Complex Numbers

A complex number z is of the form

$$z = a + ib, \text{ where } i^2 = -1,$$

and $a, b \in R$.

a=real part of z=Re z.

b=imaginary part of z=Im z.

z is real $\iff b = 0$.

z is purely imaginary $\iff a = 0$.

Let $z = a + ib$ and $w = c + id$. Then,

$$z + w = a + c + i(b + d)$$

$$z - w = a - c + i(b - d)$$

$$\begin{aligned} z.w &= (a + ib) \cdot (c + id) \\ &= ac + iad + ibc + i^2bd \\ &= ac - bd + i(ad + bc) \end{aligned}$$

$$kz = ka + i(kb), k \in R.$$

If $z = a + ib$ is any complex number, then the complex conjugate of z (also called the conjugate of z) is denoted by $\overline{z}$ (read "z bar") is defined by $\overline{z} = a - ib$.

$$\overline{z + w} = \overline{z} + \overline{w}$$

$$\overline{z - w} = \overline{z} - \overline{w}$$

$$\overline{z.w} = \overline{z} \cdot \overline{w}$$

$$\overline{\overline{z}} = z$$

$$\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$$

$$z + \overline{z} = 2 \text{ Re } z$$

$$z - \overline{z} = 2i \text{ Im } z$$

Let $z = a + ib$ and $w = c + id \neq 0$.

$$\begin{aligned}
\frac{z}{w} &= \frac{a + ib}{c + id} \\
&= \frac{a + ib}{c + id} \cdot \frac{c - id}{c - id} \\
&= \frac{ac + bd + i(bc - ad)}{c^2 + d^2} \\
&= \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2} \\
&= x + iy
\end{aligned}$$

where $x = \dfrac{ac + bd}{c^2 + d^2}$ and $y = \dfrac{bc - ad}{c^2 + d^2}$.

The absolute value (or modulus) of $z = a + ib$ is

$$|z| = \sqrt{z\ \bar{z}} = \sqrt{a^2 + b^2}$$

We have the following equalities:

$$|zw| = |z|.|w|$$

$$z^{-1} = \frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}$$

$$z^{-1} = \frac{\overline{z}}{|z|^2}$$

Example: Let $z = 9 - 8i$ and $w = 5 + 2i$. Then find $|z|$, $|w|$, $|z/w|$. Write $\dfrac{z}{w}$ in the form of $a + ib$.

Solution:

$$|z| = \sqrt{9^2 + (-8)^2} = \sqrt{81 + 64} = \sqrt{145}$$

$$|w| = \sqrt{5^2 + 2^2} = \sqrt{25 + 4} = \sqrt{29}$$

$$|z/w| = \frac{|z|}{|w|} = \frac{\sqrt{145}}{\sqrt{29}} = \frac{\sqrt{5 \cdot 29}}{\sqrt{29}} = \sqrt{5}$$

$$\frac{z}{w} = \frac{9 - 8i}{5 + 2i} = \frac{9 - 8i}{5 + 2i} \cdot \frac{5 - 2i}{5 - 2i}$$

$$= \frac{(45 - 16) + i(-40 - 18)}{25 + 4}$$

$$= \frac{29 - 58i}{29} = 1 - 2i$$

Homework

1) Let $z = 3 + 4i$ and $w = 5 - 2i$. Express the followings in the form of $a + ib$.

(i) $(z - w)^2$, **(ii)** $\dfrac{z}{w}$, **(iii)** $\dfrac{\overline{z}}{\overline{w}}$,

(iv) $\dfrac{1}{z^2}$, **(v)** $\dfrac{w}{2z}$

2) Find:

(i) Re $\dfrac{1}{2+i}$ **(ii)** Im $\dfrac{2+i}{3+4i}$

(iii) Im $\dfrac{2-i}{3-4i}$

3) Write the followings in the form of $a + ib$.

(i) $\dfrac{11+2i}{4+3i}$ **(ii)** $(3+5i)(3-5i)$

(iii) $(7-3i)-(-2+4i)$ **(iv)** $\dfrac{6+i}{7+3i}$

(v) $\dfrac{1}{(3+4i)^2}$ **(vi)** $\dfrac{\sqrt{3}+i}{(1-i)(\sqrt{3}-i)}$

4) In each part solve for z.

(i) $iz = 2-i$ **(ii)** $(4-3i)\overline{z} = i$

5) If $z = 1-5i$ and $w = 3+4i$, find

$|z|$, $|w|$, $|z/w|$, $|\overline{z/w}|$, and $|\overline{z}/\overline{w}|$,

Polar Form of a Complex Number

Let $z = a + ib$.

$$\cos\theta = \frac{a}{|z|} \Longrightarrow a = |z|\cos\theta$$

$$\sin\theta = \frac{b}{|z|} \Longrightarrow b = |z|\sin\theta$$

$$z = a + ib = |z|\cos\theta + i|z|\sin\theta$$

$$= |z|(\cos\theta + i\sin\theta) = |z|\text{cis}\theta.$$

Here θ is the angle between the positive real axis and the point z, $-\pi < \theta \leq \pi$ (all angles are measured in radians).

θ is called the argument of z, and it is denoted by $\theta = \arg z$.

$$z = |z|(\cos\theta + i\sin\theta)$$

is called the polar form of z.

Example: Let $z = 1 + i$. What is the polar form of z?

Solution:

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\left.\begin{aligned} \cos\theta &= \frac{a}{|z|} = \frac{1}{\sqrt{2}} \\ \sin\theta &= \frac{b}{|z|} = \frac{1}{\sqrt{2}} \end{aligned}\right\} \Longrightarrow \theta = \pi/4.$$

$$\begin{aligned} z &= |z|(\cos\theta + i\sin\theta) \\ &= \sqrt{2}(\cos\pi/4 + i\sin\pi/4) \end{aligned}$$

Example: What is the polar form of $z = 3 + i3\sqrt{3}$?

Solution:

$$|z| = \sqrt{3^2 + (3\sqrt{3})^2} = \sqrt{9 + 9 \cdot 3} = 6$$

$$\left.\begin{aligned} \cos\theta &= \frac{a}{|z|} = \frac{3}{6} = \frac{1}{2} \\ \sin\theta &= \frac{b}{|z|} = \frac{3}{3\sqrt{3}} = \frac{\sqrt{3}}{2} \end{aligned}\right\} \Longrightarrow \theta = \pi/3.$$

$$\begin{aligned} z &= |z|(\cos\theta + i\sin\theta) \\ &= 6(\cos\pi/3 + i\sin\pi/3) \end{aligned}$$

Example: What is the polar form of $z = \sqrt{2} - i\sqrt{2}$?

Solution:

$$|z| = \sqrt{(\sqrt{2})^2 + (-\sqrt{2})^2} = \sqrt{2+2} = 2$$

$$\left.\begin{aligned} \cos\theta &= \frac{a}{|z|} = \frac{\sqrt{2}}{2} \\ \sin\theta &= \frac{b}{|z|} = \frac{-\sqrt{2}}{2} \end{aligned}\right\} \Longrightarrow \theta = -\pi/4.$$

$$\begin{aligned} z &= |z|(\cos\theta + i\sin\theta) \\ &= 2(\cos\pi/4 - i\sin\pi/4) \end{aligned}$$

Homework

1) Write the polar form of the following complex numbers:

(i) $z = -4 + 4i$, **(ii)** $z = 4i$,

(iii) $z = -7$, **(iv)** $z = \dfrac{2+2i}{1-i}$.

2) Represent in the form of $a + ib$:

(i) $z = 4(\cos \pi/2 + i \sin \pi/2)$,

(ii) $z = \sqrt{8}(\cos \pi/4 + i \sin \pi/4)$,

(iii) $2\text{cis}(-\pi/6)$,

(iv) $\dfrac{2 \text{ cis } (-3\pi/4)}{2 \text{ cis } (5\pi/6)}$.

Complex Division in Polar Form

If $z_1 = r_1 \text{ cis } \theta_1$ and $z_2 = r_2 \text{ cis } \theta_2$,

then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \text{ cis } (\theta_1 - \theta_2),$$

and

$$\overline{z_1} = r_1 \text{ cis } (-\theta_1)$$

(complex conjugate of z in polar form).

Example: $z =$ cis $(\pi/2)$ and

$w = 2$ cis $(-\pi/3)$. Find z/w.

Solution:

$$\frac{z}{w} = \frac{\text{cis } (\pi/2)}{2 \text{ cis } (-\pi/3)}$$

$$= \frac{1}{2} \text{ cis } (\pi/2 - (-\pi/3))$$

$$= \frac{1}{2} \text{ cis } (5\pi/6)$$

$$= \frac{1}{2}(\cos(5\pi/6) + i\sin(5\pi/6))$$

$$= \frac{1}{2}(\frac{-\sqrt{3}}{2} + i\frac{1}{2})$$

$$= \frac{-\sqrt{3}}{4} + i\frac{1}{4}$$

Complex Multiplication in Polar Form

If $z_1 = r_1 \text{ cis } \theta_1$ and $z_2 = r_2 \text{ cis } \theta_2$, then

$$z_1 \cdot z_2 = |z_1| \cdot |z_2| \cdot \text{ cis } (\theta_1 + \theta_2).$$

Example: If

$z = 2 \text{ cis } \dfrac{3\pi}{8}$ and

$w = 5 \text{ cis } \dfrac{2\pi}{3}$.

Then

$$z \cdot w = \left(2 \text{ cis } \frac{3\pi}{8}\right)\left(5 \text{ cis } \frac{2\pi}{3}\right)$$

$$= 2 \cdot 5 \text{ cis } (\frac{3\pi}{8} + \frac{2\pi}{3})$$

$$= 10 \text{ cis } (\frac{25\pi}{24})$$

De Moivre's Theorem: For any positive integer n,

$$z^n = |z|^n(\cos n\theta + i \sin n\theta),$$

where $z = |z|(\cos\theta + i\sin\theta)$.

Example: Write $z = (1+i)^{20}$ in the form of $a + ib$.

Solution: $1+i = \sqrt{2}$ cis $(\pi/4)$. Hence

$$(1+i)^{20} = \left(\sqrt{2}(\cos\pi/4 + i\sin\pi/4)\right)^{20}$$

$$= (\sqrt{2})^{20}\,(\cos 20\pi/4 + i\sin 20\pi/4))$$

$$= 2^{10}(\cos 5\pi + i\sin 5\pi)$$

$$= 2^{10}(\cos(4\pi+\pi) + i\sin(4\pi+\pi))$$

$$= 2^{10}(\cos\pi + i\sin\pi)$$

$$= 2^{10}(-1 + i\cdot 0)$$

$$= -2^{10} = -1024$$

Homework

1) If $z = 2 \text{ cis } (\pi/3)$, find z^6 in the form of $a + ib$. Ans: 64.

2) Express $z = (-1 + i)^4$ in the form of $a + ib$.

Roots of a Complex Number

Let $z^n = \alpha$ cisθ. Then

$$z_k = \sqrt[n]{\alpha} \text{ cis}\left(\frac{\theta + 2k\pi}{n}\right);$$

where $k = 0, 1, 2, ..., n-1$.

Example: Let $z^3 = -8i$. Find z and write it in the standard form.

Solution: $\alpha = |-8i| = 8$, $\theta = -\pi/2$.

$$z_k = \sqrt[3]{8} \text{ cis} \left(\frac{-\pi/2 + 2k\pi}{3}\right);\ k = 0, 1, 2.$$

$$z_0 = 2 \text{cis} \left(\frac{-\pi/2}{3}\right) = 2 \text{ cis } \left(\frac{-\pi}{6}\right)$$

$$= 2\ (\cos(-\pi/6) + i \sin(-\pi/6))$$

$$= 2\ (\frac{\sqrt{3}}{2} + i\frac{-1}{2}) = \sqrt{3} - i$$

$$z_1 = 2 \text{ cis} \left(\frac{(-\pi/2) + 2\pi}{3} \right)$$

$$= 2 \text{ cis } \left(\frac{3\pi/2}{3} \right) = 2\text{cis } (\pi/2)$$

$$= 2 \ (\cos(\pi/2) + i \sin(\pi/2))$$

$$= 2(0 + i) = 2i.$$

$$z_2 = 2 \text{ cis} \left(\frac{(-\pi/2) + 4\pi}{3} \right)$$

$$= 2 \text{ cis } (7\pi/6)$$

$$= 2 \ (\cos(7\pi/6) + i \sin(7\pi/6))$$

$$= 2 \ \left(\frac{-\sqrt{3}}{2} - i\frac{1}{2} \right) = -\sqrt{3} - i.$$

Example: Find the roots of

$z^2 + z + 1 = 0.$

Solution:

$$z^2 + z + 1 = \left(z + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 = 0$$

$$\implies \quad \left(z + \frac{1}{2}\right)^2 + \frac{3}{4} = 0$$

$$\implies \quad \left(z + \frac{1}{2}\right)^2 = \frac{-3}{4}$$

$$\implies \quad z + \frac{1}{2} = \pm\frac{\sqrt{3}}{2}i$$

$$\implies \quad z = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i$$

i.e, $z_1 = \dfrac{-1}{2} + \dfrac{\sqrt{3}}{2}i\ , \quad z_2 = \dfrac{-1}{2} - \dfrac{\sqrt{3}}{2}i$