Aperiodic Substitution Tilings

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Overview

We study tilings of $\mathbb{R}^2$ which are aperiodic, but not completely random.

Tiling $T \longrightarrow$ topological space Ω_T

Elements of Ω_T are tilings, and $\mathbb{R}^2$ acts by translating them.

Compute Ω_T for some periodic examples, use this to describe it in aperiodic cases.

C^*-algebra of a tiling and invariants.

Tilings

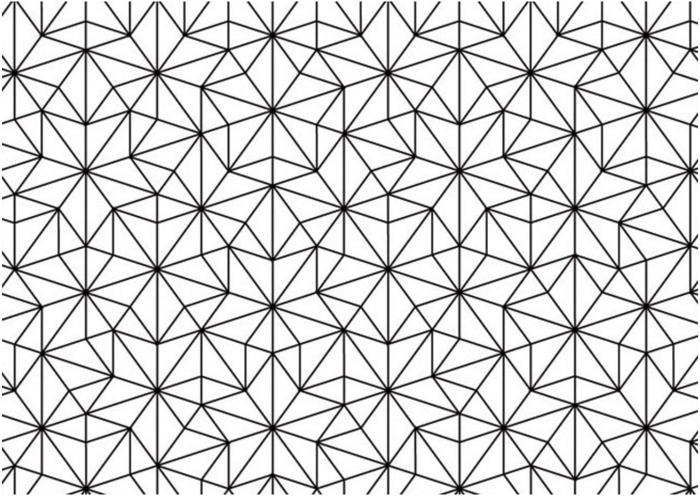
Definition

A **tiling** T of $\mathbb{R}^2$ is a countable set $T = \{t_1, t_2, \dots\}$ of subsets of $\mathbb{R}^2$, called **tiles** such that

- Each tile is homeomorphic to the closed ball (they are usually polygons),
- $t_i \cap t_j$ has empty interior whenever $i \neq j$, and
- $\cup_{i=1}^{\infty} t_i = \mathbb{R}^2$.

- A **patch** is a finite subset of T. The **support** of a patch is the union of its tiles.
- If T is a tiling, $x \in \mathbb{R}^2$, $T + x$ is the tiling formed by translating every tile in T by x.
- T is **aperiodic** if $T + x \neq T$ for all $x \in \mathbb{R}^2 \setminus \{0\}$.

Example: Penrose Tiling



Substitution Rules

Frequently we have a finite number of "tile types".

$\mathcal{P} = \{p_1, p_2, \ldots, p_N\}$ is called a set of **prototiles** for T if $t \in T \implies t = p + x$ for some $p \in \mathcal{P}$ and $x \in \mathbb{R}^2$.

Definition

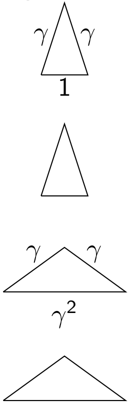
A **substitution rule** on a set of prototiles $\mathcal{P}$ consists of

- A scaling constant $\lambda > 1$
- A rule ω such that, for each $p \in \mathcal{P}$, $\omega(p)$ is a patch whose support is λp and whose tiles are translates of members of $\mathcal{P}$.

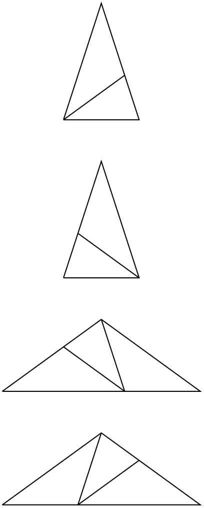
- ω can be applied to patches and tilings by applying it to each tile.
- ω can be iterated, since $\omega(p)$ is a patch.

Example: Penrose Tiling

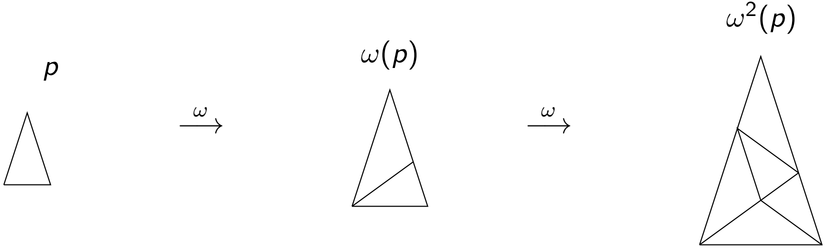
Prototiles
(+ rotates by $\frac{\pi}{5}$)
γ = golden ratio



$\overset{\omega}{\underset{\longrightarrow}{\lambda=\gamma}}$

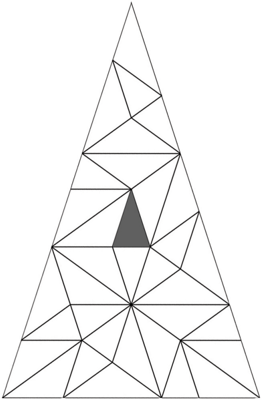


Example: Penrose Tiling



Producing a Tiling from a Substitution Rule

$\omega^4(\triangle) =$



$p \subset \omega^4(p) \subset \omega^8(p)$

$\omega^{4n}(p) \subset \omega^{4(n+1)}(p)$

Then

$$T = \bigcup_{n=1}^{\infty} \omega^{4n}(p)$$

is a tiling.

The Tiling Metric

The Tiling Metric

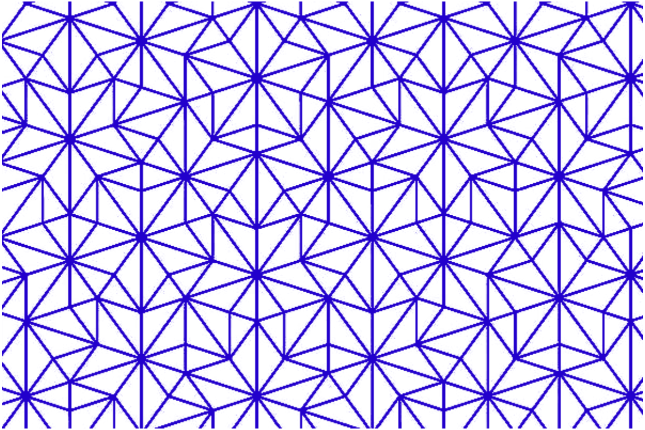
The distance between two tilings T and T' is less than ε if T and T' agree on a ball of radius $\frac{1}{\varepsilon}$ up to a small translation of at most ε. The distance $d(T, T')$ is then defined as the inf of all these ε (or 1 if no such ε exists).

There are essentially two ways that T and T' can be close:

1. $T' = T + x$ for some $|x| < \varepsilon$.
2. T' agrees with T exactly on a large ball around the origin, then disagrees elsewhere.

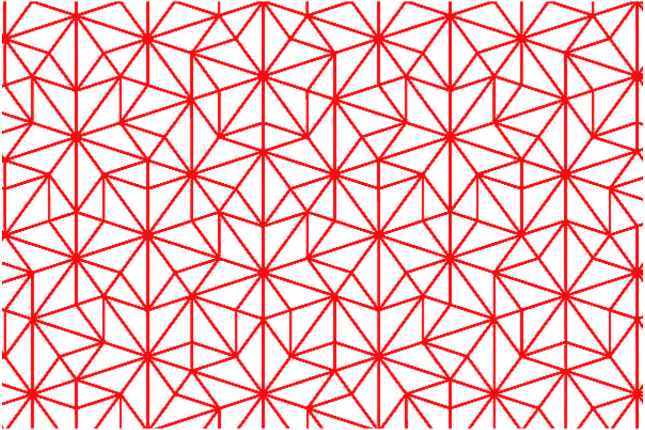
In most cases, 1 looks like a disc while 2 looks like a Cantor set (ie, totally disconnected, compact, no isolated points).

Example: Penrose Tiling



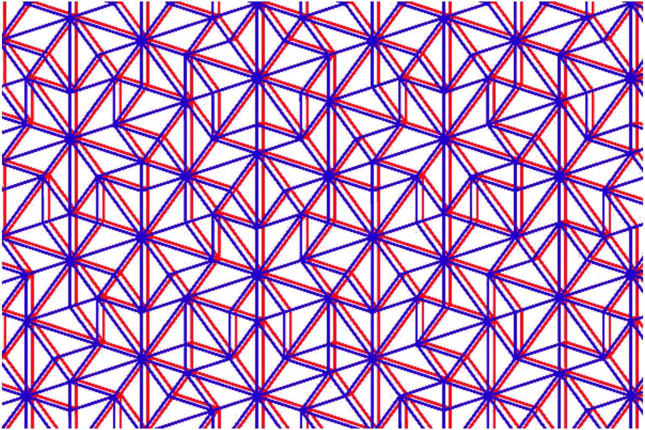
T_1

Example: Penrose Tiling



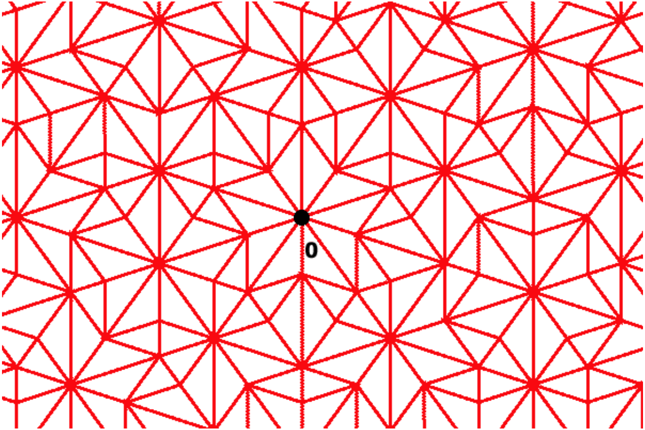
T_2

Example: Penrose Tiling



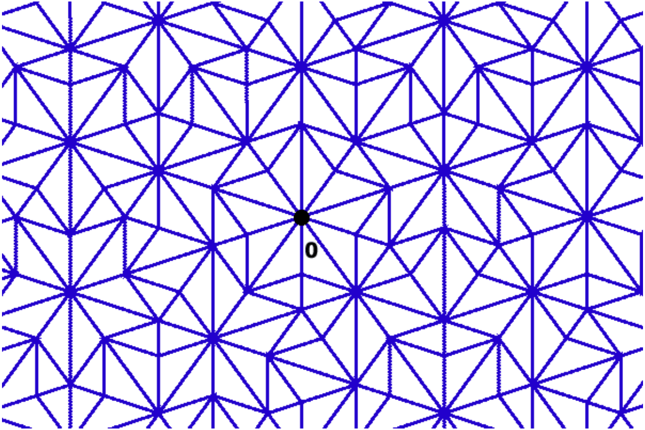
T_2 is a small shift of $T_1 \Rightarrow T_1$ is close to T_2

Example: Penrose Tiling



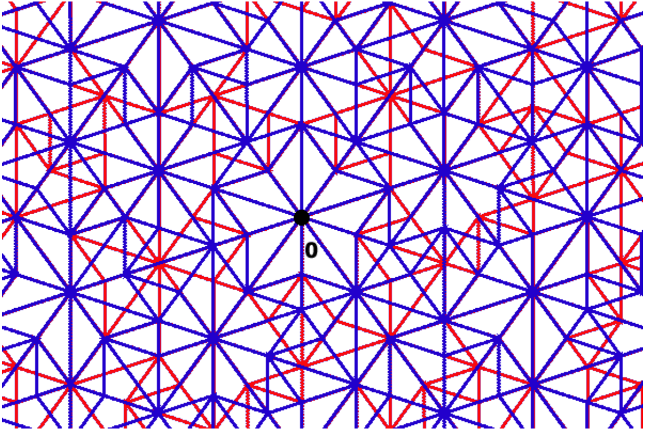
T_1

Example: Penrose Tiling



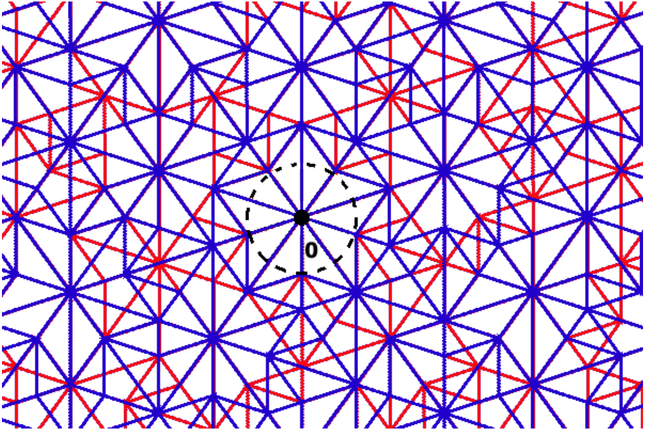
T_2

Example: Penrose Tiling



T_1 and T_2 agree around the origin, disagree elsewhere.

Example: Penrose Tiling



$d(T_1, T_2) <$ (radius of the ball above.)$^{-1}$

The Tiling Space

Definition

The **tiling space** associated with a tiling T, denoted Ω_T, is the completion of $T + \mathbb{R}^2 = \{T + x \mid x \in \mathbb{R}^2\}$ in the tiling metric. This is also called the **continuous hull** of T.

It's not obvious, but the elements of Ω_T are tilings.

Ω_T is the set of all tilings T' such that every patch in T' appears somewhere in T.

Properties of the tiling space

Definition

A tiling T is said to have **Finite Local Complexity** (FLC) if for every $r > 0$, the number of different patches (up to translation) of diameter r in T is finite .

This is always satisfied if the prototiles are polygons and meet full-edge to full-edge. If T has FLC, then Ω_T is compact.

Definition

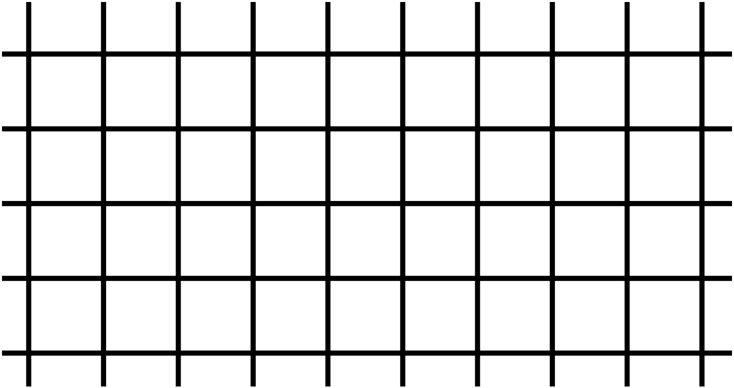
A substitution rule ω is said to be **primitive** if there exists some n such that such that $\omega^n(p_i)$ contains a copy of p_j for every $p_i, p_j \in \mathcal{P}$.

If T is formed by a primitive substitution rule, and $T' \in \Omega_T$, then $\Omega_{T'} = \Omega_T$.

T, T' both come from same primitive $\omega \implies \Omega_T = \Omega_{T'}$

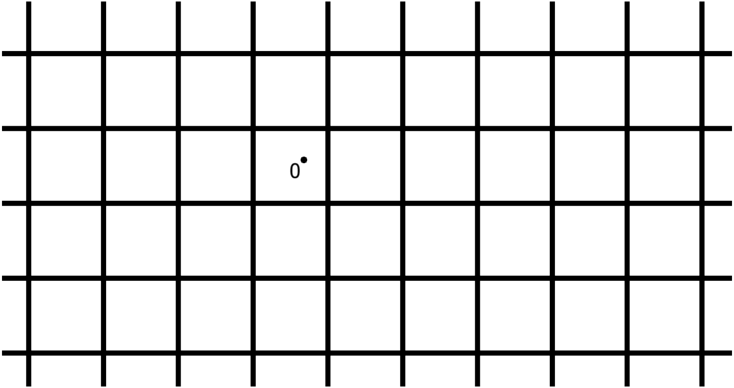
Replace $\Omega_T \to \Omega$.

Example: Grid



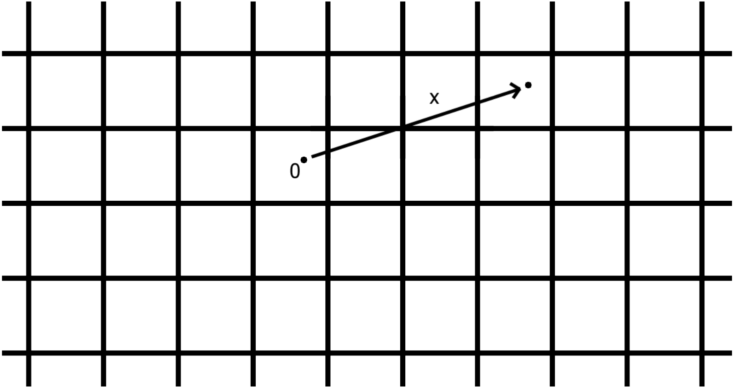
Infinite grid in $\mathbb{R}^2$

Example: Grid



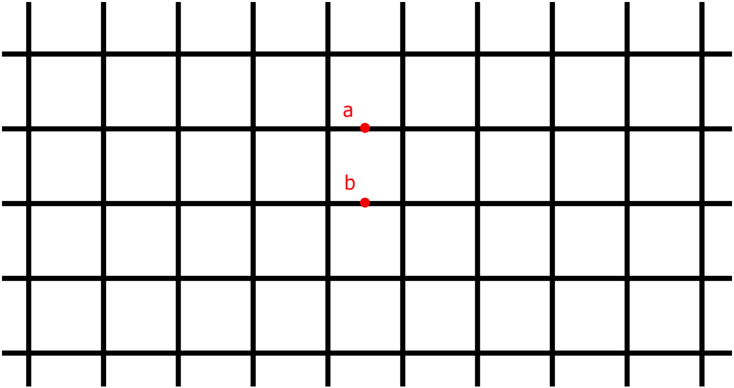
Placement of the origin in any square determines the tiling.

Example: Grid



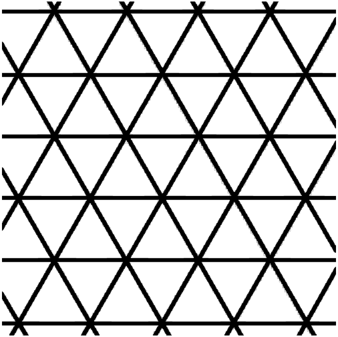
Placement of the origin in any square determines the tiling. $T = T - x$

Example: Grid



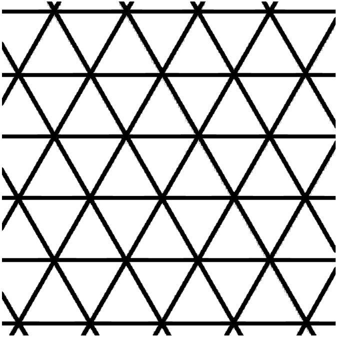
a and b are the same in the tiling space $\Longrightarrow \Omega_T \cong \mathbb{T}^2$

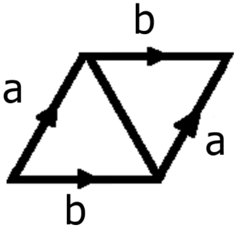
Example: Equilateral Triangles



Infinite tiling of the plane with equilateral triangles.

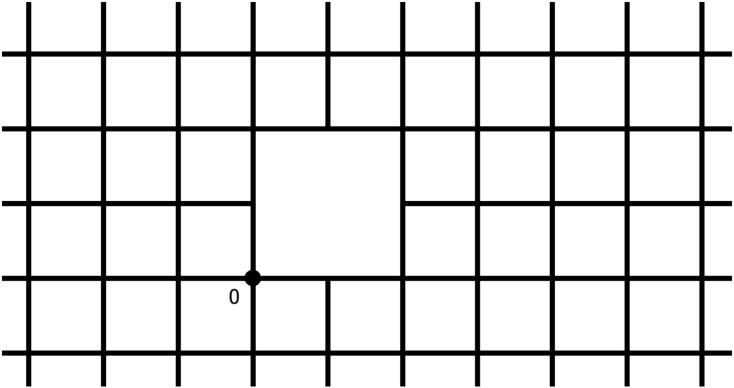
Example: Equilateral Triangles





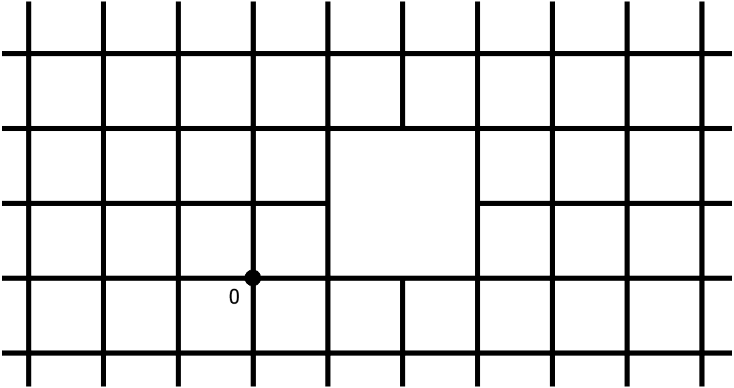
Space of “origin placements” $\Omega_T \cong \mathbb{T}^2$

Example: Modified Grid



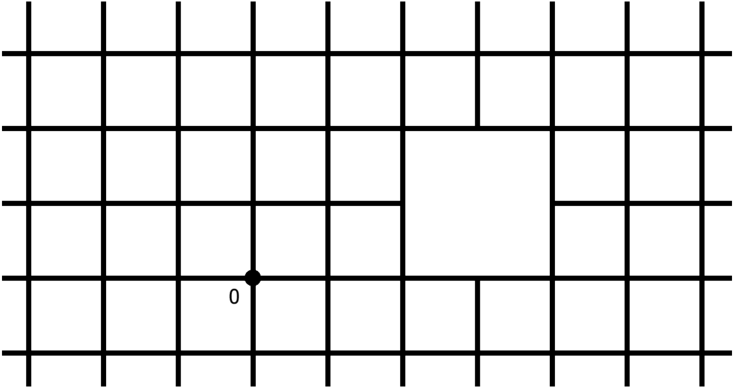
T, same as usual grid with a larger square at origin.

Example: Modified Grid



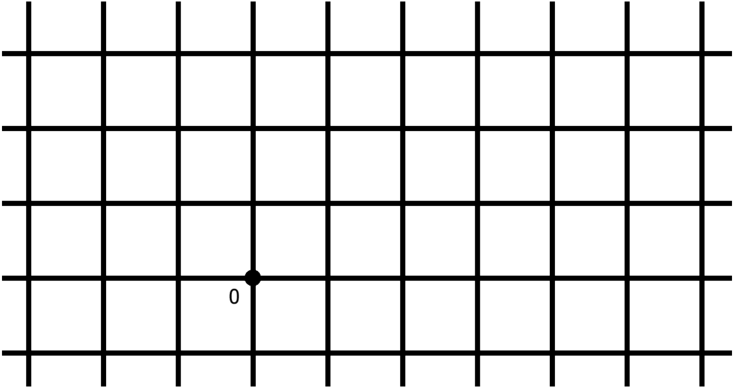
$T + (1, 0)$

Example: Modified Grid



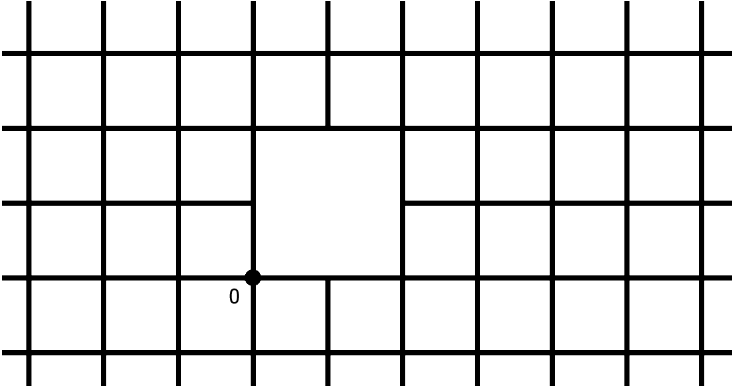
$T + (2,0)$

Example: Modified Grid



$T + (51, 0)$

Example: Modified Grid



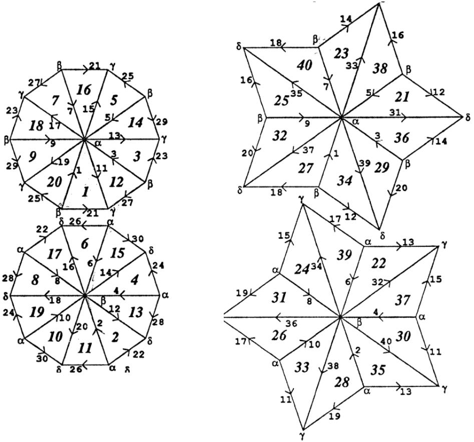
$T + (n, 0)$ is a Cauchy sequence converging to the usual grid.

Approximating the tiling space

For periodic tilings, we made Ω_T by building a space out of the prototiles. We "glued them together" along their edges if those edges could touch in the tiling.

Idea: do this for aperiodic tilings $\rightarrow$ obtain a CW-complex Γ, but not Ω_T.

Approximating the tiling space



The CW complex Γ for the Penrose tiling.

Approximating the tiling space

For periodic tilings, we made Ω_T by building a space out of the prototiles. We "glued them together" along their edges if those edges could touch in the tiling.

Idea: do this for aperiodic tilings $\rightarrow$ obtain a CW-complex Γ, but not Ω_T.

Anderson, Putnam (1996) - ω induces a homeomorphism γ on Γ, and if we form the inverse limit

$$\Omega_0 = \varprojlim \Gamma = \{(x_i)_{i \in \mathbb{N}} \mid x_i \in \Gamma \forall i, x_i = \gamma(x_{i+1})\}$$

Then if T satisfies another condition (called **forcing the border**),

$$\Omega_0 \cong \Omega$$

The discrete tiling space

Recall two ways that T and T' can be close:

1. $T' = T + x$ for some $|x| < \varepsilon$.
2. T' agrees with T exactly on a large ball around the origin, then disagrees elsewhere.

In the case our periodic examples, neighbourhoods consist of the first way only. The second way is much more interesting!

For this reason we assume finite local complexity, a primitive substitution rule, and that every tiling in Ω is aperiodic.

We produce a subspace of Ω to essentially make the first way vanish.

The discrete tiling space

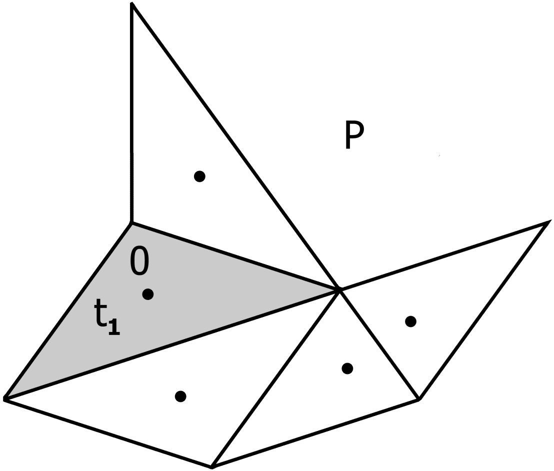
We replace each prototile $p \in \mathcal{P} \longrightarrow (p, x(p))$, where $x(p) \in$ the interior of p. The point $x(p)$ is called the **puncture** of p. If $t \in T$, then $t = p + y$ for some y and so we define $x(t) = x(p) + y$.

Define $\Omega_{punc} \subset \Omega$ as the set of all tilings $T \in \Omega$ such that the origin is on a puncture of a tile in T, ie, $x(t) = 0$ for some $t \in T$. Ω_{punc} is called the **discrete tiling space** or **discrete hull**.

Ω_{punc} is homeomorphic to a Cantor set (ie it is totally disconnected, compact, and has no isolated points).

Its topology is generated by clopen sets of the following form: if P is a patch and $t \in P$, then let

$$U(P, t) = \{T \in \Omega_{punc} \mid 0 \in t \in P \subset T\}$$



If T looks like this around the origin $0 \in \mathbb{R}^2$, then $T \in U(P, t_1)$.

The tiling algebra

Let $\mathcal{R}_{punc} = \{(T, T+x) \mid T, T+x \in \Omega_{punc}\}$. Then $\mathcal{R}_{punc}$ is an equivalence relation. Topology from $\Omega_{punc} \times \mathbb{R}^2$.

$C_c(\mathcal{R}_{punc})$ - the complex-valued compactly supported continuous functions on $\mathcal{R}_{punc}$.

For $f, g \in C_c(\mathcal{R}_{punc})$, the convolution product and involution are

$$f * g(T, T') = \sum_{T'' \in [T]} f(T, T'')g(T'', T')$$

$$f^*(T, T') = \overline{f(T', T)}$$

$C_c(\mathcal{R}_{punc})$ is a $*$-algebra, and when completed in a suitable norm becomes a C*-algebra, $C^*(\mathcal{R}_{punc})$.

The tiling algebra

K-theory is an important invariant for C*-algebras. $K_0(A)$ records the structure of the projections in A up to a generalized notion of dimension.

Anderson, Putnam (1996) - the K-theory of $C^*(\mathcal{R}_{punc})$ is isomorphic to the cohomology of Ω.

$$K_0(C^*(\mathcal{R}_{punc})) \cong \check{H}^0(\Omega) \oplus \check{H}^2(\Omega)$$

This is great news!

- Cohomology is well-behaved with respect to inverse limits.
- Cohomology of Γ is easy to compute.

Penrose: $K_0(C^*(\mathcal{R}_{punc})) \cong \mathbb{Z} \oplus \mathbb{Z}^8$.