Name (PRINT): ______________________________ Student # __________

Test #4
Mathematics 69.107
Nov 17, 2000
Full marks: 40

Answer all questions. Use non-programmable calculator if needed.

Evaluate the following integrals.

(1) [8 marks]

$$\int \frac{2}{(x+1)^2(x^2+4)}\,dx$$

Solution:

$$\frac{2}{(x+1)^2(x^2+4)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+4}$$

Thus,

$$2 = A(x+1)(x^2+4) + B(x^2+4) + (Cx+D)(x+1)^2.$$

When $x = -1$, $2 = B(1+4)$. Thus,

$$B = \frac{2}{5}.$$

Substituting B back and simplifying, we have

$$2 = (A+C)x^3 + (A+2C+D+2/5)x^2 + (4A+C+2D)x + (4A+D+8/5).$$

Equating coefficients we get

$$\begin{aligned} A+C &= 0 \\ A+2C+D &= \frac{-2}{5} \\ 4A+C+2D &= 0 \\ 4A+D &= 2-\frac{8}{5} \end{aligned}$$

We have that $C = -A$, and $D = -4A + 2/5$. We conclude that

$$4A + C + 2D = 4A - A - 8A + 4/5 = -5A + 4/5 = 0,$$

and

$$A = 4/25, \quad C = -4/25, \quad D = -6/25.$$

Finally,

$$\begin{aligned} \int \frac{2}{(x+1)^2(x^2+4)}\,dx &= \int \frac{4/25}{x+1}\,dx + \int \frac{2/5}{(x+1)^2}\,dx + \int \frac{(-4x-6)/25}{x^2+4}\,dx \\ &= \frac{4}{25}\ln|x+1| - \frac{2}{5}\frac{1}{x+1} - \frac{2}{25}\ln(x^2+4) - \frac{6}{25}\int \frac{1}{x^2+2^2}\,dx. \end{aligned}$$

Thus,

$$\int \frac{2}{(x+1)^2(x^2+4)}\,dx = \frac{4}{25}\ln|x+1| - \frac{2}{5(x+1)} - \frac{2}{25}\ln(x^2+4) - \frac{3}{25}\tan^{-1}\left(\frac{x}{2}\right).$$

(2) [8 marks]

$$\int x \tan^{-1} x \; dx$$

Solution:

$$\begin{aligned}
\int x \tan^{-1} x \; dx &= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2} \; dx = \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) \; dx \\
&= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + C.
\end{aligned}$$

(3) [8 marks]

$$\int e^x \; \sin(2x) \; dx$$

Solution:

$$\begin{aligned}
\int e^x \; \sin(2x) \; dx &= -\frac{1}{2}e^x \; \cos(2x) + \frac{1}{2}\int e^x \; \cos(2x) \; dx \\
&= -\frac{1}{2}e^x \; \cos(2x) + \frac{1}{4}e^x \; \sin(2x) - \frac{1}{4}\int e^x \; \sin(2x) \; dx.
\end{aligned}$$

Thus,

$$\frac{5}{4}\int e^x \; \sin(2x) \; dx = -\frac{1}{2}e^x \; \cos(2x) + \frac{1}{4}e^x \; \sin(2x) + C_1,$$

and finally,

$$\int e^x \; \sin(2x) \; dx = \frac{1}{5}e^x \left(\sin(2x) - 2\cos(2x)\right) + C.$$

(4) [8 marks]

$$\int_0^{\pi/2} \sin^2 x \; \cos^3 x \; dx$$

Solution:

$$\begin{aligned}
\int_0^{\pi/2} \sin^2 x \; \cos^3 x \; dx &= \int_0^{\pi/2} \sin^2 x \; \cos^2 x \; \cos x \; dx = \int_0^{\pi/2} \sin^2 x \; (1 - \sin^2 x) \; \cos x \; dx \\
&= \int_0^1 u^2(1-u^2) \; du = \left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Bigg|_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15},
\end{aligned}$$

where we used the substitution $u = \sin x$.

(5) [8 marks]

$$\int \frac{1}{x^2 - 2x + 5} \; dx$$

Solution:

$$\int \frac{1}{x^2 - 2x + 5} \; dx = \int \frac{1}{(x-1)^2 + 2^2} \; dx = \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C.$$