This test paper has 5 questions and total of 40 marks.

It can not be taken from the examination room. You may use a non-programmable calculators.

SHOW ALL YOUR WORK. Duration: 50 minutes.

NAME:

STUDENT NO:

Marks

[8] **1)** Solve the following systems of linear equations. Write down which elementary row operations you used at each stage. If there are an infinite number of solutions, find the parametric solutions in vector form and give one particular solution. Write no solution if applicable.

a)

$$x + y + z = 1$$
$$2x + 4y - 3z = 2$$
$$4x + 6y - z = 3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 4 & -3 & 2 \\ 4 & 6 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & -5 & 0 \\ 0 & 2 & -5 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -5/2 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right].$$

Therefore, the system has no solutions.

b)

$$2x - 4z = -10$$
$$3y + 9z = 12$$
$$4x + 6y + 10z = 4$$

$$\left[\begin{array}{ccc|c} 2 & 0 & -4 & -10 \\ 0 & 3 & 9 & 12 \\ 4 & 6 & 10 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 0 & 3 & 9 & 12 \\ 0 & 6 & 18 & 24 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

thus, there are an infinite number of solutions. Parametric solutions:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 + 2t \\ 4 - 3t \\ t \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

Particular solution: for instance for $t = 0$ we have $x = -5, y = 4, z = 0$.

[6] **2)** Determine for which value(s) of k the following system will have

a) no solutions, **b)** infinitely many solutions, **c)** exactly one solution:

$$\begin{aligned} x - 2y + 3z &= 1 \\ 2x - 3y + 4z &= 2 \\ 3x - 4y + 5z &= k \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 2 & -3 & 4 & 2 \\ 3 & -4 & 5 & k \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & -4 & k-3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & k-3 \end{array}\right].$$

Thus, if $k - 3 \neq 0$, that is, if $k \neq 3$, there are no solutions. For $k = 3$ there are infinitely many solutions.

Answers: a) $k \neq 3$, **b)** $k = 3$, **c)** no value of k.

[9] **3)** Determine whether the following sets of vectors are linearly independent. Briefly explain.

a) $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\7\end{bmatrix}\right\}$; **b)** $\left\{\begin{bmatrix}7\\6\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$; **c)** $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$

a): No. For instance:

$$(-5)\begin{bmatrix}1\\0\end{bmatrix} + 7\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\7\end{bmatrix}.$$

b): Yes. The following system has only the trivial solution:

$$\begin{aligned} 7x + y &= 0 \\ 6x + 2y &= 0 \end{aligned}$$

c): No. The zero vector is an element of the set so it must be linearly dependent.

[8] **4)** Let $u = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}, v = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 0 \end{bmatrix}, w = \begin{bmatrix} 0 \\ -4 \\ 7 \\ -6 \end{bmatrix}$.

a) Show that $\{u, v, w\}$ is linearly dependent.
b) Is w in the span of $\{u, v\}$? If yes, write w as a linear combination of u and v; otherwise, briefly explain.

a)

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 2 & 2 & -4 & 0 \\ -1 & 4 & 7 & 0 \\ 2 & 0 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, the are infinitely many solutions given by $x = 3t, y = -t, z = t$, for $t \in R$. For instance, when $t = 1$ we have $x = 3, y = -1, z = 1$, and we have

$$3 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 2 \\ 4 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ -4 \\ 7 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

b) Yes since $w = -3u + v$ using part **a)** above.

[9] **5)** Decide whether the following transformations from R^2 to R^3 are linear. If the transformation is linear, prove it. If it is not linear briefly explain your answer.

a) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ -x \\ 2y \end{bmatrix}$; **b)** $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ xy \\ y \end{bmatrix}$; **c)** $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+3 \\ y+3 \\ x+y \end{bmatrix}$

a) Yes. $T\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} w \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} x+w \\ y+z \end{bmatrix}\right) = \begin{bmatrix} x+w+y+z \\ -(x+w) \\ 2(y+z) \end{bmatrix}$

$$= \begin{bmatrix} x+y \\ -x \\ 2y \end{bmatrix} + \begin{bmatrix} z+w \\ -w \\ 2z \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) + T\left(\begin{bmatrix} w \\ z \end{bmatrix}\right).$$

ii) $T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = \begin{bmatrix} cx+cy \\ -cx \\ 2cy \end{bmatrix} = c\begin{bmatrix} x+y \\ -x \\ 2y \end{bmatrix} = cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right).$

By i) and ii) above, T is a linear transformation.

b) No: the component xy destroys the linearity. For instance we have:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} w \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} x+w \\ y+z \end{bmatrix}\right) = \begin{bmatrix} x+w \\ (x+w)(y+z) \\ y+z \end{bmatrix}$$

$$\neq \begin{bmatrix} x \\ xy \\ y \end{bmatrix} + \begin{bmatrix} w \\ zw \\ z \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) + T\left(\begin{bmatrix} w \\ z \end{bmatrix}\right).$$

c) No: the $+3$ destroys the linearity by a similar argument as in part **b)**.