LINEAR TRANSFORMATIONS

A transformation (or mapping) T is linear if:

i) $T(x+y) = T(x) + T(y)$ for all x, y in the domain of T.

ii) $T(cx) = cT(x)$ for all x and all scalars c.

Every matrix transformation is a linear transformation.

Example: Let $T: R^2 \to R^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \\ x+y \end{bmatrix}.$$

Show that T is a linear transformation.

Solution:

i) $T\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} w \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) + T\left(\begin{bmatrix} w \\ z \end{bmatrix}\right).$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} w \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} x+w \\ y+z \end{bmatrix}\right)$$

$$= \begin{bmatrix} y+z \\ x+w \\ x+w+y+z \end{bmatrix}$$

$$= \begin{bmatrix} y \\ x \\ x+y \end{bmatrix} + \begin{bmatrix} z \\ w \\ z+w \end{bmatrix}$$

$$= T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) + T\left(\begin{bmatrix} w \\ z \end{bmatrix}\right)$$

ii) $T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right)$

$$= \begin{bmatrix} cy \\ cx \\ cx+cy \end{bmatrix} = c\begin{bmatrix} y \\ x \\ x+y \end{bmatrix}$$

$$= cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

By i) and ii) T is a linear transformation.

Remark 1: Conditions for a linear transformation can be written as only one equation:

$$T(cx + dy) = cT(x) + dT(y)$$

for all $x, y \in R^n$, and for all scalars c and d.

Remark 2: For any linear transformation T,

$$T(0) = T(0x) = 0T(x) = 0.$$

Example: Let

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}, u = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \text{and} v = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Compute Au, Av, $A(u+v)$, and $A(5u)$.

Solution:

$$Au = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 1 + 1 \cdot (-1) + (-1) \cdot 1 \\ 0 \cdot 1 + 1 \cdot (-1) + 2 \cdot 1 \\ -2 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.$$

$$Av = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 0 + 1\cdot 1 + (-1)\cdot 2 \\ 0\cdot 0 + 1\cdot 1 + 2\cdot 2 \\ -2\cdot 0 + 0\cdot 1 + 1\cdot 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}.$$

$$A(u+v) = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+0-3 \\ 0+0+6 \\ -2+0+3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} = Au + Av.$$

$$A(5u) = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix} \left(5 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -5 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5-5-5 \\ 0-5+10 \\ -10+0+5 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 5 \\ -5 \end{bmatrix} = 5 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = 5(Au).$$

Indeed, A is a linear transformation from R^3 to R^3. It is denoted by $T(X) = AX$

Example: Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix}$. Let T be the linear transformation from R^2 to R^3 defined by $T(X) = AX$.

a) Find the image of $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ under T.

b) Find a vector X in R^2 such that $T(X) = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}$.

c) Is there another vector $Y \neq X$ such that $T(Y) = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}$.

d) Is $W = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ in the range of T?

Solution:

a) $$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4 \\ -1+0 \\ 2+2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}.$$

b) We need to find a vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}.$$

Corresponding augmented matrix:

$$\left[\begin{array}{rr|r} 1 & 2 & -1 \\ -1 & 0 & -3 \\ 2 & 1 & 4 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 + R_1 \\ R_3' = R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 2 & -4 \\ 0 & -3 & 6 \end{array}\right] \begin{array}{l} R_2{}' = (1/2)R_2 \\ R_3{}' = (1/3)R_3 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{array}\right] R_3{}' = R_3 + R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right].$$

Then, $x_2 = -2$ and

$x_1 = -1 - 2x_2 = -1 - 2(-2) = 3$, thus

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

c) No, since the system

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}$$

has a unique solution.

d) We need to check

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

$$\left[\begin{array}{rr|r} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & -2 \end{array}\right] \begin{array}{l} R_2{}' = R_2 + R_1 \\ R_3{}' = R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{rr|r} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & -3 & -4 \end{array}\right] R_2{}' = (1/2)R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -3 & -4 \end{array}\right] {R_3}' = R_3 + 3R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right],$$

which is inconsistent. So,

$$W = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

is not in the range of T.

Example: Let $T: R^2 \to R^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x+1 \end{bmatrix}.$$

Show that T is not a linear transformation.

Solution:

$$T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = \begin{bmatrix} cy \\ cx+1 \end{bmatrix}$$

$$\neq c\begin{bmatrix} y \\ x+1 \end{bmatrix} = cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right).$$

Example: Let $T: R^2 \to R^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

Find $T(u)$, $T(v)$, and $T(u+v)$, where

$$u = \begin{bmatrix} 5 \\ 1 \end{bmatrix},\ v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Solution:

$$T\left(\begin{bmatrix} 5 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 5 \end{bmatrix}, \quad T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 3 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = T\left(\begin{bmatrix} 8 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} -5 \\ 8 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 5 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 5 \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \end{bmatrix}.$$

T rotates u, v,$u+v$ 90° counterclockwise. So, T is a rotation transformation.

We can rewrite $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix}$ as

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is called the standard matrix of the linear transformation T.

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

The transformation $T : R^2 \longrightarrow R^2$ given by $T(X) = AX$ is called a shear transformation.

Find $T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$, $T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$, and $T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right)$.

Solution:

$$T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}.$$

Definition: Let $T : R^n \longrightarrow R^m$ be a linear transformation.

1) If for each vector b in R^m there is at least one vector X in R^n such that $T(X) = b$, then T is said to be onto R^m.

2) If each vector b in R^m is the image of at most one vector X in R^n, then T is said to be one-to-one.

Example: Let $T : R^n \longrightarrow R^m$ be a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \text{ and}$$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

a) Find the values of n and m.

b) Find the image of $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ under T.

c) Find the standard matrix of T.

d) Is T onto?

e) Is T one-to-one?

Solution: a) $n = 3$, $m = 2$.

b)

$$T\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$= T\left(2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$= 2T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$= 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}.$$

c) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$.

Let us do part b) in a different way:

$$T\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

d) To be onto $AX = b$ must have a solution for each $b \in R^2$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

A does not have a pivot position in-each row. So, $AX = b$ does not have a solution for at least one b in R^2. T is not onto.

In other words: The columns of A do not span R^2.

e) $AX = 0$ has infinitely many solutions. So, T is not one-to-one.
In other words: Columns of A are linearly dependent.

Theorem: Let $T : R^n \longrightarrow R^m$ be a linear transformation, and let A be the standard matrix of T.
1) T is one-to-one $\Longleftrightarrow T(X) = 0$ has only the trivial solution.
$\Longleftrightarrow$ Columns of A are linearly independent.
$\Longleftrightarrow$ Each column has a pivot.
2) T is onto $\Longleftrightarrow$ Columns of A span R^m.
$\Longleftrightarrow$ Each row has a pivot.

Example: $T : R^3 \longrightarrow R^4$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 - x_3 \\ x_1 - x_2 \\ x_2 + x_3 \\ x_1 - x_2 \end{bmatrix}.$$

i) Find the standard matrix A of the linear transformation T.

ii) Is T one-to-one?

iii) Is T onto?

Solution: i) $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$,

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Standard matrix of T is

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

ii)

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{matrix} {R_2}' = R_2 - R_1 \\ {R_4}' = R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix} R_2 \longleftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \end{bmatrix} R_4{}' = R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} R_3{}' = R_3 + 2R_2$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since each column has a pivot position, the columns of A are linearly independent, and so T is one-to-one.

iii) Since each row of REF of A does not have a pivot position, T is not onto.

(For example, for the vector $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ in R^4 there is no vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in R^3 such that $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, i.e,

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right]$$

is inconsistent.)

Example: Let $T : R^3 \longrightarrow R^3$ be a linear transformation.

i) Show that

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

are linearly dependent.

ii) Show that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right), T\left(\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}\right), T\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right)$$

are linearly dependent.

Solution: i)

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & -1 & -1 \\ 1 & 3 & 1 \end{bmatrix} R_3{}' = R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, we have

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & -1 & -1 \\ 1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

which gives that vectors are linearly dependent, and

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}.$$

ii)

$$T\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right) = T\left(-2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}\right)$$

$$= -2T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}\right),$$

and so,

$$T\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right) + 2T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

which says that vectors are linearly dependent.

Exercise: Let $T : R^n \longrightarrow R^m$ be a linear transformation and let $\{v_1, v_2, v_3\}$ be a linearly dependent set in R^n. Show that $\{T(v_1), T(v_2), T(v_3)\}$ is linearly dependent in R^m.

Remark: If $\{v_1, v_2, v_3\}$ is linearly independent set, then $\{T(v_1), T(v_2), T(v_3)\}$ does not need to be linearly independent.

For example, for the linear transformation $T : R^3 \longrightarrow R^2$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + 4x_2 + 6x_3 \end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\end{bmatrix}, T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}2\\4\end{bmatrix}, \text{and}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}3\\6\end{bmatrix}.$$

We know that the vectors

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

are linearly independent, and the vectors

$$\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}$$

are linearly dependent.