SUBSPACES OF R^n

Definition: A subset H of R^n is called a subspace if it has the following three properties:

i) The zero vector is in H,

ii) For each u and v in H, the sum $u + v$ is in H,

iii) For each u in H and each scalar c, the vector cu is in H.

Example: Let

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then, Span$\{e_1, e_2\} = H$ is a subspace of R^3.

Solution: i)

$$0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ so } 0 \in H.$$

ii) Let $u = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and

$$v = c\begin{bmatrix}1\\0\\0\end{bmatrix} + d\begin{bmatrix}0\\1\\0\end{bmatrix}.$$ Then,

$$\begin{aligned} u + v &= a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}1\\0\\0\end{bmatrix} + d\begin{bmatrix}0\\1\\0\end{bmatrix} \\ &= (a+c)\begin{bmatrix}1\\0\\0\end{bmatrix} + (b+d)\begin{bmatrix}0\\1\\0\end{bmatrix} \in H. \end{aligned}$$

iii)

$$\begin{aligned} cu &= c\left(a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix}\right) \\ &= ca\begin{bmatrix}1\\0\\0\end{bmatrix} + cb\begin{bmatrix}0\\1\\0\end{bmatrix} \in H. \end{aligned}$$

It is the xy-plane in R^3.

Remark: If a plane in R^3 does not contain origin 0, then it is not a subspace.

Example: Display the set Span$\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ in R^2. Is it a subspace of R^2?

It is a line passing through the origin and the point $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. It is a subspace.

Remark: **1)** If a line does not pass through the origin, it is not a subspace.

2) R^n is a subspace of R^n.

$\{0\}$ is a subspace of R^n.

3) If $v_1, v_2, ..., v_k$ are vectors in R^n, then Span$\{v_1, v_2, ..., v_k\}$ is a subspace of R^n. This is called the subspace spanned by $v_1, v_2, ..., v_k$.

Example: Determine which of the following are subspaces of R^3.

i) All vectors of the form $(a, 0, 0)$.

ii) All vectors of the form $(a, 1, 1)$.

iii) All vectors of the form (a, b, c), where $b = a + c$.

iv) All vectors of the form (a, b, c), where $b = a + c + 1$.

Solution:i)

$$\begin{aligned} u + v &= (a_1, 0, 0) + (a_2, 0, 0) \\ &= (a_1 + a_2, 0, 0) = (a, 0, 0), \end{aligned}$$

$cu = c(a_1, 0, 0) = (ca_1, 0, 0)$, which is of the form $(a, 0, 0)$.

If $c = 0$, then $c(a, 0, 0) = (0, 0, 0)$.

So, all vectors of the form $(a, 0, 0)$ is a subspace.

ii) $(a_1, 1, 1)+(a_2, 1, 1) = (a_1+a_2, 2, 2)$

$$= (a, 2, 2),$$

which is not of the form $(a, 1, 1)$.

So, all vectors of the form $(a, 1, 1)$, i.e, $\{(a, 1, 1) \mid a \in R \ \}$ is not a subspace.

iii)

$$(a, b, c) = (a, a + c, c)$$
$$= a(1, 1, 0) + c(0, 1, 1)$$

So,

$$\{(a, a + c, c) \mid a, \ c \in R \ \}$$

$$= \text{Span}\{(1, 1, 0), (0, 1, 1)\},$$

which is a subspace.

vi)

$$\begin{aligned}&(a_1, b_1, c_1) + (a_2, b_2, c_2)\\&= (a_1, a_1 + c_1 + 1, c_1)\\&\quad + (a_2, a_2 + c_2 + 1, c_2)\\&= (a_1 + a_2, a_1 + a_2 + c_1 + c_2 + 2, c_1 + c_2)\\&= (a, a + c + 2, c)\end{aligned}$$

which is not of the form $(a, a + c + 1, c)$. So, $\{(a, a + c + 1, c) \mid a, c \in R \}$ is not a subspace.

Example: Let H be the set of all vectors of the form $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}$. Show that H is a subspace of R^3.

Solution: **<u>First way:</u>**

i) For $t = 0$, $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H$.

ii) Let $u = \begin{bmatrix} 2t_1 \\ 0 \\ -t_1 \end{bmatrix}$ and $v = \begin{bmatrix} 2t_2 \\ 0 \\ -t_2 \end{bmatrix}$ be two vectors in H. Then,

$$u + v = \begin{bmatrix} 2t_1 \\ 0 \\ -t_1 \end{bmatrix} + \begin{bmatrix} 2t_2 \\ 0 \\ -t_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2(t_1 + t_2) \\ 0 \\ -(t_1 + t_2) \end{bmatrix} = \begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix} \in H.$$

iii)

$$cu = c\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix} = \begin{bmatrix} c2t \\ 0 \\ c(-t) \end{bmatrix}$$

$$= \begin{bmatrix} 2(ct) \\ 0 \\ -(ct) \end{bmatrix} \in H.$$

Thus, H is a subspace.

Second way:

$\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix} = t\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $t \in R$. Then,

$H = \text{Span}\left\{\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}\right\}$, and so H is a subspace.

Example: Let W be the set of all vectors of the form $\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix}$. Show that W is a subspace of R^4.

Solution:

$$\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} = \begin{bmatrix} s \\ s \\ 2s \\ 0 \end{bmatrix} + \begin{bmatrix} 3t \\ -t \\ -t \\ 4t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}.$$

Thus, $W = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}\right\}$,

and So, W is a subspace of R^4.

Example: Let W be the set of all vectors of the form $\begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}$. Is W a subspace of R^3?

Solution: The equation

$$0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}$$

is not satisfied for any a and b. So, W is not a subspace.

Note: If u and v are two vectors in W, then $u+v$ is not in W either.

Column Space and Null Space of a Matrix

Definition: The column space of a matrix A is the set, Col$A$, of all linear combinations of the columns of A, i.e, the subspace spanned by the columns of A.

The null space of a matrix A is the set, Nul$A$, of all solutions to the homogenous equation $AX = 0$.

Example: Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 7 & 6 \\ -4 & 6 & -2 \end{bmatrix}$

and $b = \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix}$.

Is b in the column space of A?

Is $\begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$ in the null space of A?

Solution:

$$\left[\begin{array}{rrr|r} 1 & -3 & -4 & 3 \\ -3 & 7 & 6 & -5 \\ -4 & 6 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -3 & -4 & 3 \\ 0 & -2 & -6 & 4 \\ 0 & -6 & -18 & 12 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|r} 1 & -3 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 5 & -3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = -3 - 5t$$

$$x_2 = -2 - 3t$$

$$x_3 = t$$

$$\begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix} = (-3-5t)\begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix} + (-2-3t)\begin{bmatrix} -3 \\ 7 \\ 6 \end{bmatrix} + t\begin{bmatrix} -4 \\ 6 \\ -2 \end{bmatrix},$$

and so b is in Col$A$.

$$A\begin{bmatrix}5\\3\\-1\end{bmatrix}=\begin{bmatrix}1&-3&-4\\-3&7&6\\-4&6&-2\end{bmatrix}\begin{bmatrix}5\\3\\-1\end{bmatrix}$$

$$=\begin{bmatrix}5-9+4\\-15+21-6\\-20+18+2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Thus, $\begin{bmatrix}5\\3\\-1\end{bmatrix}$ is in the Nul$A$.

Basis for a Subspace

Definition: A basis for a subspace H of R^n is a linearly independent set (in H) which spans H.

Example: Find a basis for the null space of the matrix $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 7 & 6 \\ -4 & 6 & -2 \end{bmatrix}$.

Solution: We need to find the set of all solutions to $AX = 0$. In the previous example, we had that

$$\left[\begin{array}{rrr|r} 1 & -3 & -4 & 0 \\ -3 & 7 & 6 & 0 \\ -4 & 6 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5t \\ -3t \\ t \end{bmatrix} = t \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}.$$

$\left\{ \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul$A$.

Example: Find a basis for Col$A$.

Solution: We know that

$$\begin{bmatrix} 1 & -3 & -4 \\ -3 & 7 & 6 \\ -4 & 6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the first and the second columns of the matrix A have pivot positions,

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 6 \end{bmatrix} \right\}$$

is a basis for Col$A$.

Dimension of a Subspace

Definition: The dimension of a nonzero subspace H, denoted by dim$H$,is the number of vectors in any basis for H.

The dimension of the zero subspace $\{0\}$ is 0.

The rank of a matrix A, denoted by rank$A$, is the dimension of the column space of A.

The Rank Theorem: If a matrix A has n columns, then

rank$A$+dim Nul$A$=n.

Example: For the matrix

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 7 & 6 \\ -4 & 6 & -2 \end{bmatrix},$$

we have found that

rank$A$=dim Col$A$=2, and dim Nul$A$=1.

$2 + 1 = 3 = \#$ of columns of A.

Example: We are given that

$$A = \begin{bmatrix} 1 & -3 & 2 & 5 \\ -2 & 6 & 0 & -3 \\ 4 & -12 & -4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 5 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

i) Find bases for Col$A$ and Nul$A$.

ii) Find dim Col$A$ and dim Nul$A$.

iii) Verify the Rank Theorem.

Solution: i) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} \right\}$

is a basis for Col$A$.

Solving $AX = 0$ is equivalent to solving $BX = 0$.

$$x_4 = t, \quad x_3 = (-7/4)t, \quad x_2 = s,$$

$$x_1 = 3s - 2x_3 - 5t = 3s - 2(-7/4)t - 5t$$

$$= 3s + (7/2)t - 5t = 3s - (3/2)t.$$

Thus,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s-(3/2)t \\ s \\ (-7/4)t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3/2 \\ 0 \\ -7/4 \\ 1 \end{bmatrix} t.$$

Then,

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 0 \\ -7/4 \\ 1 \end{bmatrix} \right\}$$

is a basis for Nul$A$.

ii) rank$A$=dim Col$A$=2, dim Nul$A$=2.

iii)

$$\begin{aligned} &\text{rank}A + \text{dim Nul}A \\ &= 2+2 \\ &= 4 \\ &= \#\text{ of columns of } A \end{aligned}$$

Example: We are given that

$$A = \begin{bmatrix} 3 & -5 & -1 & 4 & 4 \\ -2 & 4 & 2 & 7 & 8 \\ 5 & -9 & -3 & -3 & -4 \\ -2 & 6 & 6 & 5 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -5 & -1 & 4 & 4 \\ 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

i) Find bases for Col$A$ and Nul$A$.

ii) Find dim Col$A$ and dim Nul$A$.

iii) Verify the Rank Theorem.

Solution: **i)** A basis for Col$A$ is

$$\left\{ \begin{bmatrix} 3 \\ -2 \\ 5 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ 4 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ -3 \\ 5 \end{bmatrix} \right\}.$$

a basis for Nul$A$:

$AX = 0 \Longleftrightarrow BX = 0$. So, let us solve $BX = 0$.

$$x_5 = t, \quad x_4 = -t, \quad x_3 = s,$$

$$2x_2 + 4x_3 + 3x_5 = 0,$$

$$x_2 = -2x_3 - (3/2)x_5 = -2s - (3/2)t,$$

$$3x_1 - 5x_2 - x_3 = 0,$$

$$\begin{aligned} x_1 &= (5/3)x_2 + (1/3)x_3 \\ &= (5/3)[-2s - (3/2)t] + (1/3)s \\ &= -3s - (5/2)t \end{aligned}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3s - (5/2)t \\ -2s - (3/2)t \\ s \\ -t \\ t \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -5/2 \\ -3/2 \\ 0 \\ -1 \\ 1 \end{bmatrix} t.$$

Then,

$$\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5/2 \\ -3/2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

is a basis for Nul$A$.

ii) rank$A$=dim Col$A$=3, dim Nul$A$=2.

iii)

$$\begin{aligned} &\text{rank}A + \text{dim Nul}A \\ &= 3 + 2 \\ &= 5 \\ &= \#\text{ of columns of } A \end{aligned}$$

Example: Find a basis for the subspace spanned by the vectors

$$\begin{bmatrix} 1 \\ -2 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 9 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ -6 \\ -8 \\ -1 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 1 & -3 & -1 & 5 \\ -2 & 5 & 0 & -6 \\ -4 & 9 & -2 & -8 \\ 3 & -5 & 5 & -1 \end{bmatrix} \begin{array}{l} R_2' = R_2 + 2R_1 \\ R_3' = R_3 + 4R_1 \\ R_4' = R_4 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -3 & -1 & 5 \\ 0 & -1 & -2 & 4 \\ 0 & -3 & -6 & 12 \\ 0 & 4 & 8 & -16 \end{bmatrix} \begin{array}{l} R_3' = R_3 - 3R_2 \\ R_4' = R_4 + 4R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & -3 & -1 & 5 \\ 0 & -1 & -2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for the subspace spanned by the given vectors is

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 9 \\ -5 \end{bmatrix} \right\}.$$

Remark: We have

$$\begin{bmatrix} 1 & -3 & -1 & 5 \\ -2 & 5 & 0 & -6 \\ -4 & 9 & -2 & -8 \\ 3 & -5 & 5 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & -1 & 5 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1' = R_1 + 3R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 5 & -7 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and thus,}$$

$$\begin{bmatrix} -1 \\ 0 \\ -2 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ -2 \\ -4 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 5 \\ 9 \\ -5 \end{bmatrix},$$

$$\begin{bmatrix} 5 \\ -6 \\ -8 \\ -1 \end{bmatrix} = -7 \begin{bmatrix} 1 \\ -2 \\ -4 \\ 3 \end{bmatrix} - 4 \begin{bmatrix} -3 \\ 5 \\ 9 \\ -5 \end{bmatrix}.$$

Coordinate Vector

Definition: Let H be a subspace, $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ be a basis for H, and $x \in H$. Then,

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n,$$

and the vector

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is called the coordinate vector of x relative to basis $\mathcal{B}$.

Example: Let H be a subspace with a basis

$$\mathcal{B} = \left\{ b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, b_2 = \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\}.$$

Find the coordinate vector of

$X = \begin{bmatrix} -9 \\ 7 \end{bmatrix}$ relative to basis $\mathcal{B}$.

Solution: We need to find c_1 and c_2 such that

$$\begin{bmatrix} -9 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 5 \end{bmatrix}.$$

$$\left[\begin{array}{rr|r} 1 & -3 & -9 \\ -3 & 5 & 7 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -3 & -9 \\ 0 & -4 & -20 \end{array}\right]$$

$$\sim \left[\begin{array}{rr|r} 1 & -3 & -9 \\ 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & 6 \\ 0 & 1 & 5 \end{array}\right],$$

which gives $c_1 = 6$, and $c_2 = 5$. Thus,

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

Example: Let H be a subspace with a basis

$$\mathcal{B} = \left\{ \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 7 \\ 5 \\ -6 \end{bmatrix} \right\}.$$

Find the coordinate vector of

$X = \begin{bmatrix} -8 \\ -1 \\ -3 \end{bmatrix}$ relative to basis $\mathcal{B}$.

Solution:

$$\left[\begin{array}{rr|r} -3 & 7 & -8 \\ 1 & 5 & -1 \\ -4 & -6 & -3 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 5 & -1 \\ -3 & 7 & -8 \\ -4 & -6 & -3 \end{array}\right]$$

$$\sim \left[\begin{array}{rr|r} 1 & 5 & -1 \\ 0 & 22 & -11 \\ 0 & 14 & -7 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 5 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|r} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{array}\right],$$

which gives $c_1 = 3/2$, and $c_2 = -1/2$.

Thus,

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$