MATRIX OPERATIONS

Example: Let

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}.$$

$$A + B = \begin{bmatrix} -1+1 & 0+2 \\ 2+3 & 3+0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 5 & 3 \end{bmatrix}$$

$$= B + A.$$

$$AB = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \cdot 1 + 0 \cdot 3 & -1 \cdot 2 + 0 \cdot 0 \\ 2 \cdot 1 + 3 \cdot 3 & 2 \cdot 2 + 3 \cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot (-1) + 2 \cdot 2 & 1 \cdot 0 + 2 \cdot 3 \\ 3 \cdot (-1) + 0 \cdot 2 & 3 \cdot 0 + 0 \cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

We see that

$A + B = B + A$ but $AB \neq BA$.

$$(-5)A = -5 \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -5 \cdot (-1) & -5 \cdot 0 \\ -5 \cdot 2 & -5 \cdot 3 \end{bmatrix} =$$

Example: Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

$$AC = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

We have $AB = AC$ but $B \neq C$. A does not cancel.

Example: Let

$$A = \begin{bmatrix} 0 & 5 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 0 & 5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So, $AB = 0$ but $A \neq 0$, $B \neq 0$.

We denote an $m \times n$ matrix as

$$A = \begin{bmatrix} a_{1\ 1} & a_{1\ 2} & \cdots & a_{1\ n} \\ a_{2\ 1} & a_{2\ 2} & \cdots & a_{2\ n} \\ & \cdots & \cdots & \\ a_{m\ 1} & a_{m\ 2} & \cdots & a_{m\ n} \end{bmatrix} = (a_{i\ j})$$

Zero matrix is

$$O = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The identity matrix is

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

To add two matrices A and B, they must have the same size.

To multiply two matrices A and B, the number of columns of A must be the same as the number of rows of B.

Example: Let

$$A = \begin{bmatrix} 7 & 0 & -1 \\ -1 & 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 4 & 1 \\ 5 & -3 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 4 \\ -4 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 7 \\ -3 \end{bmatrix}.$$

Determine if each of the following matrices is defined:

$-2A,\ B+4A,\ AC,\ CD,\ A+B,$
$3C-E,\ CB,\ EB.$

Example: Let

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find A^2, A^3 and A^4.

Solution:

$$A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 1 & 1 \cdot (-1) + (-1) \cdot 1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$A^3 = AA^2 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Similarly, $A^4 = AA^3 = A^3A$.

Exercise: Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Find A^4. Make a guess for A^n.

Example: Let

$$A = \begin{bmatrix} 3 & -4 \\ -5 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 & 4 \\ 5 & k \end{bmatrix}.$$

For what value(s) of k, is $AB = BA$?

Solution:

$$\begin{bmatrix} 3 & -4 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ 5 & k \end{bmatrix} = \begin{bmatrix} 21-20 & 12-4k \\ -35+5 & -20+k \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 12-4k \\ -30 & -20+k \end{bmatrix}$$

$$\begin{bmatrix} 7 & 4 \\ 5 & k \end{bmatrix} \begin{bmatrix} 3 & -4 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -24 \\ 15 - 5k & -20 + k \end{bmatrix}$$

We obtain $15 - 5k = -30$. So, $k = 9$.

Example: Let

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \text{ and } AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}.$$

Determine the first column of B.

Solution:

$$\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} a \\ d \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -2 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 4 \end{array}\right].$$

So, $a = 7$ and $d = 4$, i.e,

$$\begin{bmatrix} a \\ d \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}.$$

Determine the second column of B. (Exercise).

The transpose of an $m \times n$ matrix A is the $n \times m$ matrix whose columns are the corresponding rows of A, and denoted by A^T.

Example: Let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 6 & 7 & 8 \\ 0 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}.$$

Then,

$$A^T = \begin{bmatrix} 3 & 6 & 0 \\ -1 & 7 & 1 \\ 2 & 8 & 5 \end{bmatrix}, \quad B^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}.$$

Note that

$$\left(A^T\right)^T = A,\ (A+B)^T = A^T + B^T,$$

$$(rA)^T = rA^T,\ (AB)^T = B^T A^T.$$

Example: Let

$$A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}.$$

Then,

$$(AB)^T = B^T A^T = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix},$$

$$= \begin{bmatrix} 2 & 4 \\ 10 & -1 \end{bmatrix}, \quad \text{and}$$

$$AB = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 10 \\ 4 & -1 \end{bmatrix}.$$

$$\text{Thus,} \quad (AB)^T = \begin{bmatrix} 2 & 4 \\ 10 & -1 \end{bmatrix}.$$

So, we have $(AB)^T = B^T A^T = \begin{bmatrix} 2 & 4 \\ 10 & -1 \end{bmatrix}$.