EIGENVALUES AND EIGENVECTORS

Definition: Let A be an $n \times n$ matrix. A non-zero vector X in R^n is called an eigenvector of A if AX is a scalar multiple of X, i.e,

$$AX = \lambda X$$

for some scalar λ. The scalar λ is called an eigenvalue of A, and X is said to be an eigenvector corresponding to λ.

Example: Let

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}, X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{and } Y = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Are X and Y eigenvectors of A?

Solution:

$$AX = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3X,$$

which says that $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 3$.

$$AY = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -10 \end{bmatrix} \neq \mu \begin{bmatrix} -1 \\ 2 \end{bmatrix},$$

i.e, AY is not a multiple of Y. So Y is not an eigenvector of A.

λ is an eigenvalue of A

$\iff AX = \lambda X$

for some non-zero vector X

$\iff AX - \lambda X = 0$

$\iff (A - \lambda I)X = 0$

for some non-zero vector X

$\iff \det(A - \lambda I) = 0.$

$\det(A - \lambda I) = 0$ is called the characteristic equation of A, the scalars λ satisfying this equation are the eigenvalues of A.

When expanded, the determinant $\det(A-\lambda I)$ is a polynomial in λ, it has degree n, and it is called the characteristic polynomial of A.

Definition: The set of all solutions of $(A - \lambda I)X = 0$ is the nullspace of the matrix $A - \lambda I$. This set, which is a subspace, is called the eigenspace of A corresponding to λ.

Note: Eigenspace contains eigenvectors together with the zero vector. But an eigenvector is never the zero vector.

Example: Find the eigenvalues of $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$.

Solution:

$$A - \lambda I = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{bmatrix}.$$

The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I) &= (3-\lambda)(-\lambda) - (-2) \\ &= \lambda^2 - 3\lambda + 2. \end{aligned}$$

The characteristic equation of A is

$$\lambda^2 - 3\lambda + 2 = 0.$$

The solutions of the characteristic equation are $\lambda = 1$ and $\lambda = 2$, which are the eigenvalues of A.

Example: By the previous example, we know that $\lambda = 1$, and $\lambda = 2$ are the eigenvalues of $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$. Find the eigenspace of A corresponding to $\lambda = 1$ and $\lambda = 2$.

Solution:

Eigenspace of A for $\lambda = 1$:

Form $A - \lambda I = A - I$:

$$A - \lambda I = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}.$$

Row reduce the augmented matrix of $(A - \lambda I)X = 0$ to its echelon form, i.e,

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The general solution is

$$X = \begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t,\ t \in R.$$

Eigenspace of A corresponding to the eigenvalue $\lambda = 1$ is

$$\text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

A basis for this eigenspace is $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Eigenspace of A for $\lambda = 2$:

$$A - 2I = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \sim$$

The general solution of $(A-2I)X = 0$

is

$$X = \begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} t.$$

Eigenspace of A corresponding to

$\lambda = 2$ is

Span$\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$.

A basis for this eigenspace is $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$.

An eigenvalue λ might be zero:

$AX = 0 \cdot X$ has a non-trivial solution

$\iff AX = 0$ has a non-trivial solution

$\iff A$ is not invertible.

Thus,

$\lambda = 0$ is an eigenvalue of $A \iff A$ is not invertible.

Example: Find the eigenvalues of

$$A = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}.$$

Solution:

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 6 & 2 \\ 0 & -1-\lambda & -8 \\ 1 & 0 & -2-\lambda \end{vmatrix}$$

$$= (5-\lambda)\begin{vmatrix} -1-\lambda & -8 \\ 0 & -2-\lambda \end{vmatrix}$$

$$+ \begin{vmatrix} 6 & 2 \\ -1-\lambda & -8 \end{vmatrix}$$

$$= (5-\lambda)(-1-\lambda)(-2-\lambda)$$

$$- 48 + 2(1+\lambda)$$

$$= -\lambda^3 + 2\lambda^2 + 15\lambda - 36,$$

and so

$$|A - \lambda I| = 0 \Longleftrightarrow -\lambda^3 + 2\lambda^2 + 15\lambda - 36 = 0$$

$$\Longleftrightarrow \lambda^3 - 2\lambda^2 - 15\lambda + 36 = 0$$

$$\Longleftrightarrow (\lambda - 3)(\lambda^2 + \lambda - 12) = 0$$

$$\Longleftrightarrow (\lambda - 3)(\lambda + 4)(\lambda - 3) = 0.$$

Thus, $\lambda_1 = \lambda_2 = 3$,$\lambda_3 = -4$.

Remark: The eigenvalues of a triangular matrix are the entries on its main diagonal.

Example: Eigenvalues of $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$ are $\lambda_1 = 4$, $\lambda_2 = 0$, and $\lambda_3 = -3$.

Theorem: Let A be an $n \times n$ matrix. If $v_1, v_2, ..., v_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_r$, then $\{v_1, v_2, ..., v_r\}$ is linearly independent.

Example: we know that $\lambda = 1$, and $\lambda = 2$ are the eigenvalues of

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}.$$

$X = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to $\lambda = 1$.

$Y = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to $\lambda = 2$.

So, by the above theorem,

$$\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

is linearly independent.

Example: Let A and B be two $n \times n$ matrices such that $B = PAP^{-1}$ for an invertible matrix P. (In this case, we say that A is similar to B.) Show that A and B have the same characteristic polynomial, and so the same eigenvalues.

Solution:

$$B - \lambda I = PAP^{-1} - \lambda PP^{-1} = P(A - \lambda I)P^{-1}$$

$$\det(B - \lambda I) = \det\left(P(A - \lambda I)P^{-1}\right)$$

$$= \det P \cdot \det(A - \lambda I) \cdot \det P^{-1}$$

$$= \det(A - \lambda I).$$