A) = \text{ Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \right\}.$$

Answer: (b) We have:

$$x_4 = t, \quad x_3 = -t, \quad x_2 = 0, \quad x_1 = -t,$$

and a basis for Nul(A) is

$$\left\{ \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Answer: (c) We have rank(A) $= 3$ and dim(Nul(A)) $= 1$.

[6] **4)** Let A be a 5×7 matrix.

(a) What is the largest possible rank of A?

(b) Could A have a one-dimensional null space? Briefly explain.

Answer: **(a)** The largest possible rank of A is 5.

Answer: **(b)** No, since $\text{Nul}(A)$ has dimension at least 2.

[10] **5)** Let $A = \begin{bmatrix} 4 & 0 & 9 \\ -3 & 3 & 7 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Give the characteristic polynomial of A.
(b) Find the eigenvalues of A.
(c) Find the corresponding eigenspace for the smallest eigenvalue of A.
(d) Give the dimension of the eigenspace of Part **(c)** above.

Answer: **(a)** $(4-\lambda)(3-\lambda)(1-\lambda)$

Answer: **(b)** $\lambda = 4, 3, 1$

Answer: **(c)** The smallest eigenvalue is 1. Thus, we study the null space of $A - \lambda I = A - I$.

$$A - I = \begin{bmatrix} 3 & 0 & 9 \\ -3 & 2 & 7 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 16 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, we have:

$$x_3 = t, \quad x_2 = -8t, \quad x_1 = -3t,$$

and a basis for the eigenspace corresponding to the eigenvalue 1 of A is

$$\left\{ \begin{pmatrix} -3 \\ -8 \\ 1 \end{pmatrix} \right\}.$$

Answer: **(d)** The dimension of the eigenspace of Part (c) is 1.