SYSTEMS OF LINEAR EQUATIONS

A system of linear equations (or a linear System) is a collection of one or more linear equations involving the same set of variables, say $x_1, x_2, ...x_n$.

Example:

$$2x_1 - x_2 - 3x_3 = -1$$
$$-2x_1 + 2x_2 + 5x_3 = 3.$$

$(x_1, x_2, x_3) = (1, 0, 1)$ is a solution for this system.

$(x_1, x_2, x_3) = (2, -4, 3)$ is also a solution for this system.

Verify: $2 \cdot 1 - 0 - 3 \cdot 1 = -1$ and $-2 \cdot 1 + 2 \cdot 0 + 5 \cdot 1 = 3$,

Also, $2 \cdot 2 - (-4) - 3 \cdot 3 = -1$ and $-2 \cdot 2 + 2(-4) + 5 \cdot 3 = 3$.

The set of all possible solutions is called the solution set of the linear system.
Two linear systems are called equivalent if they have the same solution set.

Geometrically, solution of two linear equations in two variables is the intersection of two lines.

Example: Give a geometric representation of the following system of equations.

$$l_1: \quad x + y = 3$$
$$l_2: \quad x - y = 1$$

Solution: $(2, 1)$ is the intersection point, and it is the only solution.
In this case, we say that there is a unique solution.

Example: Give a geometric representation of the following system of equations.

$$\begin{aligned} l_1 : &\quad -x + 2y = 1 \\ l_2 : &\quad x - 2y = -3 \end{aligned}$$

Solution: Lines are parallel, no intersection. That is, there is no solution of the given system.

Example: Give a geometric representation of the following system of equations.

$$\begin{aligned} l_1: \quad & 2x - y = 1 \\ l_2: \quad & -4x + 2y = -2 \end{aligned}$$

Solution: l_1 and l_2 coincide. There are infinitely many solutions.

$(x, y) = (1, 1)$ is a solution.
$(x, y) = (2, 3)$ is a solution.
$(x, y) = (3, 5)$ is a solution.

The general solution is
$x = t,\ y = -1 + 2t,\ t \in R$.

A system of linear equations has either

1) No solution, or

2) (Unique) Exactly one solution, or

3) Infinitely many solutions.

A system of linear equations is called consistent if it has at least one solution, and inconsistent if it has no solution.

Example: Solve the following system of linear equations.

$$
\begin{aligned}
x + 2y &= 4 \\
-x + 3y + 3z &= -2 \\
y + z &= 0.
\end{aligned}
$$

Solution: The augmented matrix for this linear system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ -1 & 3 & 3 & -2 \\ 0 & 1 & 1 & 0 \end{array}\right].$$ Its size is 3×4.

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$ is the coefficient matrix.
Its size is 3×3.

We start with the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ -1 & 3 & 3 & -2 \\ 0 & 1 & 1 & 0 \end{array}\right] R_2^{'} = R_2 + R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 5 & 3 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right] R_2 \longleftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 5 & 3 & 2 \end{array}\right] R_3^{'} = R_3 + (-5)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 2 \end{array}\right] R_3^{'} = (-\frac{1}{2})R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] R_2' = R_2 + (-1)R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] R_2' = R_2 + (-1)R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] R_1' = R_1 + (-2)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \text{ (RREF)}$$

The system is consistent and has a unique solution $(x, y, z) = (2, 1, -1)$.

Example: Determine whether the following linear system has a solution.

$$
\begin{aligned}
x + 2y + z &= 3 \\
x - y + z &= 1 \qquad (*) \\
-2x - 4y - 2z &= 4
\end{aligned}
$$

Solution: Its augmented matrix is

$$
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ -2 & -4 & -2 & 4 \end{array}\right]
\begin{array}{l} R_2^{'} = R_2 + (-1)R_1 \\ R_3^{'} = R_3 + 2R_1 \end{array}
$$

$$
\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & 0 & -2 \\ 0 & 0 & 0 & 10 \end{array}\right]
$$

In equation notation:

$$\begin{aligned} x + 2y + z &= 3 \\ 0 \cdot x - 3y + 0 \cdot z &= -2 \\ 0 \cdot x + 0 \cdot y + 0 \cdot z &= 10 \end{aligned}$$

It gives $0 = 10$, which is impossible. So the system is inconsistent, i.e, the system does not have any solutions.

Example:

$$\begin{aligned} x + 2y - 3z &= 3 \\ -2x - 5y + 4z &= 5 \\ -5x - 13y + 9z &= 18 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ -2 & -5 & 4 & 5 \\ -5 & -13 & 9 & 18 \end{array}\right] \begin{array}{l} R_2' = R_2 + 2R_1 \\ R_3' = R_3 + 5R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -1 & -2 & 11 \\ 0 & -3 & -6 & 33 \end{array}\right] R_2' = (-1)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & 1 & 2 & -11 \\ 0 & -3 & -6 & 33 \end{array}\right] R_3' = R_3 + 3R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & 1 & 2 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right] \; R_1' = R_1 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -7 & 25 \\ 0 & 1 & 2 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

In equation form:

$$\begin{aligned} x - 7z &= 25 \\ y + 2z &= -11 \end{aligned}$$

Set $z = t$, where t is any real number.

Then,

$y = -11 - 2t$ and $x = 25 + 7t$.

So, the general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 25 + 7t \\ -11 - 2t \\ t \end{bmatrix} = \begin{bmatrix} 25 \\ -11 \\ 0 \end{bmatrix} + \begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix} t \;,$$

$t \in R$. (Here, the general solution is given in parametric vector form.)

The system has infinitely many solutions, consistent.

We can write the system (*) in the form

$$\begin{bmatrix} 1 & 2 & -3 \\ -2 & -5 & 4 \\ -5 & -13 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 18 \end{bmatrix} \quad (**)$$

OR

$$x \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + y \begin{bmatrix} 2 \\ -5 \\ -13 \end{bmatrix} + z \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 18 \end{bmatrix} (***).$$

Solving the linear system (*) is the same as solving the matrix equation (**), or vector equation (***).

Example: Find the value of the constant k such that the following system has

i) no solution,

ii) infinitely many solutions,

iii) unique solution.

$$\begin{aligned} x + 2y - z &= 1 \\ -2x - 3y + 2z &= -1 \\ -5x - 8y + 5z &= k \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ -2 & -3 & 2 & -1 \\ -5 & -8 & 5 & k \end{array}\right] \begin{array}{l} R_2' = R_2 + 2R_1 \\ R_3' = R_3 + 5R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 5+k \end{array}\right] \begin{array}{l} R_1^{'} = R_1 - 2R_2 \\ R_3^{'} = R_3 - 2R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3+k \end{array}\right]$$

i) If $k \neq -3$, there is no solution.

ii) If $k = -3$, there are infinitely many solutions,

iii) There is no k such that the system has a unique solution.

Definition: If $v_1, v_2, v_3, ..., v_p$ are vectors in R^n, and $c_1, c_2, c_3, ..., c_p$ are scalars, then the vector x defined by

$$x = c_1 v_1 + c_2 v_2 + c_3 v_3 + \cdots + c_p v_p$$

is called a linear combination of $v_1, v_2, v_3, ..., v_p$ using weights $c_1, c_2, c_3, ..., c_p$.

Definition: The set of all linear combinations of $v_1, v_2, v_3, ..., v_p$ is called the span of $v_1, v_2, v_3, ..., v_p$
and denoted by Span$\{v_1, v_2, v_3, ..., v_p\}$.

Note: Every scalar multiple of v_1 (for example) is in Span$\{v_1, v_2, v_3, ..., v_p\}$, because

$$cv_1 = cv_1 + 0 \cdot v_2 + 0 \cdot v_3 + \cdots + 0 \cdot v_p.$$

The zero vector is always in

Span$\{v_1, v_2, v_3, ..., v_p\}$.

The only vectors in Span$\{v_1\}$ are multiples of v_1.

Example: Determine if the vector $b = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$ is a linear combination of the vectors $a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $a_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$, $a_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}$.

Solution:

$$x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} + z \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{array}\right] R_3{}' = R_3 + (-1)R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & -2 \end{array}\right]. \quad (REF)$$

By Back-Substitution:

$11z = -2$ gives $z = -2/11$.

$3y + 7z = -5,$

$3y = -5 - 7z$

$= -5 - 7 \cdot (-2/11)$

$= -5 + (14/11). \implies y = -41/33$

$x - 2y - 6z = 11 \implies x = 11 + 2y + 6z.$

$x = 11 + 2(-41/33) + 6(-2/11)$

$\implies x = 245/33.$

Check:

$$\frac{245}{33}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{-41}{33}\begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} + \frac{-2}{11}\begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}.$$

Example: Determine if $b = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ is a linear combination of the column vectors of the matrix

$$A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}.$$

Solution: b is a linear combination of the columns of A if and only if the equation

$$x \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + z \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} \quad (*)$$

has a solution. We know that solving the equation (*) is the same as solving the matrix equation

$$\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}. \quad (**)$$

So,

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right] \; R_2' = R_2 + 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{array}\right] \; R_3' = R_3 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad\qquad .$$

There are infinitely many solutions, and so the vector b is a linear combination of the column vectors of the matrix A. The general solution is

$$z = t, y = 3 - 4t, x = 2 - 5t, t \in R.$$

Choose $t = 0$. Then,

$$2\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}.$$

Choose t=1. Then,

$$-3\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}.$$

Example: For what value(s) of h is $y = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}$ in the plane generated by $v_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{cc|c} 1 & -2 & h \\ 0 & 1 & -3 \\ -2 & 7 & -5 \end{array}\right] R_3{}' = R_3 + 2R_1$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 3 & 2h-5 \end{array}\right] R_3{}' = R_3 + (-3)R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 0 & 2h+4 \end{array}\right],$$

which is consistent if and only if $2h + 4 = 0$. So, if $h = -2$, then y is in the plane generated by v_1 and v_2.

Note: If $h = -2$, then $-8v_1 - 3v_2 = y$, that is,

$$-8\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - 3\begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ -5 \end{bmatrix}.$$

SOME TERMS THAT WE USE

pivot position: a position of a leading entry in an echelon form of a matrix.

pivot: a non-zero number that is in a pivot position.

pivot column: a column that contains a pivot position.

basic variable: a variable that corresponds to a pivot column.

free variable: a variable that is not a basic variable, i.e, a variable that corresponds to a non-pivot column.

Example: Solve the following two systems of linear equations

$$\begin{aligned} 2x + 4y - 6z &= 2 \\ y + 3z &= 5 \\ -3x - 5y + 7z &= -3 \end{aligned}$$

and

$$\begin{aligned} 2x + 4y - 6z &= 0 \\ y + 3z &= 4 \\ -3x - 5y + 7z &= -1, \end{aligned}$$

i.e, solve the two matrix equations

$$AX = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} \text{ and } AX = \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}, \text{ where}$$

$$A = \begin{bmatrix} 2 & 4 & -6 \\ 0 & 1 & 3 \\ -3 & -5 & 7 \end{bmatrix}.$$

Solution: The augmented matrix for the first system is

$$\left[\begin{array}{ccc|c} 2 & 4 & -6 & 2 \\ 0 & 1 & 3 & 5 \\ -3 & -5 & 7 & -3 \end{array}\right]$$

and the augmented matrix for the second system is

$$\left[\begin{array}{ccc|c} 2 & 4 & -6 & 0 \\ 0 & 1 & 3 & 4 \\ -3 & -5 & 7 & -1 \end{array}\right].$$

We can combine these two augmented matrices as follows:

$$\left[\begin{array}{ccc|cc} 2 & 4 & -6 & 2 & 0 \\ 0 & 1 & 3 & 5 & 4 \\ -3 & -5 & 7 & -3 & -1 \end{array}\right].$$

$$\left[\begin{array}{ccc|cc} 2 & 4 & -6 & 2 & 0 \\ 0 & 1 & 3 & 5 & 4 \\ -3 & -5 & 7 & -3 & -1 \end{array}\right] R_1{}' = (1/2)R_1$$

$$\sim \left[\begin{array}{ccc|cc} 1 & 2 & -3 & 1 & 0 \\ 0 & 1 & 3 & 5 & 4 \\ -3 & -5 & 7 & -3 & -1 \end{array}\right] R_3{}' = R_3 + 3R_1$$

$$\sim \left[\begin{array}{ccc|cc} 1 & 2 & -3 & 1 & 0 \\ 0 & 1 & 3 & 5 & 4 \\ 0 & 1 & -2 & 0 & -1 \end{array}\right] \begin{array}{l} R_1{}' = R_1 + (-2)R_2 \\ R_3{}' = R_3 + (-1)R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|cc} 1 & 0 & -9 & -9 & -8 \\ 0 & 1 & 3 & 5 & 4 \\ 0 & 0 & -5 & -5 & -5 \end{array}\right] R_3{}' = (-1/5)R_3$$

$$\sim \left[\begin{array}{ccc|cc} 1 & 0 & -9 & -9 & -8 \\ 0 & 1 & 3 & 5 & 4 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} R_1{}' = R_1 + 9R_3 \\ R_2{}' = R_2 + (-3)R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right].$$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ is the solution for the first system and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the solution for the second system.

Example: solve the two matrix equations $Ax = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$ and $Ax = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -2 \\ -2 & 3 & 3 \end{bmatrix}.$$

Solution:

$$\left[\begin{array}{ccc|cc} 1 & 2 & -1 & 4 & -2 \\ 3 & 7 & -2 & 1 & 5 \\ -2 & 3 & 3 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|cc} 1 & 0 & 0 & -16 & 15 \\ 0 & 1 & 0 & 3 & -2 \\ 0 & 0 & 1 & -14 & 13 \end{array}\right].$$

The solution for the first system is $x = \begin{bmatrix} -16 \\ 3 \\ -14 \end{bmatrix}$ and the solution for the second system is $x = \begin{bmatrix} 15 \\ -2 \\ 13 \end{bmatrix}$.

Theorem: The following statements are equivalent for any $m \times n$ matrix A.

i) For each b in R^m, the equation $Ax = b$ has a solution.

ii) The columns of A span R^m.

iii) A has a pivot position in every row.

"The columns of A span $R^m \iff$ Any vector in R^m can be written as a linear combination of the columns of A."