INVERSE OF A MATRIX

Definition: Let A be an $n \times n$ square matrix. If there is a matrix B such that

$AB = BA = I_n$

then A is said to be invertible and B is called the inverse of A.

Theorem: Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a 2×2 matrix.

A is invertible $\Leftrightarrow ad - bc \neq 0$.

And,

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad-bc=0$, then A is not invertible. $ad-bc$ is called the determinant of A, and denoted by
$|A| = \text{det}A = ad-bc$.

Theorem: If A is an invertible $n \times n$ matrix then for each $b \in R^n$, the equation $AX = b$ has the unique solution

$$X = A^{-1}b.$$

Example: Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

We want to have a matrix, A^{-1}, such that $AA^{-1} = I_2$.

$ad - bc = 1 \cdot 4 - 3 \cdot 2 = 4 - 6 = -2 \neq 0,$

so A is invertible, and

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}, \quad \text{and}$$

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Example: Solve the equation

$$AX = \begin{bmatrix} -4 & -5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Solution:

$$\begin{aligned} |A| &= ad - bc \\ &= -4 \cdot 6 - (-5) \cdot 5 \\ &= -24 + 25 \\ &= 1 \neq 0, \end{aligned}$$

so A is invertible, and

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 & -5 \\ 5 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 5 \\ -5 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -9 \end{bmatrix}.$$

ELEMENTARY MARTRICES

An elementary matrix is obtained by performing a single row operation on an identity matrix.

Example: Consider the following matrices:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} R_2 \longleftrightarrow R_3$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 \longleftrightarrow R_3$$

Then,

$$A \sim \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix} \text{ and } I \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_1.$$

$$\begin{aligned} E_1 A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \\ &= \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}. \end{aligned}$$

ii) $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \; R_2' = R_2 + 2R_1$

$$A \sim \begin{bmatrix} a & b & c \\ 2a+d & 2b+e & 2c+f \\ g & h & i \end{bmatrix}.$$

Corresponding elementary matrix:

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

($R_2' = R_2 + 2R_1$: a_{21} entry in I_3 is 2).

$$E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$
$$= \begin{bmatrix} a & b & c \\ 2a + d & 2b + e & 2c + f \\ g & h & i \end{bmatrix}.$$

iii) $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $R_1' = 2R_1$. Then

$$A \sim \begin{bmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & i \end{bmatrix},$$

and the corresponding elementary matrix is

$$E_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

(E_3 is obtained by multiplying the first row of I_3 by 2), and

$$E_3 A = \begin{bmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Example:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \; E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$E_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_3^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example: Consider the following elementary matrices.

$$R_1' = R_1 - 3R_2: \quad E_1 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix},$$

$$R_2' = 4R_2: \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix},$$

$$R_1 \leftrightarrow R_2: \quad E_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then, $E_1^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, $E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}$

and $E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Example: Find the inverse of
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 4 & 2 & 0 \end{bmatrix}.$$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 4 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_3' = R_3 - 4R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -4 & 0 & 1 \end{array}\right] \quad R_2 \longleftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \quad {R_2}' = \frac{1}{2} R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right].$$

Thus $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$.

$E_1 : R_3' = R_3 - 4R_1, \quad a_{31} = -4$, and
$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$.

$E_2 : R_2 \longleftrightarrow R_3$, and
$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

$E_3 : R_2{}' = \frac{1}{2}R_2$. We multiply the second row of I by 1/2. So

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Indeed, $E_3E_2E_1A = I$, and
$A^{-1} = E_3E_2E_1$.

$$E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} = A^{-1}.$$

Example: Write $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and A^{-1}

as products of elementary matrices.

Solution: First we compute A^{-1}.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \quad (R_2{}' = R_2 - R_1)$$

$$\sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \quad (R_1{}' = R_1 - R_2)$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right].$$

Thus $A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

$$R_2{}' = R_2 - R_1 : \quad E_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

$$R_1{}' = R_1 - R_2 : \quad E_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} A &= (E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} A^{-1} &= E_2 E_1 \\ &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}. \end{aligned}$$

The Invertible Matrix Theorem

Let A be a square nxn matrix. Then the following statements are equivalent.

1) A is an invertible matrix.

2) A is row equivalent to I_n.

3) A has a pivot position in each row.

4) A has a pivot position in each column.

5) The equation $Ax = 0$ has only the trivial solution.

6) The columns of A form a linearly independent set.

7) The equation $Ax = b$ is consistent for every b in R^n.

8) The columns of A span R^n.

9) A^T is an invertible matrix.

10) There is an nxn matrix C such that $AC = CA = I$.

11) The linear transformation $T(x) = Ax$ is one-to-one.

12) The linear transformation $T(x) = Ax$ is onto.

Example: Use Invertible matrix theorem to decide if the matrix

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & 7 & 1 \\ 0 & 0 & 9 \end{bmatrix}$$

is invertible.

Solution: A has n pivot positions. A is row equivalent to I_3. So, A is invertible.

Remark: Invertible matrix theorem applies only to square matrices.

Example: Let

$$A = \begin{bmatrix} 1 & 3 & -5 \\ 0 & 2 & -3 \\ 0 & -4 & 7 \\ -1 & 5 & -8 \end{bmatrix} \quad R_4{}' = R_4 + R_1$$

$$\sim \begin{bmatrix} 1 & 3 & -5 \\ 0 & 2 & -3 \\ 0 & -4 & 7 \\ 0 & 8 & -13 \end{bmatrix} \quad \begin{matrix} R_3{}' = R_3 + 2R_2 \\ R_4{}' = R_4 - 4R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -5 \\ 0 & 2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} R_4{}' = R_4 + R_3$$

$$\sim \begin{bmatrix} 1 & 3 & -5 \\ 0 & 2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

There are three basic variables and no free variables. So, columns of A are linearly independent.

$AX = 0$ has only trivial solution. But for

$$b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

we have

$$\left[\begin{array}{rrr|r} 1 & 3 & -5 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & -4 & 7 & 0 \\ -1 & 5 & -8 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 3 & -5 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right],$$

which does not have any solutions.

Columns of A are linearly independent but they do not span R^4.

Exercise: Let

$$A = \begin{bmatrix} 2 & 5 \\ -1 & -3 \\ 2 & 4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Verify that $CA = I_2$. Is A invertible? Explain your answer.

Example: Let A be a matrix such that $A^3 = 0$. Use this to simplify $(I - A)(I + A + A^2)$, and then express $(I - A)^{-1}$ in terms of I and A.

Solution:

$$
\begin{aligned}
&(I - A)(I + A + A^2) \\
&= I + A + A^2 - A - A^2 - A^3 \\
&= I - A^3 = I.
\end{aligned}
$$

So, $(I - A)^{-1} = I + A + A^2$.

Properties of Invertible Matrices

Let A and B be $n \times n$ invertible matrices. Then,

$(A^{-1})^{-1} = A$, $(AB)^{-1} = B^{-1}A^{-1}$, $(kA)^{-1} = (1/k)A^{-1}$, $(A^T)^{-1} = (A^{-1})^T$.

Example: Suppose $PXQ = R$, where P and Q are invertible matrices. Express X^T in terms of P^T, Q^T and R^T.

Solution: From $(PXQ)^T = R^T$ we obtain $Q^T X^T P^T = R^T$. Then,
$X^T = (Q^T)^{-1} R^T (P^T)^{-1}$.

Exercise: Solve the equation $C^{-1}(A+X)B^{-1} = I$ for X, assuming that A, B and C are all $n \times n$ matrices, and that B and C are invertible.

INVERSE OF A LINEAR TRANSFORMATION

Let $T : R^n \longrightarrow R^n$ be a linear transformation. T is said to be invertible if there exists a linear transformation $S : R^n \longrightarrow R^n$ such that

$S(T(X)) = X$ and $T(S(X)) = X$

for all X in R^n.

Theorem: Let $T : R^n \longrightarrow R^n$ be a linear transformation, and let A be the standard matrix for T. Then, T is invertible if and only if A is invertible, and the inverse of T is given by $S(X) = A^{-1}X$.

Example: Let $T : R^2 \longrightarrow R^2$ be given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 6x_1 - 8x_2 \\ -5x_1 + 7x_2 \end{bmatrix}.$$

Show that T is invertible and find a formula for T^{-1}.

Solution:

$$A = \begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix}.$$

$$\begin{aligned} detA &= 6 \cdot 7 - (-5) \cdot (-8) \\ &= 42 - 40 = 2 \neq 0. \end{aligned}$$

A is invertible, and so T is invertible.

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 7/2 & 4 \\ 5/2 & 3 \end{bmatrix},$$ thus

$$S(X) = A^{-1}X = \begin{bmatrix} 7/2 & 4 \\ 5/2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} (7/2)x_1 + 4x_2 \\ (5/2)x_1 + 3x_2 \end{bmatrix}.$$

Verify:

$$T(S(X)) = T(S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$
$$= T\left(\begin{bmatrix} (7/2)x_1 + 4x_2 \\ (5/2)x_1 + 3x_2 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix} \begin{bmatrix} (7/2)x_1 + 4x_2 \\ (5/2)x_1 + 3x_2 \end{bmatrix}$$
$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = X.$$

Similarly, $S(T(X)) = X$ as well.