Name (PRINT): ____________________ Student # __________

Test #1
Mathematics 69.107
Sept 29, 2000
Full marks: 40

Answer all questions. Use non-programmable calculator if needed.

(1) [10 marks]

(a) [5 marks] If $f(x) = \frac{x^2+1}{\cos(2x)}$, find $f^{'}(x)$.

$$\begin{aligned} f^{'}(x) &= \frac{2x(\cos(2x)) - (x^2+1)(2)(-\sin(2x))}{\cos^2(2x)} = \frac{2x(\cos(2x)) + 2(x^2+1)(\sin(2x))}{\cos^2(2x)} \\ &= 2x\frac{1}{\cos(2x)} + 2(x^2+1)\frac{\sin(2x)}{\cos(2x)}\frac{1}{\cos(2x)} = 2x\sec(2x) + 2(x^2+1)\tan(2x)\sec(2x). \end{aligned}$$

(b) [5 marks] Evaluate $\frac{d}{dx}\cos(x\sin(x))$.

$$\begin{aligned} \frac{d}{dx}\cos(x\sin(x)) &= -\sin(x\sin(x))\frac{d}{dx}(x\sin(x)) \\ &= -\sin(x\sin(x))(\sin(x) + x\cos(x)). \end{aligned}$$

(2) [10 marks]

Consider the function $2(x^2+y^2)^2 = 25(x^2-y^2)$.

(a) [6 marks] Find $\frac{dy}{dx}$ by the method of implicit differentiation.

Using the method of implicit differentiation we have

$$\begin{aligned} 4(x^2+y^2)(2x+2yy^{'}) &= 25(2x-2yy^{'}) \\ 8x(x^2+y^2) + 8y(x^2+y^2)y^{'} &= 50x - 50yy^{'}) \\ (50y + 8y(x^2+y^2))y^{'} &= 50x - 8x(x^2+y^2). \end{aligned}$$

Therefore,

$$y^{'} = \frac{25x - 4x(x^2+y^2)}{25y + 4y(x^2+y^2)}.$$

(b) [4 marks] Find the equation of the tangent line to the curve at the point $(3,1)$.

At $(3,1)$, the slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{(3,1)} = \frac{25(3) - 4(3)((3)^2+(1)^2)}{25(1) + 4(1)((3)^2+(1)^2)} = -\frac{9}{13}.$$

Thus, the equation of the tangent line at $(3,1)$ is

$$y - 1 = -\frac{9}{13}(x-3) \qquad \Rightarrow \qquad 9x + 13y = 40.$$

(3) [10 marks]

(a) [6 marks] If $f(x) = \sqrt{\sin^{-1}(\frac{2}{x})}$, find $f^{'}(x)$.

$$\begin{aligned} f^{'}(x) &= \frac{1}{2}\left(\sin^{-1}\left(\frac{2}{x}\right)\right)^{-1/2} \frac{1}{\sqrt{1-(\frac{2}{x})^2}}\left(-\frac{2}{x^2}\right) \\ &= -\frac{1}{\sqrt{\sin^{-1}(\frac{2}{x})}} \frac{x}{\sqrt{x^2-4}} \frac{1}{x^2} \\ &= -\frac{1}{x\sqrt{x^2-4}\sqrt{\sin^{-1}(\frac{2}{x})}}. \end{aligned}$$

(b) [4 marks] Using the result: $\lim_{t\to 0} \frac{\sin(t)}{t} = 1$, evaluate $\lim_{t\to 0} \frac{\sin^2(t)}{3t^2}$.

$$\lim_{t\to 0} \frac{\sin^2(t)}{3t^2} = \frac{1}{3}\lim_{t\to 0}\left(\frac{\sin(t)}{t}\right)^2 = \frac{1}{3}\left(\lim_{t\to 0}\frac{\sin(t)}{t}\right)^2 = \frac{1}{3}(1)^2 = \frac{1}{3}.$$

(4) [10 marks]

Consider the equation $x^4 + x - 1 = 0$. Use Newton's method to find a solution, correct to two decimal places, starting with $x_0 = 0$.

We have

$$f(x) = x^4 + x - 1, \qquad f^{'}(x) = 4x^3 + 1, \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Then,

$$\begin{aligned} x_0 &= 0 \\ x_1 &= 0 - \frac{f(0)}{f'(0)} = -\frac{-1}{1} = 1 \\ x_2 &= 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{5} = 0.8 \\ x_3 &= 0.8 - \frac{f(0.8)}{f'(0.8)} = 0.8 - 0.0687664 = 0.7312336 \\ x_4 &= 0.7312336 - \frac{(0.7312336)^4 + 0.7312336 - 1}{4(0.7312336)^3 + 1} = 0.7245486 \\ x_5 &= 0.7245486 - \frac{(0.7245486)^4 + 0.7245486 - 1}{4(0.7245486)^3 + 1} = 0.7244921. \end{aligned}$$

The root to two decimal points is 0.72.