Complex Eigenvalues

For a complex scalar λ, $\det(A - \lambda I) = 0$ if and only if there is a non-zero vector x in C^n such that $Ax = \lambda x$. We call λ a (complex) eigenvalue and x a (complex) eigenvector. The eigenspace of λ is the set of all solutions of the equation $(A - \lambda I)x = 0$.

If x is a complex vector in C^n, then the vector $\overline{x}$, whose entries are the complex conjugates of the entries in x, is called the complex conjugate of x. Thus, if $x =$Re $x + i$ Im x, then $\overline{x} =$Re $x - i$ Im x.

Let A be an $n \times n$ matrix whose entries are real. Then

$$\overline{Ax} = \overline{A}\overline{x} = A\overline{x}.$$

If λ is an eigenvalue of A with a corresponding eigenvector x in C^n, then

$$\overline{Ax} = \overline{\lambda x} \Longrightarrow A\overline{x} = \overline{\lambda}\overline{x}.$$

That is $\overline{\lambda}$ is also an eigenvalue of A. This shows that if A is a real matrix, its complex eigenvalues occur in conjugate pairs.

Example: Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then,

$$|A - \lambda I| = 0$$

$$\Longleftrightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\Longleftrightarrow \quad (\lambda + i)(\lambda - i) = 0$$

$$\Longleftrightarrow \quad \lambda_1 = i \ , \quad \lambda_2 = -i.$$

<u>Eigenvectors for $\lambda_1 = i$:</u>

$$(A - \lambda_1 I)X = 0 \Longleftrightarrow (A - iI)X = 0$$

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \quad R_1' = iR_1$$

$$\sim \left[\begin{array}{cc|c} 1 & i & 0 \\ -1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, $x_2 = t$, $x_1 = -it$ and

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix},\ t \in C.$$

Complex eigenspace corresponding to $\lambda_1 = i$ is $E_1 =$Span $\left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$.

A basis for E_1 is $\left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$.

<u>Eigenvectors for $\lambda_2 = -i$:</u>

$$(A - \lambda_2 I)X = 0 \Longleftrightarrow (A + iI)X = 0$$

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \quad R_1' = -iR_1$$

$$\sim \left[\begin{array}{cc|c} 1 & -i & 0 \\ -1 & i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$, $t \in C$.

Complex eigenspace corresponding to $\lambda_2 = -i$ is $E_2 =$Span $\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$.

A basis for E_1 is $\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$.

For the matrices $P = \begin{bmatrix} -i & 1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $A = PDP^{-1}$.

<u>check:</u> $AP = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -i & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$
and

$$PD = \begin{bmatrix} -i & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}.$$

$\det P = -i - i = -2i \neq 0$.

P is invertible.

Example: Let $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$.

a) Find the eigenvalues of A, and a basis for each eigenspace in C^2.

b) Diagonalize A.

Solution:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix},$$

$$\det(A - \lambda I) = (1-\lambda)(3-\lambda) + 2$$

$$= \lambda^2 - 4\lambda + 5 = 0,$$

$$\Longleftrightarrow \lambda_{1,2} = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 5}}{2}$$

$$= \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i.$$

Eigenvectors for $\lambda_1 = 2 + i$:

$$(A - \lambda_1 I)X = 0 \Longleftrightarrow (A - (2+i)I)X = 0$$

$$\begin{bmatrix} 1-(2+i) & -2 \\ 1 & 3-(2+i) \end{bmatrix} = \begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1-i & -2 & 0 \\ 1 & 1-i & 0 \end{array}\right] R_1 \longleftrightarrow R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & 1-i & 0 \\ -1-i & -2 & 0 \end{array}\right] R_2' = R_2 + (1+i)R_1$$

$$\sim \left[\begin{array}{cc|c} 1 & 1-i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (-1+i)t \\ t \end{bmatrix} = t \begin{bmatrix} -1+i \\ 1 \end{bmatrix}.$$

Complex eigenspace corresponding to

$\lambda_1 = 2+i$ is $E_1 =$Span $\left\{ \begin{bmatrix} -1+i \\ 1 \end{bmatrix} \right\}$.

A basis for E_1 is $\left\{ \begin{bmatrix} -1+i \\ 1 \end{bmatrix} \right\}$.

<u>Eigenvectors for $\lambda_2 = 2-i$:</u>

$$(A - \lambda_1 I)X = 0 \Longleftrightarrow (A - (2-i)I)X = 0$$

$$\begin{bmatrix} 1-(2-i) & -2 \\ 1 & 3-(2+i) \end{bmatrix} = \begin{bmatrix} -1+i & -2 \\ 1 & 1+i \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1+i & -2 & 0 \\ 1 & 1+i & 0 \end{array}\right] R_1 \longleftrightarrow R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & 1+i & 0 \\ -1+i & -2 & 0 \end{array}\right] R_2' = R_2 + (1-i)R_1$$

$$\sim \left[\begin{array}{cc|c} 1 & 1+i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (-1-i)t \\ t \end{bmatrix} = t \begin{bmatrix} -1-i \\ 1 \end{bmatrix}.$$

Complex eigenspace corresponding to $\lambda_1 = 2 - i$ is $E_2 =$Span $\left\{ \begin{bmatrix} -1-i \\ 1 \end{bmatrix} \right\}$.

A basis for E_2 is $\left\{ \begin{bmatrix} -1-i \\ 1 \end{bmatrix} \right\}$.

For the matrices $P = \begin{bmatrix} -1+i & -1-i \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}$, $A = PDP^{-1}$.

Check:

$$PD = \begin{bmatrix} -1+i & -1-i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}$$

$$= \begin{bmatrix} -3+i & -3-i \\ 2+i & 2-i \end{bmatrix}$$

$$AP = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1+i & -1-i \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3+i & -3-i \\ 2+i & 2-i \end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$. Show that the entries of A^k approach zero for large k.

Solution: If A can be diagonalized, then $A = PDP^{-1}$, for an invertible matrix P and a diagonal matrix D. Then $A^k = (PDP^{-1})^k = PD^kP^{-1}$, $D^k = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$. So, if $|\lambda_1|$, $|\lambda_2| < 1$, then $D^k \longrightarrow 0$, and thus $A^k \longrightarrow 0$.

$$|A - \lambda I| = \begin{vmatrix} \frac{1}{2} - \lambda & -1/2 \\ 1/2 & \frac{1}{2} - \lambda \end{vmatrix}$$

$$= (\frac{1}{2} - \lambda)^2 + \frac{1}{4} = \frac{1}{4} - \lambda + \lambda^2 + \frac{1}{4}$$

$$= \lambda^2 - \lambda + \frac{1}{2}$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1-2}}{2} = \frac{1 \pm \sqrt{-1}}{2} = \frac{1 \pm i}{2}$$

$$\left.\begin{array}{l} \lambda_1 = \dfrac{1+i}{2} \\ \lambda_2 = \dfrac{1-i}{2} \end{array}\right\} \Longrightarrow \lambda_1 \neq \lambda_2$$

$\Longrightarrow A$ diagonalizable.

$$|\lambda_1| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} < 1$$

$$|\lambda_2| = \sqrt{\frac{1}{2}} < 1.$$

Thus, $A^k \longrightarrow 0$ as $k \longrightarrow \infty$.

Example: Let $A = \begin{bmatrix} 0 & 3+4i \\ 3-4i & 0 \end{bmatrix}$.

a) Find the eigenvalues of A, and a basis for each eigenspace in C^2.

b) Diagonalize A.

Solution:

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 3+4i \\ 3-4i & -\lambda \end{vmatrix}$$

$$= \lambda^2 - (3+4i)(3-4i) = \lambda^2 - (9+16)$$

$$= \lambda^2 - 25 = 0,$$

Thus, $\lambda_1 = 5, \quad \lambda_2 = -5.$

Eigenvectors for $\lambda_1 = 5$:

$$(A - 5I)X = 0 \Longleftrightarrow \left[\begin{array}{cc|c} -5 & 3+4i & 0 \\ 3-4i & -5 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & \frac{-3-4i}{5} & 0 \\ 3-4i & -5 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & \frac{-3-4i}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3+4i}{5}t \\ t \end{bmatrix} = \frac{1}{5}t \begin{bmatrix} 3+4i \\ 5 \end{bmatrix}.$$

Complex eigenspace corresponding to $\lambda_1 = 5$ is $E_1 =$Span $\left\{ \begin{bmatrix} 3+4i \\ 5 \end{bmatrix} \right\}$.

A basis for E_1 is $\left\{ \begin{bmatrix} 3+4i \\ 5 \end{bmatrix} \right\}$.

<u>Eigenvectors for $\lambda_2 = -5$:</u>

$$(A+5I)X = 0 \Longleftrightarrow \left[\begin{array}{cc|c} 5 & 3+4i & 0 \\ 3-4i & 5 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & \frac{3+4i}{5} & 0 \\ 3-4i & 5 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & \frac{3+4i}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{3+4i}{5}t \\ t \end{bmatrix} = \frac{1}{5}t \begin{bmatrix} -(3+4i) \\ 5 \end{bmatrix}.$$

Complex eigenspace corresponding to $\lambda_2 = -5$ is $E_2 =$Span $\left\{ \begin{bmatrix} -3-4i \\ 5 \end{bmatrix} \right\}$.

A basis for E_2 is $\left\{ \begin{bmatrix} -3-4i \\ 5 \end{bmatrix} \right\}$.

Note that A has complex entries, eigenvalues of A are real, and corresponding eigenvectors are complex.

For the matrices

$$P = \begin{bmatrix} 3+4i & -3-4i \\ 5 & 5 \end{bmatrix}, \text{ and}$$

$$D = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}, A = PDP^{-1}.$$

Check:

$$PD = \begin{bmatrix} 3+4i & -3-4i \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 15+20i & 15+20i \\ 25 & -25 \end{bmatrix}$$

$$AP = \begin{bmatrix} 0 & 3+4i \\ 3-4i & 0 \end{bmatrix} \begin{bmatrix} 3+4i & -3-4i \\ 5 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 15+20i & 15+20i \\ 25 & -25 \end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} i & 0 & 1 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}$.

a) Find the eigenvalues of A, and a basis for each eigenspace.

b) If possible, diagonalize A.

Solution: Since A is a triangular matrix, eigenvalues of A are the entries on the main diagonal, i.e, $\lambda = i$.

$$A - iI = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and so,}$$

$$(A - iI)X = 0 \iff X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The eigenspace for $\lambda = i$ is

$$E = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

A basis for the eigenspace E is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Since A is a 3×3 matrix and has only two linearly independent eigenvectors, A is not diagonalizable.

Example: Let $A = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$.

If possible, diagonalize A.

Solution:

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & -4 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 2-\lambda & -4 \\ 2 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(2-\lambda)(-2-\lambda) + 8]$$

$$= (1-\lambda)(\lambda^2 + 4)$$

$$= (1-\lambda)(\lambda + 2i)(\lambda - 2i) = 0,$$

which gives $\lambda_1 = 1, \lambda_2 = -2i, \lambda_3 = 2i$.

Eigenvectors for $\lambda_1 = 1$:

$$A - I = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 0 & 0 \\ 2 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A - I)X = 0 \Longleftrightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Eigenvectors for $\lambda_2 = -2i$:

$$A + 2iI = \begin{bmatrix} 2+2i & 0 & -4 \\ 0 & 1+2i & 0 \\ 2 & 0 & -2+2i \end{bmatrix}$$

$$\sim \begin{bmatrix} 1+i & 0 & -2 \\ 0 & 1+2i & 0 \\ 1 & 0 & -1+i \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1+i \\ 0 & 1+2i & 0 \\ 1+i & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1+i \\ 0 & 1+2i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A-I)X = 0 \Longleftrightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1-i \\ 0 \\ 1 \end{bmatrix}$$

Since A is a real matrix and $\lambda_3 = \overline{\lambda_2}$,

an eigenvector for $\lambda_3 = 2i$ is

$$Y = \overline{\begin{bmatrix} 1-i \\ 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} 1+i \\ 0 \\ 1 \end{bmatrix}.$$

For the matrices $P = \begin{bmatrix} 0 & 1-i & 1+i \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & 2i \end{bmatrix}$, $A = PDP^{-1}$.