DIAGONALIZATION

Definition: An $n \times n$ matrix A is said to be diagonalizable if A is similar to a diagonal matrix D, i.e, if
$A = PDP^{-1}$ for some invertible matrix P and a diagonal matrix D.

$$A = PDP^{-1} \Longleftrightarrow P^{-1}AP = D.$$

Example: For $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$, we know that eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$, and corresponding eigenvectors are $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, respectively.

Find an invertible 2×2 matrix P, and a diagonal matrix D such that $A = PDP^{-1}$.

Solution: For the matrices

$$P = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}, \text{ and } \ D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

$$A = PDP^{-1}$$

$$= \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}^{-1}.$$

Example: By using the result of the previous example compute A^4.

Solution:

$$A = PDP^{-1} \Longleftrightarrow A^4 = (PDP^{-1})^4.$$

$$A^4 = (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1})$$

$$= PD(P^{-1}P)D(P^{-1}P)D(P^{-1}P)DP^{-1}$$

$$= PDIDIDIDP^{-1} = PDDDDP^{-1}$$

$$= PD^4P^{-1}$$

$$= \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^4 \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^4 & 0 \\ 0 & 2^4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -32 \\ 1 & 16 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 30 \\ -15 & -14 \end{bmatrix}$$

Example: Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -4 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution: Since A is a triangular matrix, eigenvalues of A are the entries on the main diagonal, i.e, $\lambda_1 = 1$, $\lambda_2 = -4$, and $\lambda_3 = -2$.

<u>Eigenvectors corresponding to $\lambda_1 = 1$:</u>

$$A - \lambda_1 I = A - I$$

$$= \begin{bmatrix} 1-1 & -1 & 0 \\ 0 & -4-1 & 2 \\ 0 & 0 & -2-1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 0 & -5 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - I)x = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_1 \in R.$$

Eigenspace for $\lambda_1 = 1$ is $E_1 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

<u>Eigenvectors corresponding to $\lambda_2 = -4$</u>

$$A - \lambda_2 I = A - (-4)I = A + 4I$$

$$= \begin{bmatrix} 1+4 & -1 & 0 \\ 0 & -4+4 & 2 \\ 0 & 0 & -2+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A+4I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1/5)x_2 \\ x_2 \\ 0 \end{bmatrix}$$

$$= (1/5)x_2 \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}, t \in R.$$

Eigenspace for $\lambda_2 = -4$ is $E_2 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} \right\}$.

Eigenvectors corresponding to $\lambda_3 = -2$

$$A - \lambda_3 I = A - (-2)I = A + 2I$$

$$= \begin{bmatrix} 1+2 & -1 & 0 \\ 0 & -4+2 & 2 \\ 0 & 0 & -2+2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A+2I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1/3)x_3 \\ x_3 \\ x_3 \end{bmatrix}$$

$$= (1/3)x_3 \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, t \in R.$$

Eigenspace for $\lambda_3 = -2$ is $E_3 = \text{Span}\left\{\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}\right\}$.

Then,

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \text{ and } P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

$A = PDP^{-1}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1/5 & -2/15 \\ 0 & 1/5 & -1/5 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

Remark: We check whether

$AP = PD$ instead of checking

$A = PDP^{-1}$.

Theorem (Criterion for Diagonalization): Let A be an $n \times n$ matrix such that A has n distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ with corresponding eigenvectors $p_1, p_2, ..., p_n$.

Then A is diagonalizable and

$$P = [p_1 \quad p_2 \quad ... \ p_n]$$

and

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Warning: This theorem gives a sufficient condition for a matrix to be diagonalizable.

It is not necessary for an $n \times n$ matrix to have n distinct eigenvalue to be diagonalizable, as we will see in the following example:

Example: $A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

has only two distinct eigenvalues
$\lambda_1 = 5$ and $\lambda_2 = 1$.
$(|A - \lambda I| = -(\lambda - 5)^2(\lambda - 1).)$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix}.$$

Eigenvectors for $\lambda_1 = 5$:

$$(A - 5I)x = 0$$
$$\iff \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$ Thus,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So, for $\lambda_1 = 5$, we have two (linearly independent) eigenvectors

$p_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $p_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Eigenvectors for $\lambda_2 = 1$:

$$(A - 1I)x = 0$$

$$\iff \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$ Thus,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, t \in R.$$

So, an eigenvector for $\lambda_2 = 1$ is

$p_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Thus,

$$P = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = PDP^{-1}.$$

The eigenspace for $\lambda_1 = 5$ is

$$E_1 = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The eigenspace for $\lambda_2 = 1$ is

$$E_2 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Note that

$$E_1 \cup E_2 = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

form a basis for R^3.

Also note that for the matrices

$$P' = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } D' = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A = P'D'P'^{-1}.$$

Procedure to diagonalize an $n \times n$ matrix:

1) Find the eigenvalues of A.

2) Find a basis for the eigenspace of each eigenvalue.

3) Let $\mathcal{B} = \{p_1, p_2, ..., p_m\}$ be the union of all the bases in 2).

4) If $m < n$, then A is not diagonalizable.

If $m = n$, then A is diagonalizable and

$$D = (d_{ij})_{\substack{1 \leq i \leq n \\ i \leq j \leq n}},$$

$$d_{ii} = \lambda_i, \quad d_{ij} = 0 \text{ if } i \neq j, \text{ and}$$

$$P = [p_1\ p_2\ ...\ p_n],$$

where p_i is an eigenvector corresponding to eigenvalue λ_i.

Example: Is $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ diagonalizable?

Solution: Since A is upper triangular, the eigenvalues of A are $\lambda_1 = \lambda_2 = \lambda = 0$.

$A - \lambda I = A$ and

$$Ax = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Eigenvectors are non-zero multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. A does not have 2 linearly independent eigenvectors, so A is not diagonalizable.

Example: Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$.

Is A diagonalizable?

Solution: Step 1:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 3 \\ 1 & -1-\lambda & 2 \\ -1 & 1 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -1-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} + 3\begin{vmatrix} 1 & -1-\lambda \\ -1 & 1 \end{vmatrix}$$

$$= (1-\lambda)(1+\lambda)(2+\lambda) - 2(1-\lambda) - 3\lambda$$

$$= (1-\lambda^2)(2+\lambda) - 2 - \lambda$$

$$= -\lambda^3 - 2\lambda^2 + \lambda + 2 - 2 - \lambda$$

$$= -\lambda^3 - 2\lambda^2 = -\lambda^2(\lambda+2) = 0$$

$$\Longleftrightarrow \lambda_1 = \lambda_2 = 0, \;\; \lambda_3 = -2.$$

Step 2: Eigenvectors for $\lambda = 0$:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$Ax = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ -x_3 \\ x_3 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = t \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}.$$

So, $p_1 = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 0$ and corresponding eigenspace is span$\left\{\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}\right\}$.

Eigenvector for $\lambda_3 = -2$:

$$A - (-2)I = A + 2I = \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A + 2I)x = 0 \iff \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

So, $p_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_3 = -2$ and corresponding eigenspace is Span$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Step 3: $\mathcal{B} = \{p_1, p_2\} = \left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Step 4: $\mathcal{B}$ has only two elements, A is a 3×3 matrix and $2 < 3$. Thus A is not diagonalizable.

Example: Let $A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$.

Find matrices, if possible, P and D such that $P^{-1}AP = D$ is a diagonal matrix.

Solution:

Step 1: The eigenvalues of A:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 & 2 \\ 0 & 4-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & 0 \\ 2 & 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= (4-\lambda) \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 3-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{vmatrix}$$

$$= (4-\lambda)(3-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$$

$$= (4-\lambda)(3-\lambda)\left[(1-\lambda)^2 - 4\right]$$

$$= (4-\lambda)(3-\lambda)(\lambda^2 - 2\lambda - 3)$$

$$= (4-\lambda)(3-\lambda)(\lambda - 3)(\lambda + 1)$$

$$= (\lambda-4)(\lambda-3)(\lambda - 3)(\lambda + 1)$$

$$= 0 \iff \lambda = -1, 3, 4.$$

Step 2: A basis for the eigenspace of each eigenvalue:

$\underline{\lambda = -1:}$

$$A - (-1)I = A + I = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$(A + I)x = 0 \Longleftrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_4 \\ 0 \\ 0 \\ x_4 \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

So, $p_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -1$ and corresponding eigenspace

$$\text{is Span}\left\{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

<u>$\lambda = 3$:</u>

$$A - 3I = \begin{bmatrix} -2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$(A - 3I)x = 0 \Longleftrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

So, $p_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $p_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are two linearly independent eigenvectors for $\lambda = 3$ and corresponding eigenspace is $\text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$\underline{\lambda = 4:}$

$$A - 4I = \begin{bmatrix} -3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -5/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(A - 4I)x = 0 \Longleftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$x = \begin{bmatrix} 0 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

So, $p_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvectors for $\lambda = 4$ and corresponding eigenspace is $\text{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$.

Step 3:

$$\begin{aligned} \mathcal{B} &= \{p_1, p_2, p_3, p_4\} \\ &= \left\{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}. \end{aligned}$$

Step 4: Since A is 4×4 and $\mathcal{B}$ has 4 elements, A is diagonalizable.

$A = PDP^{-1}$, where

$$P = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

Check: $AP = PD$

Note: Also, $A = PDP^{-1}$, for the matrices

$$P = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Check: $AP = PD$

Example: Diagonalize if possible:

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}.$$

Solution: Step 1: Find Eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix}$$

cofactor expansion along with the first column

$$= (2-\lambda)\begin{vmatrix} -2-\lambda & 1 \\ -3 & 2-\lambda \end{vmatrix} - \begin{vmatrix} -3 & 1 \\ -3 & 2-\lambda \end{vmatrix}$$

$$+ \begin{vmatrix} -3 & 1 \\ -2-\lambda & 1 \end{vmatrix}$$

$$= (2-\lambda)(\lambda^2 - 4 + 3) - [-3(2-\lambda) + 3]$$

$$+(-3 + 2 + \lambda)$$

$$= (2-\lambda)(\lambda^2-1) - (3\lambda-3) + (\lambda-1)$$

$$= (2-\lambda)(\lambda-1)(\lambda+1) - 2(\lambda-1)$$

$$= (\lambda-1)\left[(2-\lambda^2)(\lambda+1) - 2\right]$$

$$= (\lambda-1)(-\lambda^2+\lambda) = -\lambda(\lambda-1)^2 = 0$$

$$\Longleftrightarrow \lambda = 0, 1.$$

Step 2: Eigenvectors for $\lambda = 0$:

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Ax = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

So, $p_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 0$ and corresponding eigenspace is span$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Eigenvector for $\lambda = 1$:

$$A - I = \begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A - I)x = 0 \Longleftrightarrow \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

So, $p_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $p_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

are two linearly independent eigenvectors for $\lambda = 1$ and corresponding

eigenspace is Span$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Step 3:

$$\mathcal{B} = [p_1, p_2, p_3] = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Step 4: Since A is 3×3 and $\mathcal{B}$ has 3 elements, A is diagonalizable.

$A = PDP^{-1}$, where

$$P = [p_1 \quad p_2 \quad p_3] = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Check: $AP = PD$