DETERMINANTS

Recall that for a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

determinant of A is $\det(A) = ad - bc$.

A is invertible $\Leftrightarrow ad - bc \neq 0$,

If $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

For the matrix $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$,

$\det B = 1 \cdot 4 - 2 \cdot 2 = 0$. Hence, B is not invertible.

Definition: Let $A = (a_{ij})$, $(1 \le i \le n, \quad 1 \le j \le n)$ be an $n \times n$ matrix, and let A_{ij} denote the submatrix formed by deleting the i th row and j th column of A. Then,

$$\begin{aligned}\det(A) &= a_{11}\det(A_{11}) - a_{12}\det(A_{12}) \\ &\quad + \cdots + (-1)^{1+n} a_{1n}\det(A_{1n}) \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j}\det(A_{1j}).\end{aligned}$$

Example: If $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$, what is $\det(A) =$?

Solution: $A_{11} = \begin{bmatrix} 5 & 6 \\ -8 & 9 \end{bmatrix}$,

$A_{12} = \begin{bmatrix} -4 & 6 \\ 7 & 9 \end{bmatrix}$, and $A_{13} = \begin{bmatrix} -4 & 5 \\ 7 & -8 \end{bmatrix}$.

$\det(A_{11}) = 5 \cdot 9 - 6 \cdot (-8) = 93$;

$\det(A_{12}) = (-4) \cdot 9 - 6 \cdot 7 = -78$;

$\det(A_{13}) = (-4) \cdot (-8) - 7 \cdot 5 = -3$.

$$\begin{aligned} \det(A) &= a_{11} \det(A_{11}) - a_{12} \det(A_{12}) \\ &\qquad + a_{13} \det(A_{13}) \\ &= 1 \cdot 93 - 2 \cdot (-78) + 3 \cdot (-3) \\ &= 93 + 156 - 9 = 240. \end{aligned}$$

And, also $A_{21} = \begin{bmatrix} 2 & 3 \\ -8 & 9 \end{bmatrix}$,

$A_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$, and $A_{23} = \begin{bmatrix} 1 & 2 \\ 7 & -8 \end{bmatrix}$.

$\det(A_{21}) = 18 + 24 = 42$;

$\det(A_{22}) = 9 - 21 = -12$;

$\det(A_{23}) = -8 - 14 = -22$.

Thus,

$$\begin{aligned}
&\det(A) \\
&= -a_{21}\det(A_{21}) + a_{22}\det(A_{22}) \\
&\qquad - a_{23}\det(A_{23}) \\
&= -(-4)(42) + 5(-12) - 6(-22) \\
&= 168 - 60 + 132 \\
&= 240.
\end{aligned}$$

If A is a 3×3 matrix, we can calculate $\det(A)$ in the following way as well. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix} \begin{matrix} 1 & 2 \\ -4 & 5 \\ 7 & -8 \end{matrix}$$

$$\begin{aligned} &\det(A) \\ &= 1\cdot 5\cdot 9 + 2\cdot 6\cdot 7 + 3\cdot(-4)\cdot(-8) \\ &\quad -(7\cdot 5\cdot 3 + (-8)\cdot 6\cdot 1 + 9\cdot(-4)\cdot 2) \\ &= 45 + 84 + 96 - (105 - 48 - 72) \\ &= 225 - (-15) = 240. \end{aligned}$$

Example: If $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ -5 & -8 & 4 & -3 \end{bmatrix}$,

find $\det(A)$.

Solution: We can use <u>cofactor expansion</u> along the first row.

$$\det(A) = 4(-1)^{1+1}\det(A_{11})$$

$$= 4\begin{vmatrix} -1 & 0 & 0 \\ 6 & 3 & 0 \\ -8 & 4 & -3 \end{vmatrix}$$

$$= 4(-1)\cdot(-1)^{1+1}\begin{vmatrix} 3 & 0 \\ 4 & -3 \end{vmatrix}$$

$$= -4(-9) = 36$$

A square matrix is called upper triangular if all the entries below the main diagonal are zero, and it is called lower triangular if all the entries above the main diagonal are zero. A matrix is called triangular if it is either upper triangular or lower triangular. The identity matrix is both upper triangular and lower triangular.

If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.

So, for the above matrix A,

$\det(A) = 4 \cdot (-1) \cdot 3 \cdot (-3) = 36.$

Example: If $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 1 & 4 & 1 \\ -1 & 2 & 1 & 0 \\ 1 & 0 & 3 & 1 \end{bmatrix}$,

what is $\det(A)$?

Solution:

$$\begin{aligned} \det(A) \; &= a_{11} \det(A_{11}) - a_{12} \det(A_{12}) \\ &\quad + a_{13} \det(A_{13}) - a_{14} \det(A_{14}). \end{aligned}$$

$$\det(A_{11}) = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} - 2 \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix}$$

$$= (1 - 0) - 2(4 - 3) = 1 - 2 = -1.$$

$$\det(A_{12}) = \begin{vmatrix} 2 & 4 & 1 \\ -1 & 1 & 0 \\ 1 & 3 & 1 \end{vmatrix}$$

$$= (-1)(-1)^{2+1} \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix}$$

$$+ 1 \cdot (-1)^{2+2} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= (4 - 3) + (2 - 1) = 1 + 1 = 2$$

We do not need to calculate $\det(A_{13})$ since $a_{13} = 0$.

$$\det(A_{14}) = \begin{vmatrix} 2 & 1 & 4 \\ -1 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}$$

$$= (1 - 8) + 3(4 + 1) = -7 + 15 = 8.$$

$$\begin{aligned} \det(A) \; &= a_{11} \det(A_{11}) - a_{12} \det(A_{12}) \\ &\quad + a_{13} \det(A_{13}) - a_{14} \det(A_{14}) \end{aligned}$$

$$= 1 \cdot (-1) - 2 \cdot 2 + 0 \cdot \det(A_{13}) - (-1) \cdot 8$$

$$= -1 - 4 + 8 = 3.$$

Properties of Determinants

Let A and B be two $n \times n$ square matrices.

i) $\det(A) = \det(A^T)$

ii) $\det(AB) = \det(A)\det(B)$

iii) If A is invertible, then
$\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Proof of iii): If A is invertible, then $I = AA^{-1}$, and by ii)

$$\det(I) = \det(AA^{-1})$$
$$1 = \det(A)\det(A^{-1}),$$

which gives that $\det(A^{-1}) = \dfrac{1}{\det(A)}$

Note: In general,

$\det(A+B) \neq \det(A)+\det(B)$.

Example: $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

$\det(A) = 5 - 4 = 1$,

$\det(B) = 9 - 1 = 8$,

$\det(A) + \det(B) = 1 + 8 = 9$,

$A + B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$, and

$\det(A+B) = 32 - 9 = 23$.

Thus,

$\det(A+B) \neq \det(A)+\det(B)$.

The effect of row operations on determinants

Let A be an $n \times n$ square matrix.

i) If B is the matrix that results when two rows of A are interchanged, then $\det(B) = -\det(A)$.

Example:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad R_1 \longleftrightarrow R_2$$

$$\sim \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = B.$$

Then, $\det(B) = -\det(A)$, i.e,

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

ii) If B is the matrix that results when a single row of A is multiplied by a scalar k, then $\det(B) = k\det(A)$.

Example:

If $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ and

$$B = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$ then

$\det(B) = k\det(A)$, i.e,

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

iii) If B is the matrix that results when a multiple of one row is added to another row, then $\det(B) = \det(A)$.

Example:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad R_1' = R_1 + kR_2$$

$$\sim \begin{bmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= B$$

Then, $\det(B) = \det(A)$, i.e,

$$\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Remark: Let I_n be the $n \times n$ identity matrix and E be an $n \times n$ elementary matrix.

i) If E results from multiplying a row of I_n by k, then $\det(E) = k$.

ii) If E results from interchanging two rows of I_n, then $\det(E) = -1$.

iii) If E results from adding a multiple of one row of I_n to another row, then $\det(E) = 1$.

Example:

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 5, \quad R_2' = 5R_2$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -1, \quad R_3 \longleftrightarrow R_4$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1, \quad R_2' = R_2 + 3R_4$$

Example: Evaluate $\det(A)$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 5 & 3 & 4 \\ -2 & 0 & 1 \end{bmatrix}.$$

Solution:

$$\det(A) = \begin{vmatrix} 1 & 2 & -1 \\ 5 & 3 & 4 \\ -2 & 0 & 1 \end{vmatrix} \begin{matrix} R_2' = R_2 - 5R_1 \\ R_3' = R_3 + 2R_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & 2 & -1 \\ 0 & -7 & 9 \\ 0 & 4 & -1 \end{vmatrix}$$

$$= 1 \cdot (-1)^{1+1} \begin{vmatrix} -7 & 9 \\ 4 & -1 \end{vmatrix}$$

$$= 7 - 36$$

$$= -29.$$

Example: If $\begin{bmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{bmatrix}$,

what is $\det(A)$?

Solution:

$$\det(A)$$

$$= \begin{vmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{vmatrix} \quad R_1^{'} = R_1 + 2R_3$$

$$= \begin{vmatrix} 7 & 0 & 1 & 3 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 7 & 1 & 3 \\ -1 & 2 & -2 \\ 2 & -1 & 2 \end{vmatrix} \begin{array}{l} R_2' = R_2 - 2R_1 \\ R_3' = R_3 + R_1 \end{array}$$

$$= \begin{vmatrix} 7 & 1 & 3 \\ -15 & 0 & -8 \\ 9 & 0 & 5 \end{vmatrix}$$

$$= 1 \cdot (-1)^{1+2} \begin{vmatrix} -15 & -8 \\ 9 & 5 \end{vmatrix}$$

$$= -(-15 \cdot 5 - 9(-8))$$

$$= -(-75 + 72)$$

$$= 3$$

Note: Column operations have the same effect on determinants as row operations.

Example: If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -3$, find

$$\begin{vmatrix} a_1 + b_1 & a_1 - b_1 & c_1 \\ a_2 + b_2 & a_2 - b_2 & c_2 \\ a_3 + b_3 & a_3 - b_3 & c_3 \end{vmatrix}.$$

Solution:

$$\begin{vmatrix} a_1 + b_1 & a_1 - b_1 & c_1 \\ a_2 + b_2 & a_2 - b_2 & c_2 \\ a_3 + b_3 & a_3 - b_3 & c_3 \end{vmatrix} C_2' = C_2 + C_1$$

$$= \begin{vmatrix} a_1 + b_1 & 2a_1 & c_1 \\ a_2 + b_2 & 2a_2 & c_2 \\ a_3 + b_3 & 2a_3 & c_3 \end{vmatrix}$$

$$= 2 \begin{vmatrix} a_1 + b_1 & a_1 & c_1 \\ a_2 + b_2 & a_2 & c_2 \\ a_3 + b_3 & a_3 & c_3 \end{vmatrix} \quad C_1' = C_1 - C_2$$

$$= 2 \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} \quad C_1 \longleftrightarrow C_2$$

$$= -2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= (-2) \cdot (-3)$$

$$= 6.$$

Example: Let $A = \begin{bmatrix} a & b & c \\ u & v & w \\ x & y & z \end{bmatrix}$, and $\det(A) = -6$. Evaluate the determinants of the following matrices.

i) $\begin{bmatrix} a & b & 2a + c \\ u & v & 2u + w \\ x & y & 2x + z \end{bmatrix}$ ii) $\begin{bmatrix} 3a & 3b & 3c \\ 4x & 4y & 4z \\ -u & -v & -w \end{bmatrix}$

iii) $2A$ vi) $A^{-1}A^TA$

Solution: i)

$$\begin{vmatrix} a & b & 2a + c \\ u & v & 2u + w \\ x & y & 2x + z \end{vmatrix} \quad C_3' = C_3 - 2C_1$$

$$= \begin{vmatrix} a & b & c \\ u & v & w \\ x & y & z \end{vmatrix} = \det(A) = -6$$

ii)

$$\begin{vmatrix} 3a & 3b & 3c \\ 4x & 4y & 4z \\ -u & -v & -w \end{vmatrix}$$

$$= 3\begin{vmatrix} a & b & c \\ 4x & 4y & 4z \\ -u & -v & -w \end{vmatrix}$$

$$= 3 \cdot 4\begin{vmatrix} a & b & c \\ x & y & z \\ -u & -v & -w \end{vmatrix}$$

$$= 12 \cdot (-1)\begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix}$$

$$= (-12)(-1)\begin{vmatrix} a & b & c \\ u & v & w \\ x & y & z \end{vmatrix}$$

$$= 12 \cdot (-6) = -72.$$

iii)

$$\det(2A) = 2 \cdot 2 \cdot 2 \cdot \det A$$
$$= 2^3 \det(A) = 8(-6) = -48.$$

iv) $\det(A^{-1}A^TA)$

$$= (\det A^{-1})(\det A^T)(\det A)$$
$$= \frac{1}{\det A}(\det A)(\det A)$$
$$= \frac{1}{-6}(-6)(-6)$$
$$= -6$$

CRAMER'S RULE

Let A be an invertible $n \times n$ matrix, $x = [x_1, x_2, ..., x_n]^T$. For any b in R^n, $Ax = b$ has a unique solution, and the entries of x are given by

$$x_i = \frac{\det A_i(b)}{\det A}, \; i = 1, 2, ..., n,$$

where $A_i(b)$ is the matrix obtained from A by replacing column i by the vector b.

Example: Let $A = \begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Solve $Ax = b$.

Then $A_1(b) = \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}$, $A_2(b) = \begin{bmatrix} 7 & 3 \\ 3 & 5 \end{bmatrix}$.

The solution $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of $Ax = b$ is given by

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix}}{\begin{vmatrix} 7 & -2 \\ 3 & 1 \end{vmatrix}} = \frac{3+10}{7+6} = 1$$

$$
\begin{aligned}
x_2 &= \frac{\det A_2(b)}{\det A} = \frac{\begin{vmatrix} 7 & 3 \\ 3 & 5 \end{vmatrix}}{13} \\
&= \frac{35-9}{13} = \frac{26}{13} = 2.
\end{aligned}
$$

Thus $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Check:

$$Ax = b \text{ ?}$$

$$\begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Example: Use the Cramer's rule to solve

$$x_1 + 2x_3 = 6$$
$$-3x_1 + 4x_2 + 6x_3 = 30$$
$$-x_1 - 2x_2 + 3x_3 = 8.$$

Soultion: $A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, b = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix},$

$$A_1(b) = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix},$$

$$A_2(b) = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix},$$

$$A_3(b) = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}.$$

$$\begin{aligned} \det A &= \begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix} \quad C_3' = C_3 - 2C_1 \\ &= \begin{vmatrix} 1 & 0 & 0 \\ -3 & 4 & 12 \\ -1 & -2 & 5 \end{vmatrix} \\ &= (-1)^{1+1} 1 \begin{vmatrix} 4 & 12 \\ -2 & 5 \end{vmatrix} \\ &= 20 + 24 = 44 \end{aligned}$$

$$\det A_1(b) = \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix} \quad C_1' = C_1 - 3C_3$$

$$= \begin{vmatrix} 0 & 0 & 2 \\ 12 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix}$$

$$= (-1)^{1+3} 2 \begin{vmatrix} 12 & 4 \\ -1 & -2 \end{vmatrix}$$

$$= 2(-24 + 4) = -40.$$

Thus

$$x_1 = \frac{\det A_1(b)}{\det A}$$

$$= \frac{-40}{44} = \frac{-10}{11}.$$

Similarly

$$x_2 = \frac{\det A_2(b)}{\det A} = \frac{72}{44} = \frac{18}{11},$$

and

$$x_3 = \frac{\det A_3(b)}{\det A} = \frac{152}{44} = \frac{38}{11}$$

Example: Use Cramer's rule to solve for x without solving for y, z, and w.

$$\begin{aligned} -y+z+3w &= 1 \\ x+2y-z+w &= 2 \\ 3z+3w &= 0 \\ y+8z \qquad &= 1 \end{aligned}$$

Solution:

$$A = \begin{bmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{bmatrix} \text{ and}$$

$$A_1(b) = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 1 & 1 & 8 & 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} \quad R_3^{'} = R_3 + R_1$$

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix}$$

$$= -(-1)^{1+1}(-1) \begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix}$$

$$= 9 - 27 = -18.$$

$$|A_1(b)| = \begin{vmatrix} 1 & -1 & 1 & 3 \\ 2 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 1 & 1 & 8 & 0 \end{vmatrix} \quad C_3' = C_3 - C_4$$

$$= \begin{vmatrix} 1 & -1 & -2 & 3 \\ 2 & 2 & -2 & 1 \\ 0 & 0 & 0 & 3 \\ 1 & 1 & 8 & 0 \end{vmatrix}$$

$$= (-1)^{3+4}\ 3 \begin{vmatrix} 1 & -1 & -2 \\ 2 & 2 & -2 \\ 1 & 1 & 8 \end{vmatrix} \quad \begin{matrix} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - R_1 \end{matrix}$$

$$= -3 \begin{vmatrix} 1 & -1 & -2 \\ 0 & 4 & 2 \\ 0 & 2 & 10 \end{vmatrix}$$

$$= -3(-1)^{1+1} \cdot 1 \begin{vmatrix} 4 & 2 \\ 2 & 10 \end{vmatrix}$$

$$= -3(40 - 4) = -108.$$

So, $x = \frac{|A_1(b)|}{|A|} = \frac{-108}{-18} = 6.$

Similarly,

$$y = \frac{|A_2(b)|}{|A|} = \frac{30}{-18} = \frac{-5}{3},$$

$$z = \frac{|A_3(b)|}{|A|} = \frac{-6}{-18} = \frac{1}{3},$$

$$w = \frac{|A_4(b)|}{|A|} = \frac{6}{-18} = \frac{-1}{3}.$$