# INNER PRODUCT, LENGTH, and ORTHOGONALITY

**Definition:** Let

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \text{ and } v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

be two vectors in $R^n$. Then the inner product of $u$ and $v$ is

$$u \cdot v = u^T v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

**Example:** Let $u = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$, $v = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$.

Then

$$u \cdot v = u^T v = [3 \ \ -1 \ \ -5] \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

$$= 3 \cdot 6 + (-1) \cdot (-2) + (-5) \cdot 3$$

$$= 18 + 2 - 15 = 5, \text{ and}$$

$$v \cdot u = v^T u = [6 \ \ -2 \ \ 3] \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$$

$$= 6 \cdot 3 + (-2)(-1) + 3 \cdot (-5)$$

$$= 18 + 2 - 15 = 5.$$

So, $u \cdot v = v \cdot u$

**Properties of the inner product**

Let $u$, $v$, and $w$ be vectors in $R^n$ and $c$ be a scalar. Then,

**1)** $u \cdot v = v \cdot u$

**2)** $(u + v) \cdot w = u \cdot w + v \cdot w$

**3)** $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$

**4)** $u \cdot u \geq 0$ and $u \cdot u = 0 \iff u = 0$

**Definition (the length of a vector):**

If $v = (v_1, v_2, \cdots, v_n)$ in $R^n$, then the length (or norm) of $v$ is defined by

$$||v|| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Then, $||v||^2 = v \cdot v$. A vector whose length is one unit is called a unit vector.

**Example:** Let $v = (2, -3, 1)$. Find the length of $v$ and a unit vector in the direction of $v$.

**Solution:**

$$\begin{aligned} ||v||^2 &= v \cdot v \\ &= 2^2 + (-3)^2 + 1^2 = 4 + 9 + 1 = 14, \end{aligned}$$

and so, $||v|| = \sqrt{14}$. A unit vector in the direction of $v$ is

$$\frac{1}{\sqrt{14}}(2, -3, 1) = \left(\frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right) = u.$$

The process of creating $u$ from $v$ is called normalizing $v$.

**Example:** Let $W$ =Span$\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right\}$. Find a unit vector $v$ which is a basis for $W$.

**Solution:** let $u = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Then,

$$||u|| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}.$$

$$v = \frac{1}{\sqrt{6}} u = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}.$$

Another unit vector is

$$-v = \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}.$$

**Definition (Distance between $u$ and $v$):** Let $u, v$ in $R^n$. Then the distance between $u$ and $v$ is the length of the vector $u - v$. That is

$$dist(u, v) = ||u - v||.$$

**Example:** Let $u = (1, 2, -1)$ and $v = (2, 1, 1)$ in $R^3$. Then

$$dist(u, v) = ||u - v|| = ||(-1, 1, -2)||$$

$$= \sqrt{(-1)^2 + 1^2 + (-2)^2} = \sqrt{6}.$$

**Definition (Orthogonal vectors):** Let $u$ and $v$ be two vectors in $R^n$. Then $u$ and $v$ are orthogonal to each other ($u \perp v$) if $u \cdot v = 0$. Note that 0=zero vector is orthogonal to every vector.

**Example:** Let $u$ and $v$ in $R^n$. Show that

$$u \perp v \Longleftrightarrow ||u+v||^2 = ||u||^2 + ||v||^2.$$

**Solution: (1)** Let $u \perp v$. Then $u \cdot v = 0$, and so,

$$||u+v||^2 = (u+v)\cdot(u+v)$$

$$= u \cdot u + u \cdot v + v \cdot u + v \cdot v$$

$$= ||u||^2 + ||v||^2$$

**(2)** Let $||u+v||^2 = ||u||^2 + ||v||^2$.

Then we need to show that $u \perp v$, i.e, $u \cdot v = 0$.

$$||u+v||^2 = (u+v)\cdot(u+v)$$

$$= u\cdot u + u\cdot v + v\cdot u + v\cdot v$$

$$= ||u||^2 + ||v||^2 + 2u\cdot v$$

So,

$$||u+v||^2 = ||u||^2 + ||v||^2 \Longleftrightarrow u\cdot v = 0.$$

By (1) and (2),

$$u \perp v \Longleftrightarrow ||u+v||^2 = ||u||^2 + ||v||^2.$$

"The Phythagorean Theorem"

**Definition:** Let $W$ be a subspace of $R^n$.

**1)** If $z \in R^n$ is orthogonal to every vector in $W$, then $z$ is said to be orthogonal to $W$.

**2)** The set of all $z$ that are orthogonal to $W$ is called the orthogonal complement of $W$, and is denoted by $W^T$. That is ,

$$W^T = \{z \in R^n \mid z \perp W\}.$$

**Example:** Let $S = \{(1,2,1),(1,-1,-1)\}$. Find $S^{\perp}$.

**Solution:** A vector

$$v = (x, y, z) \in S^{\perp} \Longleftrightarrow v \cdot (1,2,1) = 0 \text{ and } v \cdot (1,-1,-1) = 0.$$

So,

$$(x, y, z) \cdot (1,2,1) = 0 \text{ and } (x, y, z) \cdot (1,-1,-1) =$$

$$x + 2y + z = 0 \text{ and } x - y - z = 0;$$

which gives that

$$z = t, \ , y = -2/3t, \ x = 1/3t.$$

$$v = (x, y, z) = (\frac{1}{3}t, \frac{-2}{3}, t) = \frac{1}{3}t(1,-2,3)$$

$$= s(1,-2,3),$$

Thus, $S^{\perp} = \text{Span}\,\{(1,-2,3)\}$

**Example:** Suppose that a vector $y$ is orthogonal to vectors $u$ and $v$. Show that $y$ is orthogonal to $u + v$.

**Solution:** let $y \perp u$ and $y \perp v$. Then, $y \cdot u = 0$ and $y \cdot v = 0$.

In order to show that $y$ is orthogonal to $u + v$, we need to verify $y \cdot (u + v) = 0$.

$$y \cdot (u + v) = y \cdot u + y \cdot v = 0 + 0 = 0.$$

So, $y$ is orthogonal to $u + v$.

**Example:** Suppose the vectors $u$ and $v$ are orthogonal to the vector $z$. Show that $u + v$ is orthogonal to $z$.

**Solution:** Let $u \perp z$ and $v \perp z$. Then,

$$(u + v) \cdot z = u \cdot z + v \cdot z = 0 + 0 = 0,$$

which means that $u + v \perp z$.

**Theorem:** **1)** A vector $x$ is in $W^{\perp}$ if and only if $x$ is orthogonal to every vector in a set that spans $W$.

**2)** $W^{\perp}$ is a subspace of $R^n$.

**Theorem:** Let $A$ be an $n \times n$ matrix. Then,

$$(\text{Row}A)^{\perp} = \text{Null}A$$

$$(\text{Col}A)^{\perp} = \text{Null}A^{T}$$

**Proof:** **ii)** In i), replacing $A$ by $A^T$ gives,

$$(\text{Row}A^{T})^{\perp} = \text{Null}A^{T}$$

$$(\text{Col}A)^{\perp} = \text{Null}A^{T}.$$

**Definition:** A set of non-zero vectors $S$ is called an orthogonal set if each vector in $S$ is orthogonal to other vectors in $S$, i.e,

$$S = \{v_1, v_2, \cdots, v_p\} \text{ is an orthogonal set}$$

$$\iff v_i \cdot v_j = 0 \text{ if } i \neq j.$$

An orthogonal set in which each vector has length 1 is called an orthonormal set.

A basis consisting of orthogonal vectors is called an orthogonal basis.

A basis consisting of orthonormal vectors is called an orthonormal basis.

**Example:** $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

is an orthonormal basis for $R^3$.

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 5/7 \\ 4/7 \\ 1/7 \end{bmatrix} \right\}$$

is an orthogonal basis for $R^3$.

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\}$$

is an orthogonal basis for $R^3$.

$$\left\{ \begin{bmatrix} 2 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is an orthogonal set in $R^4$.

**Theorem:** If $S = \{u_1, u_2, ..., u_p\}$ is an orthogonal set of nonzero vectors in $R^n$, then $S$ is linearly independent.

**Proof:** Let $0 = c_1u_1 + c_2u_2 + \cdots + c_pu_p$ for some scalars $c_1, c_2, ..., c_p$. Then,

$$0 \cdot u_1 = (c_1u_1 + c_2u_2 + \cdots + c_pu_p) \cdot u_1$$

$$0 = c_1u_1 \cdot u_1 + c_2u_2 \cdot u_1 + \cdots + c_pu_p \cdot u_1,$$

$$0 = c_1u_1 \cdot u_1 + 0 + \cdots + 0,$$

since $u_i \cdot u_j = 0$ if $i \neq j$.

$$0 = c_1||u_1||^2 \text{which gives } c_1 = 0$$

since $u_1 \neq 0$. Similarly, we can show that $c_2 = 0,\ldots,c_p = 0$. Thus, $S$ is linearly independent.

**Theorem:** Let $\{u_1, u_2, ..., u_p\}$ be an orthogonal basis for $R^n$. Then for each $x \in R^n$,

$$x = c_1u_1 + c_2u_2 + \cdots + c_nu_n \text{ where}$$

$$c_1 = \frac{x \cdot u_i}{u_i \cdot u_i}, \quad i = 1, 2, ..., n.$$

**Proof:** For a fixed $i$, $1 \le i \le n$,

$$\begin{aligned} x \cdot u_i &= (c_1u_1 + c_2u_2 + \cdots + c_nu_n) \cdot u_i \\ &= c_i(u_i \cdot u_i), \end{aligned}$$

which gives

$$c_i = \frac{x \cdot u_i}{u_i \cdot u_i}$$

(since $u_i \neq 0$, $u_i \cdot u_i \neq 0$)

**Example:** Show that the set

$$S = \left\{ u_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, u_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\}$$

is an orthogonal basis for $R^3$. Express $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as a linear combination of the vectors in $S$.

**Solution:** Check that $u_1 \cdot u_2 = 0$, $u_1 \cdot u_3 = 0$, $u_2 \cdot u_3 = 0$. Then,

$$x = c_1 u_1 + c_2 u_2 + c_3 u_3$$

$$c_1 = \frac{x \cdot u_1}{u_1 \cdot u_1} = \frac{-1+4+6}{1+4+4} = \frac{9}{9} = 1,$$

$$c_2 = \frac{x \cdot u_2}{u_2 \cdot u_2} = \frac{2-2+6}{4+1+4} = \frac{6}{9} = \frac{2}{3},$$

$$c_3 = \frac{x \cdot u_3}{u_3 \cdot u_3} = \frac{2+4-3}{4+4+1} = \frac{3}{9} = \frac{1}{3}.$$

Thus,

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}.$$