Inverse Semigroups in C*-algebras

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C*-algebras

A C*-algebra is a set A which:

1. is an algebra over $\mathbb{C}$,
2. has an involution $a \mapsto a^*$ which is conjugate linear, and $(ab)^* = b^* a^*$,
3. has a norm $\| \cdot \|$ with which it is complete normed algebra (i.e. it is a Banach algebra)
4. for all $a \in A$, $\|a^* a\| = \|a\|^2$ (the C*-condition).

Examples:

1. $\mathbb{C}$
2. $\mathbb{M}_n(\mathbb{C})$ the $n \times n$ matrices over $\mathbb{C}$
3. $B(\mathcal{H})$, the bounded operators on a Hilbert space $\mathcal{H}$.

Commutative C*-algebras

X – compact Hausdorff space

$$C(X) = \{f : X \to \mathbb{C} \mid f \text{ is continuous}\}$$

This is a C*-algebra with pointwise sum, product, and complex conjugate, with

$$\|f\| = \sup_{x \in X} |f(x)|$$

If X is only locally compact, $C_0(X)$ is a C*-algebra, but without a unit.

Bedrocks of C*-algebra theory

Theorem (Gelfand-Naimark)

1. *Every commutative C*-algebra is isomorphic to $C_0(X)$ for some locally compact X.*
2. *$C_0(X)$ and $C_0(Y)$ are isomorphic if and only if X and Y are homeomorphic.*

C*-algebras are "noncommutative geometry"

Theorem (Gelfand-Naimark-Segal construction)

Every C-algebra is isomorphic to a norm-closed subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

Inverse semigroups in C*-algebras

A projection is an element $p \in A$ such that

$$p = p^2 = p^*$$

An isometry is an element $s \in A$ such that

$$s^* s = 1$$

A partial isometry is an element $s \in A$ such that

$$ss^* s = s$$

equivalently, $s^* s$ and ss^* are both projections

Any set of partial isometries $S \subset A$ closed under multiplication and involution is an inverse semigroup, and $E(S)$ is a commuting set of projections.

Example: 2×2 matrices

$\mathbb{M}_2(\mathbb{C})$ – 2×2 matrices over $\mathbb{C}$.

$$S_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

All matrix units, together with identity and zero.

S_2 is an inverse semigroup which generates $\mathbb{M}_2(\mathbb{C})$

$$\mathcal{I}_2 = S_2 \cup \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

All rook matrices, also an inverse semigroup which generates $\mathbb{M}_2(\mathbb{C})$

$\mathcal{I}_2$ is isomorphic to the symmetric inverse monoid on the two element set.

Inverse semigroups in C*-algebras

Question: can every inverse semigroup be realized as a set of partial isometries in some C*-algebra?

Answer: Yes – Paterson (99)

$\pi : S \to A$ is a representation if $\pi(st) = \pi(s)\pi(t)$ and $\pi(0) = 0$.

There exists $C^*(S)$ which is universal for representations of S.

$C^*(S) = C^*(\mathcal{G}_u(S))$ for an étale groupoid $\mathcal{G}_u(S)$ constructed from S.

$\mathcal{G}_u(S)^{(0)}$ is homeomorphic to the space of filters in $E(S)$, and $C_0(\mathcal{G}_u(S)^{(0)}) = C^*(E(S))$ is always a commutative subalgebra of $C^*(S)$.

Example: 2×2 matrices

$\mathbb{M}_2(\mathbb{C})$ – 2×2 matrices over $\mathbb{C}$

e_{ij} = matrix with 1 in (i, j) entry, 0 elsewhere.

$$E(S_2) = \{1_2, e_{11}, e_{22}, 0_2\}$$

Set of filters $= \{\{1_2\}, \{1_2, e_{11}\}, \{1_2, e_{22}\}\}$

$C(\mathcal{G}_u(S_2)^{(0)}) = \mathbb{C}^3$

Even though it "feels like" $C^*(S_2)$ should be $\mathbb{M}_2(\mathbb{C})$, it cannot be.

$C^*(S_2) \cong \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{C}$, with universal representation given by

$$\pi_u\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \qquad \pi_u\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = \begin{bmatrix}a & b & 0\\c & d & 0\\0 & 0 & 0\end{bmatrix} \textit{else}$$

Example: 2×2 matrices

$$\pi_u\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad \pi_u\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right) = \begin{bmatrix}a & b & 0\\ c & d & 0\\ 0 & 0 & 0\end{bmatrix} \textit{ else}$$

$\pi_u(E(S_2))$ is a commuting set of projections, and two commuting projections in a C*-algebra always have a join:

$$e \vee f = e + f - ef$$

$$\pi_u(e_{11}) \vee \pi_u(e_{22}) = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} \neq \pi_u(1_2) = \pi_u(e_{11} \vee e_{22})$$

If we want to recover $\mathbb{M}_2(\mathbb{C})$, we would like to look at representations which preserve joins.

Exel's tight representations

The above example was special, $E(S_2)$ is a Boolean algebra, and has joins.

In general, $E(S)$ won't have joins.

$C \subset_{\text{fin}} E(S)$ is a cover for $e \in E(S)$ if for all $0 \neq f \leqslant e$, there is a $c \in C$ such that $fc \neq 0$.

Exel (08) introduced the notion of a tight representation.

π is tight if whenever C is a cover for e, we have $\bigvee_{c \in C} \pi(c) = \pi(e)$ $(\pm\epsilon)$

$C^*_{\text{tight}}(S)$ universal for tight representations.

$C^*_{\text{tight}}(S_2) \cong \mathbb{M}_2(\mathbb{C})$

Example: Cuntz algebras

Example: Cuntz algebras

ℓ^2: Hilbert space of square-summable complex sequences.

Define $s_0, s_1 \in B(\ell^2)$ by

$$s_0(x_1, x_2, x_3, \dots) = (x_1, 0, x_2, 0, x_3, 0, \dots)$$
$$s_1(x_1, x_2, x_3, \dots) = (0, x_1, 0, x_2, 0, x_3, \dots)$$

$$s_0^*(x_1, x_2, x_3, \dots) = (x_1, x_3, x_5, \dots)$$
$$s_1^*(x_1, x_2, x_3, \dots) = (x_2, x_4, x_6, \dots)$$

$$s_0^* s_0 = 1 = s_1^* s_1$$

$s_0 s_0^* =$ projection onto odd coordinates
$s_1 s_1^* =$ projection onto even coordinates

Example: Cuntz algebras

Then

$$s_0^* s_0 = 1 = s_1^* s_1$$

$$s_0 s_0^* s_1 s_1^* = 0$$

$$s_0 s_0^* + s_1 s_1^* = 1$$

Cuntz (77) showed that the C*-algebra generated by s_0, s_1, denoted $\mathcal{O}_2$ depended only on the relations above.

Analogous construction for $\mathcal{O}_n$ – these are the Cuntz algebras. They were the first examples of separable simple C*-algebras which are infinite (ie, contain a proper isometry).

Example: Cuntz algebras

$\{0,1\}^* =$ (possibly empty) words in $\{0,1\}$

For $\alpha \in \mathcal{A}^*$, let $s_\alpha := s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{|\alpha|}}$, and note $s_\alpha^* = s_{\alpha_{|\alpha|}}^* \cdots s_{\alpha_2}^* s_{\alpha_1}^*$

Let $s_\emptyset = 1$

$s_\alpha s_\beta = s_{\alpha\beta}$ and $s_\alpha^* s_\beta^* = s_{\beta\alpha}^*$

$P_2 = \{s_\alpha s_\beta^* \mid \alpha, \beta \in \{0,1\}^*\} \cup \{0\}$ polycyclic monoid

$$(s_\alpha s_\beta^*)(s_\gamma s_\nu^*) = \begin{cases} s_{\alpha\gamma'} s_\nu^* & \text{if } \gamma = \beta\gamma' \\ s_\alpha s_{\nu\beta'}^* & \text{if } \beta = \gamma\beta' \\ 0 & \text{otherwise} \end{cases}$$

$E(P_2) = \{s_\alpha s_\alpha^* \mid \alpha \in \{0,1\}^*\} \cup \{0\}$

Example: Cuntz algebras

$$s_0^* s_0 = 1 = s_1^* s_1$$

$$s_0 s_0^* s_1 s_1^* = 0$$

$$s_0 s_0^* + s_1 s_1^* = 1$$

$C^*(P_2) \cong \mathcal{T}_2$. This is the universal C*-algebra generated by elements as above, with the last relation removed.

$C^*_{\text{tight}}(P_2) \cong \mathcal{O}_2$

The last relation is the one which involves more than the multiplicative semigroup structure.

Boolean inverse monoids

$s, t \in S$ are called compatible if $s^*t, st^* \in E(S)$. $F \subset S$ is compatible if its elements are pairwise compatible.

Definition

S is a Boolean inverse monoid if

1. for all compatible $F \subset_{\text{fin}} S$, the join $\bigvee F$ exists, and for all $s \in S$,

$$s \bigvee F = \bigvee sF \qquad \left(\bigvee F\right) s = \bigvee Fs$$

2. $E(S)$ is a Boolean algebra

Boolean inverse monoids

Pairs of elements in P_2 may have a join outside of P_2

Eg: in P_2, $s_{00}s_{00}^*$ and $s_{11}s_{11}^*$ don't have a join in P_2, but in $\mathcal{O}_2$, we have $s_{00}s_{00}^* + s_{11}s_{11}^*$

Lawson, Scott – create a Boolean inverse monoid which contains all possible joins P_2.

View P_2 as a subsemigroup of $\mathcal{I}(\{0,1\}^{\mathbb{N}})$:

$$s_\alpha s_\beta^* : \beta\{0,1\}^{\mathbb{N}} \to \alpha\{0,1\}^{\mathbb{N}}$$

$$s_\alpha s_\beta^*(\beta x) = \alpha x$$

Let C_2 be the set of all joins of finite compatible sets in P_2, this is a Boolean inverse monoid called the Cuntz monoid.

C*-algebras of Boolean inverse monoids

If S is a Boolean inverse monoid, a Boolean inverse monoid representation of S is a representation $\pi : S \to A$ such that for all $e, f \in E(S)$

$$\pi(e \vee f) = \pi(e) \vee \pi(f) = \pi(e) + \pi(f) - \pi(ef)$$

This is equivalent to saying that, for all compatible $s, t \in S$, we have

$$\pi(s \vee t) = \pi(s) + \pi(t) - \pi(ss^* t)$$

$C^*_B(S)$ – universal C*-algebra for Boolean inverse monoid representations of S.

Observation

1. *a representation is a Boolean inverse monoid representation if and only if it is a tight representation*
2. $C^*_B(S) = C^*_{tight}(S)$.

C*-algebras of inverse semigroups

Many C*-algebras have been identified as the tight C*-algebra of a generating inverse semigroup:

- Graph C*-algebras (Exel 2008)
- Tiling C*-algebras (Exel-Gonçalves-S 2012)
- Self-similar group C*-algebras (Exel-Pardo 2014)
- Katsura algebras (Exel-Pardo 2014)
- C*-algebras of right LCM semigroups (S 2015)
- Carlsen-Matsumoto subshift algebras (S 2015)
- AF C*-algebras (Lawson-Scott 2014, S 2016)
- Any C*-algebra of an ample étale groupoid (Exel 2010)
- C*-algebras of Boolean dynamical systems (Carlsen-Ortega-Pardo 2016)
- C*-algebras of labeled spaces (Boava-de Castro-Mortari 2016)

Simplicity of C*-algebras of inverse semigroups

General question: given two C*-algebras A, B, how can we tell if $A \cong B$?

Recall that commutative C*-algebras $\leftrightarrow$ locally compact Hausdorff spaces.

In topology, the problem of deciding when two spaces are homeomorphic are aided by invariants like homology.

X – topological space, $H_*(X) = \bigoplus_{n\geq 0} H_n(X)$ finitely generated abelian groups.

$X \cong Y \Rightarrow H_*(X) \cong H_*(Y)$.

For some classes of spaces, isomorphism of homology implies isomorphism of spaces – ie homology is a complete invariant of surfaces.

Simplicity of C*-algebras of inverse semigroups

In C*-algebras, we have K-theory, $K_0(A), K_1(A)$, abelian groups.

For some classes of C*-algebras, the K-theory ($\pm\epsilon$) is a complete invariant.

Determining which classes can be classified by K-theory is the Elliott program.

Most classes known to be classified by K-theoretical data consist of simple C*-algebras (simple = no closed two-sided ideals).

We would like to determine when $C^*_{\text{tight}}(S)$ is simple (for example), in terms of properties of S.

Simplicity of C*-algebras of inverse semigroups

Theorem (Renault, Brown-Clark-Farthing-Sims)

Let $\mathcal{G}$ be a Hausdorff étale groupoid. Then $C^*(\mathcal{G})$ is simple if and only if

1. *$\mathcal{G}$ is minimal (every orbit is dense)*
2. *$\mathcal{G}$ is effective (the interior of the isotropy group bundle is the unit space), and*
3. *$\mathcal{G}$ satisfies weak containment ($C_r^*(\mathcal{G}) \cong C^*(\mathcal{G})$).*

Exel-Pardo and Steinberg (2015) gave conditions on an inverse semigroup to ensure that $\mathcal{G}_{\text{tight}}$ satisfies the conditions above (except weak containment).

Simplicity of C*-algebras of inverse semigroups

$\mathcal{G}_{\text{tight}}$ Hausdorff:

$\Leftrightarrow$ For all $s \in S$, the set $\mathcal{J}_s = \{e \in E(S) \mid e \leqslant s\}$ has a finite cover.

Boolean inverse monoid case – $\Leftrightarrow$ S is a meet Boolean inverse monoid.

$\mathcal{G}_{\text{tight}}$ Minimal:

$\Leftrightarrow$ For every nonzero $e, f \in E(S)$, there exist $F \subset_{\text{fin}} S$ such that $\{esfs^* \mid s \in F\}$ is a cover for $\{e\}$.

Boolean inverse monoid case – $\Leftrightarrow$ for every nonzero $e, f \in E(S)$, there exist $F \subset_{\text{fin}} S$ such that $e \leqslant \bigvee_{s \in F} sfs^*$.

Simplicity of C*-algebras of inverse semigroups

Say an idempotent $e \leqslant s^*s$ is

1. fixed by s if $se = e$
2. weakly fixed by s if for all $0 \neq f \leqslant e$, $fsfs^* \neq 0$

$\mathcal{G}_{\text{tight}}$ Effective:

(if Hausdorff) $\Leftrightarrow$ For every $s \in S$ and every $e \in E(S)$ weakly fixed by s, there exists a finite cover for $\{e\}$ by fixed idempotents.

Boolean inverse monoid case – (if Hausdorff) $\Leftrightarrow$ for every $s \in S$, e weakly fixed by s implies e is fixed by s.

Remark about Boolean inverse monoids

Remark

If $A \cong C^*_{tight}(S)$ for some inverse semigroup S, it can be realized as $C^*_B(T)$ for some Boolean inverse monoid T.

$C^*_{\text{tight}}(S) = C^*(\mathcal{G}_{\text{tight}}(S))$ and $\mathcal{G}_{\text{tight}}(S)$ is ample. Its ample semigroup is a Boolean inverse monoid whose C*-algebra is exactly $C^*(\mathcal{G}_{\text{tight}}(S))$. This is in fact true for all C*-algebras of ample étale groupoids (Exel 2010).

Often, it is easier to describe a generating inverse semigroup combinatorially.

Inverse Semigroup – C*-algebra dictionary

Inverse semigroup	**C*-algebra**	**Groupoid**
$s \in S$	partial isometry	compact open bisection
$e \in E(S)$	projection	compact open set of units
Green's relation $\mathcal{D}$	Murray-von Neumann equivalence	
Type monoid	K-theory	
Boolean inverse monoid		**all** compact open bisections
Invariant mean	trace	invariant measure
Coffee	Coffee	Coffee