A FORMULA FOR A^{-1}

Let A be an invertible $n \times n$ matrix.

Let A_{ij} be the submatrix of A formed by deleting row i and column j.

$C_{ij} = (-1)^{i+j} \det A_{ij}$

is called the $(i, j) - cofactor$ of A.

The transpose of the matrix of cofactors from A is called the adjoint of A, i.e,

$$\text{adj}A = \begin{bmatrix} C_{11} & C_{21} & . & . & . & C_{n1} \\ C_{12} & C_{22} & . & . & . & C_{n2} \\ . & & & & & . \\ . & & & & & . \\ . & & & & & . \\ C_{1n} & C_{2n} & . & . & . & C_{nn} \end{bmatrix}.$$

$A^{-1} = \frac{1}{\det A}\text{adj}A$

Example: Find the inverse of the matrix

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}.$$

Solution:

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix} = 12,$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 6,$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 6 \\ 2 & -4 \end{vmatrix} = -16,$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 2 & -1 \\ -4 & 0 \end{vmatrix} = 4,$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = 2,$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 2 \\ 2 & -4 \end{vmatrix} = 16,$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 2 & -1 \\ 6 & 3 \end{vmatrix} = 12,$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} = -10,$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} = 16,$$

and so

$$\text{adj}A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}^T$$

$$= \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}.$$

$$\begin{aligned} \det A &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 3 \cdot 12 + 2 \cdot 6 + (-1) \cdot (-16) \\ &= 36 + 12 + 16 \\ &= 64. \end{aligned}$$

$$A^{-1} = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}.$$