The Dynamics of Inverse Semigroups

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Basics

A semigroup S is called an *inverse semigroup* if for every element $s \in S$ there is a unique element s^* such that

$$ss^*s = s \qquad \text{and} \qquad s^*ss^* = s^*$$

$E(S) =$ set of idempotents, that is, elements e such that $e^2 = e$.

It is true that

- Idempotents are self-inverse ($e^* = e$)
- If $e, f \in E(S)$, then $ef \in E(S)$ and $ef = fe$
- Every element of the forms s^*s and ss^* are idempotent
- $(s^*)^* = s$
- $(st)^* = t^*s^*$

Examples:

1. G group, then for each $g \in G$, $g^* = g^{-1}$.

   An inverse semigroup S is a group if and only if $E(S)$ contains only one element.

2. Let X be a set. Let

   $$\mathcal{I}(X) = \{f \mid f \text{ is a bijection between two subsets of } X\}$$

   Theorem: (Wagner-Preston) Every inverse semigroup is isomorphic to a subinverse semigroup of $\mathcal{I}(X)$ for some X.

   Here, idempotents are identity functions on subsets of X.

Example: Polycyclic monoids

Example: Polycyclic monoids

Let P_n be the inverse semigroup generated by elements $s_0, s_1, \ldots, s_{n-1}, 0, 1$ such that

$$s_i^* s_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We can always write elements of P_n with stars on the right.

For example:

$s_0 s_1 s_3^* s_3 s_1^* s_2^* = s_0 s_1 s_1^* s_2^*$

$s_1 s_1 s_2^* s_1 s_3 s_2 s_0^* = 0$

Example: Polycyclic monoids

$A = \{0, 1, \ldots, n-1\}$

$A^* =$ (possibly empty) words in A

$\alpha \in A^*$

Let $s_\alpha := s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{|\alpha|}}$, and note $s_\alpha^* = s_{\alpha_{|\alpha|}}^* \cdots s_{\alpha_2}^* s_{\alpha_1}^*$

Let $s_\emptyset = 1$

$s_\alpha s_\beta = s_{\alpha\beta}$ and $s_\alpha^* s_\beta^* = s_{\beta\alpha}^*$

Example: Polycyclic monoids

$P_n = \{s_\alpha s_\beta^* \mid \alpha, \beta \in A^*\} \cup \{0\}$

$$(s_\alpha s_\beta^*)(s_\gamma s_\nu^*) = \begin{cases} s_{\alpha\gamma'} s_\nu^* & \text{if } \gamma = \beta\gamma' \\ s_\alpha s_{\nu\beta'}^* & \text{if } \beta = \gamma\beta' \\ 0 & \text{otherwise} \end{cases}$$

$E(P_n) = \{s_\alpha s_\alpha^* \mid \alpha \in A^*\}$

Inverse semigroup actions

X – topological space
S – inverse semigroup

An *action* of S on X is a semigroup homomorphism

$$\theta : S \to \mathcal{I}(X)$$

such that

- each $\theta_s : D_{s^*s} \to D_{ss^*}$ is continuous, with open domain
- the union of all the domains coincides with X

Inverse semigroup actions

Normally for group actions, we are given the space.

Each inverse semigroup has intrinsic spaces on which it naturally acts.

We construct these spaces from a *natural order* on $E(S)$.

For $e, f \in E(S)$, say $e \leqslant f$ if $ef = e$. (In $\mathcal{I}(X)$, this is set inclusion).

A *filter* in $E(S)$ is a nonempty set $\xi \subset E(S)$ such that

1. $0 \notin \xi$
2. If $e \in \xi$ and $e \leqslant f$, then $f \in \xi$
3. If $e, f \in \xi$, then $ef \in \xi$.

Inverse semigroup actions

ξ filter $\longleftrightarrow$ characteristic function $\phi_\xi \in \{0,1\}^{E(S)}$ (compact, Hausdorff).

$\hat{E}_0 := \{\phi_\xi \mid \xi \text{ is a filter }\}$ – called the *spectrum* of S.
$\hat{E}_\infty := \{\phi_\xi \mid \xi \text{ is an ultrafilter }\}$
$\hat{E}_{\text{tight}} :=$ closure of $\hat{E}_\infty$ in $\hat{E}_0$ – called the *tight spectrum* of S.

S will act on these subspaces.

If $s^*s \in \xi$, then

$$s\xi s^* = \{ses^* \mid e \in \xi\}$$

is a filter containing ss^*. If ξ is ultra, then so is $s\xi s^*$.

Inverse semigroup actions

Define $D_{s^*s} \subset \hat{E}_0$ to be

$$D_{s^*s} = \{\phi_\xi \in \hat{E}_0 \mid s^*s \in \xi\}$$

Then the map $\theta : S \to \mathcal{I}(\hat{E}_0)$ defined by

$$\theta_s : D_{s^*s} \to D_{ss^*}$$

$$\theta_s(\phi_\xi) = \phi_{s\xi s^*}$$

is an action. It restricts to an action of $\hat{E}_{\text{tight}}$.

Example: Polycyclic monoids

$P_n = \{s_\alpha s_\beta^* \mid \alpha, \beta \in A^*\} \cup \{0\}$

$$(s_\alpha s_\beta^*)(s_\gamma s_\nu^*) = \begin{cases} s_{\alpha\gamma'} s_\nu^* & \text{if } \gamma = \beta\gamma' \\ s_\alpha s_{\nu\beta'}^* & \text{if } \beta = \gamma\beta' \\ 0 & \text{otherwise} \end{cases}$$

$E(P_n) = \{s_\alpha s_\alpha^* \mid \alpha \in A^*\}$

Suppose $s_\alpha s_\alpha^* \leqslant s_\beta s_\beta^* \quad \Rightarrow s_\alpha s_\alpha^* s_\beta s_\beta^* = s_\alpha s_\alpha^*$

$\Rightarrow \alpha$ starts with β.

Example: Polycyclic monoids

If $\alpha \in A^* \setminus \{\emptyset\}$, then $\xi_\alpha = \{s_\beta s_\beta^* \mid \alpha \text{ starts with } \beta\}$ is a filter.

If α is an <u>infinite</u> word, then $\xi_\alpha = \{s_\beta s_\beta^* \mid \alpha \text{ starts with } \beta\}$ is an ultrafilter.

$\hat{E}_0 \longleftrightarrow \{\text{finite words}\} \cup \{\text{infinite words}\}$

If α is finite, $\{\xi_\alpha\} \subset \hat{E}_0$ is clopen.

$\Rightarrow \overline{\hat{E}_\infty} = \hat{E}_\infty$

$\hat{E}_{\text{tight}} \cong A^{\mathbb{N}}$, with the usual product topology (cylinder sets).

Example: Polycyclic monoids

The natural action of P_n on $\hat{E}_{\text{tight}}$ mimics the shift through partially defined homeomorphisms.

For example,

$$s_0 : D_{s_0^* s_0} \to D_{s_0 s_0^*}$$

If $s_0 s_0^*$ is in an ultrafilter ξ_α, α must start with the symbol 0.

$$s_0 : A^{\mathbb{N}} \to \{0\gamma \in A^{\mathbb{N}}\}$$

$$s_0(\beta) = 0\beta$$

In general,

$$s_\alpha s_\beta^* : \{\beta\gamma \in A^{\mathbb{N}}\} \to \{\alpha\gamma \in A^{\mathbb{N}}\}$$

$$s_\alpha s_\beta^*(\beta\gamma) = \alpha\gamma$$

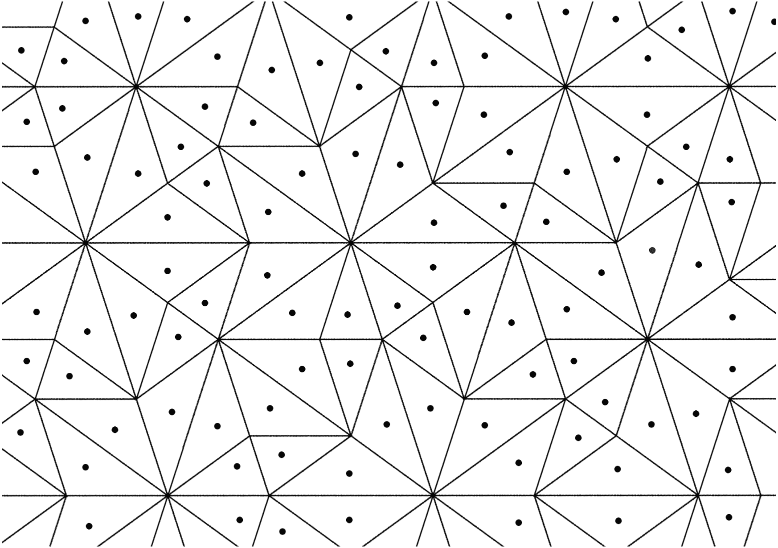
Example: Polycyclic monoids

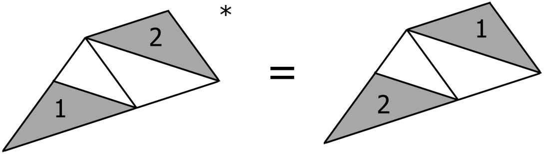
It's possible to associate an étale groupoid $\mathcal{G}(S, X, \theta)$ to an action θ of an inverse semigroup S on a space X.

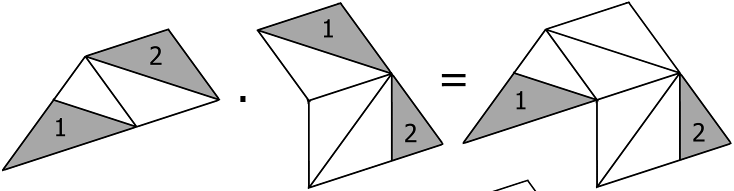
Étale groupoid $\longrightarrow$ C*-algebra $C^*(S, X, \theta)$ (Renault).

(Exel) $C^*(P_n, \hat{E}_{\text{tight}}, \theta) \cong \mathcal{O}_n$

$C^*(P_n, \hat{E}_0, \theta) \cong \mathcal{T}_n$







Idempotents: