LINEAR INDEPENDENCE
(AND LINEAR DEPENDENCE)

Definition: A set of vectors $\{v_1, v_2, ..., v_k\}$ in R^n is said to be linearly independent if the vector equation
$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0$
has only the trivial solution,
i.e, $x_1 = x_2 = \cdots = x_k = 0$.

The set $\{v_1, v_2, ..., v_k\}$ in R^n is said to be linearly dependent if there exist scalars $c_1, c_2, ..., c_k$, not all zero, such that
$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0$.

Example: Decide if the following set of vectors is linearly independent.

$$\left\{ \begin{bmatrix} 12 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}$$

Solution: Since

$$\begin{bmatrix} 12 \\ -3 \end{bmatrix} + 3 \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

the vectors are linearly dependent.

Example: Decide if the vectors

$$\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -6 \end{bmatrix}$$

are linearly independent.

Solution: Consider the equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 5 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

i.e, the system of equations

$$\begin{aligned} x_1 - 3x_2 + 0x_3 &= 0 \\ 3x_1 - 5x_2 + 5x_3 &= 0 \\ -2x_1 + 6x_2 - 6x_3 &= 0. \end{aligned}$$

The augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & -3 & 0 & 0 \\ 3 & -5 & 5 & 0 \\ -2 & 6 & -6 & 0 \end{array}\right] \begin{array}{l} R_2' = R_2 + (-3)R_1 \\ R_3' = R_3 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & -3 & 0 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right].$$

Then we have,

$x_3 = 0$, $x_2 = 0$, and $x_1 = 0$,

which means that the vectors are linearly independent.

Example: Decide if the vectors

$$\begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -2 \\ -6 \end{bmatrix}$$

are linearly independent.

Solution: Consider the equation

$$c_1 \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -7 \\ 3 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 7 \\ -2 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the corresponding system is

$$\left[\begin{array}{rrr|r} 3 & 4 & 3 & 0 \\ -1 & -7 & 7 & 0 \\ 1 & 3 & -2 & 0 \\ 0 & 2 & -6 & 0 \end{array}\right].$$

Let us find an echelon form (REF) of this matrix:

$$\left[\begin{array}{rrr|r} 3 & 4 & 3 & 0 \\ -1 & -7 & 7 & 0 \\ 1 & 3 & -2 & 0 \\ 0 & 2 & -6 & 0 \end{array}\right] R_1 \longleftrightarrow R_3$$

$$\sim \left[\begin{array}{rrr|r} 1 & 3 & -2 & 0 \\ -1 & -7 & 7 & 0 \\ 3 & 4 & 3 & 0 \\ 0 & 2 & -6 & 0 \end{array}\right] \begin{array}{l} R_2{}' = R_2 + R_1 \\ R_3{}' = R_3 + (-3)R_1 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 3 & -2 & 0 \\ 0 & -4 & 5 & 0 \\ 0 & -5 & 9 & 0 \\ 0 & 2 & -6 & 0 \end{array}\right] \begin{array}{l} R_4{}' = (1/2)R_4 \\ R_2 \longleftrightarrow R_4 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 3 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -5 & 9 & 0 \\ 0 & -4 & 5 & 0 \end{array}\right] \begin{array}{l} R_3{}' = R_3 + 5R_2 \\ R_4{}' = R_4 + 4R_2 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 3 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & -7 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 3 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Then we have, $c_3 = 0$, $c_2 = 0$, and $c_1 = 0$, which means that the vectors are linearly independent. OR

Since each column has a pivot position, vectors are linearly independent.

Example: Decide if the vectors

$$\begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 3 \\ 10 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \end{bmatrix}$$

are linearly independent.

Solution: Since the number of vectors is more than the number of entries in one vector, vectors are linearly dependent.

In R^3, the maximum number of linearly independent vectors is 3.

Example: Decide if the vectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

are linearly dependent. If yes, express one of the vectors as a linear combination of the other vectors.

Solution: Consider the equation

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \end{bmatrix} + c_4 \begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Corresponding augmented matrix and its (REF):

$$\left[\begin{array}{rrrr|r} 1 & 1 & 0 & 4 & 0 \\ -1 & 0 & 3 & -1 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 1 & 0 & -1 & 3 & 0 \end{array}\right] \begin{array}{l} R_2{}' = R_2 + R_1 \\ R_4{}' = R_4 + (-1)R_1 \end{array}$$

$$\sim \left[\begin{array}{rrrr|r} 1 & 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 3 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 \end{array}\right] \begin{array}{l} R_3{}' = R_3 + 2R_2 \\ R_4{}' = R_4 + R_2 \end{array}$$

$$\sim \left[\begin{array}{rrrr|r} 1 & 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{array}\right] \begin{array}{l} R_3{}' = (1/7)R_3 \\ R_4{}' = (1/2)R_4 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] R_4{}' = R_4 + (-1)R_3$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since the fourth column is not a pivot column, c_4 is a free variable, and thus the vectors are linearly dependent.

Say $c_4 = t$, $t \in R$. Since the first, second, and third columns are pivot columns, c_1, c_2 and c_3 are basic variables. From the third row, we have $c_3 + c_4 = 0$ which gives $c_3 = -t$. From the second row, we have

$c_2 + 3c_3 + 3c_4 = 0$ which gives

$c_2 = -3c_3 - 3c_4 = -3(-t) - 3t = 0$.

From the first row, we have

$c_1 + c_2 + 4c_4 = 0$ which gives

$c_1 = -c_2 - 4c_4 = -0 - 4t = -4t$.

If we choose $t = 1$, then we have:

$$-4\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \end{bmatrix}$$

OR

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = (-1/4) \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \end{bmatrix} + (1/4) \begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

OR

$$\begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \end{bmatrix} = -4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

Note that we can not express the vector $\begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}$ as a linear combination of the vectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix}.$$

Remark: A set S with two or more vectors is linearly dependent $\Longleftrightarrow$ at least one of the vectors in S is expressible as a linear combination of other vectors in S.

Example: Find the value(s) of h for which the vectors

$$\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ h \end{bmatrix}$$

are linearly dependent.

Solution:

$$\left[\begin{array}{rrr|r} 1 & -2 & -1 & 0 \\ 3 & -4 & 1 & 0 \\ -3 & 1 & h & 0 \end{array}\right] \begin{array}{l} R_2{}' = R_2 + (-3)R_1 \\ R_3{}' = R_3 + 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -5 & -3+h & 0 \end{array}\right] R_2{}' = (1/2)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -5 & -3+h & 0 \end{array}\right] R_3{}' = R_3 + 5R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 7+h & 0 \end{array}\right].$$

For $h = -7$ the given vectors are linearly dependent.

Note that if $h \in R$ and $h \neq -7$, then the vectors are linearly independent.

Example: Find the value(s) of h for which the vectors

$$\begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 8 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ h \\ -8 \end{bmatrix}$$

are linearly dependent.

Solution:

$$\left[\begin{array}{rrr|r} 1 & -3 & 4 & 0 \\ -5 & 8 & h & 0 \\ -2 & 6 & -8 & 0 \end{array}\right] \begin{array}{l} R_2{}' = R_2 + 5R_1 \\ R_3{}' = R_3 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 0 & -7 & 20+h & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

For all h in R, vectors are linearly dependent.

Remarks: (1) A set S with two or more vectors is linearly independent $\Longleftrightarrow$ no vector in S is expressible as a linear combination of other vectors in S.

(2) Zero vector is linearly dependent. Any set of vectors containing zero vector is linearly dependent.

(3) Any set $\{v_1, v_2, ..., v_k\}$ in R^n is linearly dependent if $k > n$.

(4) Columns of a matrix A are linearly independent $\Longleftrightarrow AX = 0$ has only the trivial solution.