CHAPTER 5. VECTORS IN DIMENSION THREE OR HIGHER

Part 1. Vectors in 3D and the Cross Product

Recall that the dot product (or inner product) $\mathbf{u} \cdot \mathbf{v}$ of two planar vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2.$$ 

For vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ in the Euclidean 3-dimensional space $\mathbb{R}^3$, the dot product (or inner product) $\mathbf{u} \cdot \mathbf{v}$ is defined by putting

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$ 

and the magnitude of $\mathbf{v}$ is given by

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$ 

The identities $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$ and $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$ still hold. Also, two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, or $\mathbf{u} \perp \mathbf{v}$, if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. Let us also recall that the area of the parallelogram spanned by vectors $\mathbf{u} = (a, b)$ and $\mathbf{v} = (c, d)$ is equal to the absolute value of the determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$ 

A $3 \times 3$ determinant is defined via the following recipe:

$$\begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = w_1u_2v_3 + w_2u_3v_1 + w_3u_1v_2 - w_3u_2v_1 - w_2u_1v_3 - w_1u_3v_2. \quad (5.1)$$

This recipe is not readable. One must recognize a pattern of the right hand side in order to use it efficiently. Notice that the right hand side can be written as

$$w_1u_2v_3 + w_2u_3v_1 + w_3u_1v_2 - w_3u_2v_1 - w_2u_1v_3 - w_1u_3v_2$$

$$= w_1(u_2v_3 - u_3v_2) + w_2(u_3v_1 - u_1v_3) + w_3(u_1v_2 - u_2v_1)$$

$$= w_1(u_2v_3 - u_3v_2) - w_2(u_1v_3 - u_3v_1) + w_3(u_1v_2 - u_2v_1)$$

$$= w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}.$$ 

(5.2)
The absolute value of the $3 \times 3$ determinant in (5.1) gives us the volume of the parallelepiped spanned by vectors $\mathbf{w} = (w_1, w_2, w_3)$, $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Identity (5.2) suggests the following definition of the cross product $\mathbf{u} \times \mathbf{v}$ of vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$:

$$\mathbf{u} \times \mathbf{v} = \left( \begin{array}{ccc} u_2 & u_3 \\ v_2 & v_3 \\ \end{array} \right), - \left( \begin{array}{ccc} u_1 & u_3 \\ v_1 & v_3 \\ \end{array} \right), - \left( \begin{array}{ccc} u_1 & u_2 \\ v_1 & v_2 \\ \end{array} \right).$$

To give a quick example, let $\mathbf{u} = (1, 1, 2)$ and $\mathbf{v} = (2, -1, 3)$. Then

$$\mathbf{u} \times \mathbf{v} = \left( \begin{array}{ccc} 1 & 2 \\ -1 & 3 \\ \end{array} \right), - \left( \begin{array}{ccc} 1 & 2 \\ 2 & 3 \\ \end{array} \right), - \left( \begin{array}{ccc} 1 & 1 \\ 2 & -1 \\ \end{array} \right) = (5, 1, -3).$$

Notice that

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (1, 1, 2) \cdot (5, 1, -3) = 5 + 1 - 6 = 0, \quad \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = (2, -1, 3) \cdot (5, 1, -3) = 10 - 1 - 9 = 0.$$

In other words, $\mathbf{u}, \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$. In the future, we shall see that this is not accidental.

The right hand side of (5.2) is just $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$. So we can rewrite (5.2) as

$$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \quad (5.3)$$

We give a quick example:

$$(1, 3, 2) \cdot \{(1, 1, 2) \times (2, -1, 3)\} = \begin{vmatrix} 1 & 3 & 2 \\ 1 & 1 & 2 \\ 2 & -1 & 3 \end{vmatrix} = 3 + 12 - 2 - 4 - 9 + 2 = 2$$

which is the volume of the parallelepiped spanned by vectors $(1, 3, 2), (1, 1, 2), (2, -1, 3)$.

Here are the basic properties about cross products:

(C1) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$, and $\mathbf{v} \times \mathbf{v} = 0$.

(C2) $(a\mathbf{u} + b\mathbf{v}) \times \mathbf{w} = a \mathbf{u} \times \mathbf{w} + b \mathbf{v} \times \mathbf{w}$

(C3) $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

(C4) The triple $\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}$ obeys the right hand rule.

(C5) $|\mathbf{u} \times \mathbf{v}|$ is the area of the parallelogram spanned by $\mathbf{u}, \mathbf{v}$, that is, $|\mathbf{u}||\mathbf{v}| \sin \theta$, where $\theta$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
Properties (C1) and (C2) are more or less easy to check. I leave their checking to you as an exercise. Now we check (C3). In order to show that \( \mathbf{u} \times \mathbf{v} \) is perpendicular to \( \mathbf{u} \), it is enough to check that their dot product is zero. Indeed, by (5.3), we have

\[
\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0
\]

because the first two rows of the determinant are identical. Hence \( \mathbf{u} \perp (\mathbf{u} \times \mathbf{v}) \). In the same way, we can show that \( \mathbf{v} \perp (\mathbf{u} \times \mathbf{v}) \). To see (C5), let \( \phi \) be the angle between \( \mathbf{w} \) and \( \mathbf{u} \times \mathbf{v} \). Then \( \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = |\mathbf{w}| |\mathbf{u} \times \mathbf{v}| \cos \phi \equiv (|\mathbf{w}| \cos \phi) |\mathbf{u} \times \mathbf{v}| \). Now, think of \( \mathbf{u} \) and \( \mathbf{v} \) as vectors in horizontal directions. Then, by (C3), we see that \( \mathbf{u} \times \mathbf{v} \) is in the vertical direction. So \( |\mathbf{w}| |\cos \phi| \) is equal to the height \( h \) of the parallelepiped spanned by \( \mathbf{u}, \mathbf{v}, \mathbf{w} \); see the following figure of a “side view” of this parallelepiped.

![Figure 1.](image)

As we know, \( |\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})| \) is the volume \( V \) of the parallelepiped. On the other hand, the volume \( V \) is just \( bh \), where \( h \), as before, is the height the parallelepiped and \( b \) is the area of its base, which is the (spatial) parallelogram spanned by \( \mathbf{u} \) and \( \mathbf{v} \). Thus we have

\[
bh = V = |\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})| = |\mathbf{w}| |\mathbf{u} \times \mathbf{v}| |\cos \phi| = (|\mathbf{w}| |\cos \phi|) |\mathbf{u} \times \mathbf{v}| = h |\mathbf{u} \times \mathbf{v}|.
\]

Cancelling \( h \), we obtain \( b = |\mathbf{u} \times \mathbf{v}| \). The last identity tells us the validity of property (C5). Finally, (C6) follows from (5.3) and the geometric interpretation of a 3 \( \times \) 3 determinant: its absolute value is the volume of the parallelepiped spanned by its row vectors.
The cross product, especially its Properties (C3) and (C5), is very useful in spatial geometry, as well as in physics (such as torque, Lorentz force, etc).

Property (C3) tells us $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta$, which gives $|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2 |\mathbf{v}|^2 \sin^2 \theta$. On the other hand, $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta$ gives $(\mathbf{u} \cdot \mathbf{v})^2 = |\mathbf{u}|^2 |\mathbf{v}|^2 \cos^2 \theta$. Thus

$$(\mathbf{u} \cdot \mathbf{v})^2 + |\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2 |\mathbf{v}|^2 (\cos^2 \theta + \sin^2 \theta) = |\mathbf{u}|^2 |\mathbf{v}|^2.$$ 

We have arrived at a rather amazing and beautiful identity:

$$(\mathbf{u} \cdot \mathbf{v})^2 + |\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2 |\mathbf{v}|^2.$$ 

Writing down in its full glory, we have

$$(u_1v_1+u_2v_2+u_3v_3)^2+\left|\begin{array}{cc} u_2 & u_3 \\ v_2 & v_3 \end{array}\right|^2+\left|\begin{array}{cc} u_1 & u_3 \\ v_1 & v_3 \end{array}\right|^2+\left|\begin{array}{cc} u_1 & u_2 \\ v_1 & v_2 \end{array}\right|^2 = (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2). \quad (5.4)$$

Certainly you can verify this identity by using brute force of algebraic manipulation.

Exercise 5.1. Expand both sides of (5.4) and then check that they are identical.

Notice the resemblance of (5.4) to (4.19):

$$(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2$$

Indeed, (5.4) is the “3–dimensional version” of (4.19).

Exercise 5.2. Use the vectors $\mathbf{u} = (2, 5, 3)$ and $\mathbf{v} = (7, -1, -3)$ to fabricate

$$(2^2 + 5^2 + 3^2)(7^2 + 1^2 + 3^2) = 12^2 + 27^2 + 37^2.$$ 

Now you can challenge your friend (who is not taking MATH0107) or your parents to find three positive integers $x, y, z$ so that $(2^2 + 5^2 + 3^2)(7^2 + 1^2 + 3^2) = x^2 + y^2 + z^2$. You can use more complicated numbers to fabricate other “amazing identities” to surprise them.

Note: the cross product $\mathbf{u} \times \mathbf{v}$ is defined only for vectors in the three dimensional space, while the dot product $\mathbf{u} \cdot \mathbf{v}$ can be defined for vectors in a space of any dimension.

Part 2. Vectors in Higher Dimensional Spaces
In our previous sections we have seen that, by introducing the Cartesian coordinate system, a vector \( u \) in a 2 dimensional space can be identified with an ordered pair \((u_1, u_2)\) of numbers and a vector \( v \) in a 3 dimensional space can be identified with an ordered triple \((v_1, v_2, v_3)\). Naturally we can regard a vector in an \( n \)-dimensional space as an ordered \( n \)-tuple: \( v = (v_1, v_2, \ldots, v_n) \). Here, \( n \) is any positive number.

Let \( u = (u_1, u_2, \ldots, u_n) \) and \( v = (v_1, v_2, \ldots, v_n) \) be two vectors in the \( n \)-dimensional space. According to the usual convention, we require

1. \( u = v \) if and only if \( u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n \); that is, two vectors are the same if and only if their corresponding components are equal;

2. (addition) \( u + v = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \); that is, adding two vectors is carried out by adding their corresponding components;

3. (scalar multiplication) \( au = (au_1, au_2, \ldots, au_n) \); that is, multiplying a vector by a scalar is carried out by multiplying each component by the same scalar;

4. (negation) \( -u = (-u_1, -u_2, \ldots, -u_n) \).

5. (magnitude, or norm, or length) \( |u| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} \).

6. (inner product, or dot product) \( u \cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n \).

To give a quick example, suppose that \( u = (1, 3, -3, -1, -4) \) and \( v = (-5, 2, 1, 5, -3) \) are given vectors in the 5–dimensional space. Then we have

\[
\begin{align*}
\mathbf{u} + \mathbf{v} &= (-4, 5, -2, 4, -7) \\
-3\mathbf{u} &= (-3, -9, 9, 3, 12) \\
|\mathbf{u}| &= \sqrt{1^2 + 3^2 + (-3)^2 + (-1)^2 + (-4)^2} = 6 \\
|\mathbf{v}| &= \sqrt{(-5)^2 + 2^2 + 1^2 + 5^2 + (-3)^2} = 8 \\
\mathbf{u} \cdot \mathbf{v} &= 1 \times (-5) + 3 \times 2 + (-3) \times 1 + (-1) \times 5 + (-4) \times (-3) = 5.
\end{align*}
\]

The usual identities such as \( \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \), \( \mathbf{u} \cdot (a\mathbf{v} + b\mathbf{w}) = a \mathbf{u} \cdot \mathbf{v} + b \mathbf{u} \cdot \mathbf{w} \), \( |\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u} \), etc. still hold.

The \( n \)-dimensional space, consisting of all \( n \)–tuples of the form \( \mathbf{u} = (u_1, u_2, \ldots, u_n) \) regarded as vectors, is called the \( n \)-dimensional (real) Euclidean space and is denoted by \( \mathbb{R}^n \). Why are we interested in vectors from higher dimensions? A blunt answer is: in applying vector algebra to natural or social sciences, we must allow such vectors.
Example (a fairy tale). In this example we consider 5 characters in a fairy tale: Princess Barbe, Prince Ken, the evil Prime Minister Mandragora, the witch Fata Morgana, and some girl called Ligneous. Each character is endowed with a vector consisting of 4 components, representing look, personality, social status and color of dressing, with 10 marks as maximum and −10 as minimum. For example, in the “look department” both Barbe and Ken earn a perfect 10. Mandragora is also good looking (after all, he believes his look could win the princess’ heart), but not as good looking as the prince: so he gets a mark of 9. Fata Morgana is the ugliest person: so she receives a −10. Ligneous has a “buttered face”: she gets 0. The Princess also earns a perfect 10 in personality. But her social status is only a 6 even though she is a princess, because she lived in an era 400 years before the word “feminists” was invented. For color of dressing, 10 is for white and −10 is for dark. Here is a summary of characters and their respective vectors of their attributes.

Princess Barbe: \( b = (10, 10, 6, 10) \)
Prince Ken: \( k = (10, 6, 10, 10) \)
Mandragora: \( m = (9, -10, 9, -10) \)
Fata Morgana: \( f = (-10, -10, -10, -10) \)
Ligneous: \( L = (0, 0, 0, 0) \)

Now we consider the dot products of various pairs of vectors to see if two persons are compatible. For Ken and Barbe, the high score \( b \cdot k = 340 \) tells us that they will get married at the end of the story. For Mandragora and Barbe, \( m \cdot b = -56 \), showing that they are not compatible. For Mandragora and Fata Morgana, \( m \cdot f = 20 \), which is small but still positive. This shows that Mandragora and Fata Morgana can get along with each other and work together, but they would not get married. Mandragora should be advised to change his dress from black to white, because this will greatly improve his chance of getting close to the princess. Indeed, after changing the last component −10 in \( m \) to 10, we get \( m \cdot b = 144 \), a substantial improvement. As for Ligneous, \( L \cdot v = 0 \) for any vector \( v \): she is indifferent to any one. For all except Ligneous, the corresponding vectors have large magnitude, indicating that they have strong characters. For Ligneous, \( |L| = 0 \), showing that she has no character and should not even be in the story.

Example (another fairy tale). For Cinderella, \( C = (10, 10, -10, -10) \), her two sisters, \( s = (-1, 0, 10, 10) \), \( t = (1, -1, 10, 10) \), and the prince, \( P = (10, 8, 10, 10) \), we
have $C \cdot P = -20$, $s \cdot P = 190$ and $t \cdot P = 202$. It seems that Cinderella cannot compete with her sisters. However, the Fairy Mother came and transformed $C$ into $T(C) = (10, 10, 10, 10)$ so that when she met the prince, $P \cdot T(C) = 380$. She won the heart of the prince by the help of the Fairy Mother who temporally changed her vector of attributes.

**Example.** Consider $n$ social, economical or political issues. For each issue a score of 10 stands for “strongly support” and $-10$ for “strongly oppose”, while 0 for “no opinion”. The scores of a person’s opinions form a vector with $n$ components. If two persons have “opinion vectors” $u$ and $v$ respectively, then the dot product $u \cdot v$ measures the degree of their congruity. A person with the zero opinion vector may be regarded as apathetic. This person never agrees nor disagrees with any one, because he simply does not care.

As usual, we say that two vectors $u$ and $v$ in $\mathbb{R}^n$ are perpendicular and we write $u \perp v$ if $u \cdot v = 0$. We can generalize the Pythagoras theorem in $n$–dimension:

**Pythagoras Theorem.** If $u \perp v$, then $|u + v|^2 = |u|^2 + |v|^2$.

The proof of this theorem (in this setting) is one line long:

$$|u + v|^2 = (u + v) \cdot (u + v) = u \cdot u + 2u \cdot v + v \cdot v = u \cdot u + v \cdot v = |u|^2 + |v|^2.$$  
Notice that $u \cdot v = 0$ follows form the assumption $u \perp v$.

**Exercise 5.3.** Prove that if $v \perp v$, then $v = 0$.

**Problem 5.4.** The unit sphere in $\mathbb{R}^n$, denoted by $S^{n-1}$, is the set of all points $P$ in $\mathbb{R}^n$ so that $|OP| = 1$; in other words, the distance between $P$ and the origin $O$ is 1. When $n = 2$ (in the case of Euclidean plane), $S^{n-1}$ becomes $S^1$ and we call it the unit circle. Using the set theoretical notation, we can write

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$  
Fix a point $A$ inside the unit sphere distinct from the origin: thus $0 < |\overrightarrow{OA}| < 1$. The so called polar of $A$, denoted by $A^*$, is the point outside of the unit sphere determined by the relation

$$\overrightarrow{OA^*} = \frac{1}{|\overrightarrow{OA}|^2} \overrightarrow{OA}.$$
You should notice that $\overrightarrow{OA}^*$ is a multiple of $\overrightarrow{OA}$ by a positive scalar and $|OA| |OA^*| = 1$. Now you are asked to verify the following highly nontrivial fact: For any point $P$ on the unit sphere $S^{n-1}$, the ratio

$$\frac{|A^*P|}{|AP|}$$

is independent of the choice of $P$. In other words, if both $P$ and $Q$ are on $S^{n-1}$, then $|A^*P|/|AP| = |A^*Q|/|AQ|$.

![Figure 2.](image)

Let us consider the following important problem: given two vectors $w$ and $v$ in the space $\mathbb{R}^n$ with $w \neq 0$, find the so-called orthogonal decomposition of $v$:

$$v = aw + h,$$

where $a$ is a scalar and $h$ is perpendicular to $w$: $h \perp w$. In other words, we need to find an explicit recipe to write $v$ as the sum of two vectors, one is parallel to $w$ and the other is perpendicular to $w$. The vector $aw$ here will be called the (orthogonal) projection of $v$ onto $w$. A visual aid of this problem is given as follows:
Take the dot product of both sides with vector $w$:

$$v \cdot w = a \cdot w + h \cdot w = a \cdot w + w \cdot w.$$  

Notice that $h \cdot w = 0$, because $h \perp w$ by assumption. Also, we may rewrite $w \cdot w$ as $|w|^2$; but here we prefer not to do so. Thus we arrive at $a = v \cdot w / w \cdot w$. In conclusion, the answer to our important question is given as follows:

$$v = p + h \quad \text{where} \quad p = \frac{v \cdot w}{w \cdot w} \cdot w,$$

the projection of $v$ onto $w$. \hfill \hspace{1cm} (5.6)

Note that, once we obtain the projection $p$, we get $h$ for free: $h = v - w$. As a quick example, take vectors $v = (3, 5, 2)$ and $w = (2, 1, -1)$ in $\mathbb{R}^3$. Then the projection of $v$ onto $w$ is given by

$$p = \frac{v \cdot w}{w \cdot w} \cdot w = \frac{6 + 5 - 2}{4 + 1 + 1} \cdot w = \frac{3}{2} \cdot w = \left( \frac{3}{2}, \frac{3}{2}, -\frac{3}{2} \right)$$

and hence $h = v - p = (0, 7/2, 7/2)$. Thus we have the following orthogonal decomposition for $v$: $(3, 5, 2) = (3, 3/2, -3/2) + (0, 7/2, 7/2)$.

Applying the Pythagoras theorem to the orthogonal decomposition $v = p + h$, we get $|v|^2 = |p|^2 + |h|^2$. Since $|h|^2 \geq 0$, we have $|v|^2 \geq |p|^2$. Upon taking square roots on both sides, we have $|v| \geq |p|$. Now

$$|v| \geq |p| = \frac{|v \cdot w|}{w \cdot w} = \frac{|v \cdot w|}{|w|^2} \cdot |w| = \frac{|v \cdot w|}{|w|}$$

Figure 3.
So we have $|v \cdot w| \leq |v||w|$, which is the celebrated Cauchy-Schwarz inequality. The argument here gives a proof of the Cauchy-Schwarz inequality.

**Exercise 5.5.** In each of the following cases, find the dot product $u \cdot v$ and the cross product $u \times v$ of the given vectors $u$ and $v$ in $\mathbb{R}^3$, and check that $u \times v$ is perpendicular to both $u$ and $v$: (a) $u = (1, 1, 2), v = (2, 3, 1)$ (b) $u = (-1, 1, 4), v = (3, -2, 2)$ (c) $u = (1, a, a^2), v = (1, 1, 1)$. (d) $u = (\cos \alpha \sin \beta, \cos \alpha \cos \beta, -\sin \alpha), v = (\sin \alpha \cos \beta, -\sin \alpha \sin \beta, 0)$.

**Exercise 5.6.** In each of the following cases, find the area of the triangle $\triangle ABC$ with the given points $A, B, C$ in $\mathbb{R}^3$. (a) $A = (1, 1, 1), B = (1, 2, 3), C = (3, 3, 1)$ (b) $A = (0, 0, 0), B = (\cos \alpha \sin \beta, \cos \alpha \cos \beta, -\sin \alpha), C = (\sin \alpha \cos \beta, -\sin \alpha \sin \beta, 0)$.

**Exercise 5.7.** In each of the following cases, find the magnitudes $|u|, |v|$ and the dot product $u \cdot v$ of the given vectors: (a) $u = (1, -1, 1), v = (1, -1, -1)$ (b) $u = (\cos \alpha + \cos \beta, \sin \alpha + \sin \beta, \cos \alpha - \cos \beta, \sin \alpha - \sin \beta)$, $v = (-\cos \alpha - \cos \beta, -\sin \alpha + \sin \beta, -\cos \alpha + \cos \beta, \sin \alpha + \sin \beta)$.

**Exercise 5.8.** Let $S = (s_1, s_2, \ldots, s_{1500})$ be vector showing the results of investigating 1500 people for smoking. The $k$ component $s_k$ is assigned a value of $-1, 0$ or $+1$ according to the following results of investigating the $k$th respondent: heavy smoker, occasion smoker, or non–smoker. Let $C = (c_1, c_2, \ldots, c_{1500})$ be a vector obtained based on the health information of these 1500 respondents: $c_k$ is $-1, 0$ or $+1$ according to the following conditions of the $k$th person: having cancer, not known, no cancer. What is your interpretation of each of the following possible outcomes? (a) $S \cdot C = 1200$, (b) $S \cdot C = 0$ (c) $S \cdot C = -1200$.

**Exercise 5.9.** In each of the following cases, find the projection of vector $v$ onto vector $w$: (a) $v = (1, 9, 7), w = (1, 1, 2)$ (b) $v = (a + 3b, 3a + b, a + 2b), w = (b, a + b, a + b)$.

**Answers to Exercises and Problems**

5.1. We have

$$(u_1v_1 + u_2v_2 + u_3v_3)^2 = u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 + 2u_1v_1u_2v_2 + 2u_1v_1u_3v_3 + 2u_2v_2u_3v_3$$
\[
\begin{vmatrix}
  u_2 & u_3 \\
  v_2 & v_3
\end{vmatrix}^2 = (u_2v_3 - u_3v_2)^2 = u_2^2v_3^2 + u_3^2v_2^2 - 2u_2v_3u_3v_2
\]
\[
\begin{vmatrix}
  u_1 & u_3 \\
  v_1 & v_3
\end{vmatrix}^2 = (u_1v_3 - u_3v_1)^2 = u_1^2v_3^2 + u_3^2v_1^2 - 2u_1v_3u_3v_1
\]
\[
\begin{vmatrix}
  u_1 & u_2 \\
  v_1 & v_2
\end{vmatrix}^2 = (u_1v_2 - u_2v_1)^2 = u_1^2v_2^2 + u_2^2v_1^2 - 2u_1v_2u_2v_1
\]

Adding all four identities and cancelling all “cross terms”, we see that the left hand side of (4.4) is equal to
\[
u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 + u_4^2v_4^2 + u_5^2v_5^2 + u_6^2v_6^2 + u_7^2v_7^2 + u_8^2v_8^2
\]

This sum of nine terms is exactly the result of expanding the right hand side of (4.4).

5.2. Notice that \( \mathbf{u} \cdot \mathbf{v} = 2 \times 7 + 5 \times (-1) + 3 \times (-3) = 0 \). So we have \( |\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 \).
Now
\[
\mathbf{u} \times \mathbf{v} = \left( \begin{array}{ccc}
  5 & 3 \\
  -1 & -3 \\
  2 & 5
\end{array} \right) = (-12, 27, -37).
\]
So \( |\mathbf{u} \times \mathbf{v}|^2 = 12^2 + 27^2 + 37^2 \). Also,
\[
|\mathbf{u}|^2 = |(2, 5, 3)|^2 = 2^2 + 5^2 + 3^2 \quad \text{and} \quad |\mathbf{v}|^2 = |(7, -1, -3)|^2 = 7^2 + 1^2 + 3^2.
\]
Hence \( |\mathbf{u}|^2|\mathbf{v}|^2 = |\mathbf{u} \times \mathbf{v}|^2 \) gives \((2^2 + 5^2 + 3^2)(7^2 + 1^2 + 3^2) = 12^2 + 27^2 + 37^2 \).

5.3. \( \mathbf{v} \perp \mathbf{v} \) means \( \mathbf{v} \cdot \mathbf{v} = 0 \). So \( |\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v} = 0 \). Thus the magnitude \( |\mathbf{v}| \) of \( \mathbf{v} \) is zero, which implies \( \mathbf{v} = 0 \).

5.4. For convenience, let us write \( \mathbf{a} = \overrightarrow{OA}, \mathbf{a}^* = \overrightarrow{OA}^2 \) and \( \mathbf{p} = \overrightarrow{OP} \). Then we have \( \mathbf{a}^* = |\mathbf{a}|^{-2}\mathbf{a} \), or \( |\mathbf{a}|^2\mathbf{a}^* = \mathbf{a} \). In particular \( |\mathbf{a}|^2|\mathbf{a}^*| = |\mathbf{a}| \), or \( |\mathbf{a}|^3|\mathbf{a}^*| = 1 \). Also, since \( P \) is a point on the unit sphere, we have \( \mathbf{p} \cdot \mathbf{p} = |\mathbf{p}|^2 = 1 \). Now
\[
|\overrightarrow{PA}^2|^2 = |\overrightarrow{OA}^2 - \overrightarrow{OP}|^2 = |\mathbf{a}^* - \mathbf{p}|^2 = (\mathbf{a}^* - \mathbf{p}) \cdot (\mathbf{a}^* - \mathbf{p})
\]
\[
= \mathbf{a}^* \cdot \mathbf{a}^* - 2\mathbf{a}^* \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{p}
\]
\[
= |\mathbf{a}|^2 - 2\mathbf{a}^* \cdot \mathbf{p} + \frac{1}{|\mathbf{a}|^2} \mathbf{a} \cdot \mathbf{p} - 1 = \frac{1}{|\mathbf{a}|^2} (1 - 2\mathbf{a} \cdot \mathbf{p} + |\mathbf{a}|^2)
\]
On the other hand,
\[
|\overrightarrow{PA}|^2 = |\overrightarrow{OA} - \overrightarrow{OP}|^2 = |\mathbf{a} - \mathbf{p}|^2 = (\mathbf{a} - \mathbf{p}) \cdot (\mathbf{a} - \mathbf{p}) = \mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{p} = |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{p} + 1
\]

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and hence $|\overrightarrow{PA^2}|^2/|\overrightarrow{PA}|^2 = 1/|\mathbf{a}|^2$, or $|\overrightarrow{PA^2}|/|\overrightarrow{PA}| = 1/|\mathbf{a}|$, which is clearly independent of $P$.

5.5. (a) $\mathbf{u} \cdot \mathbf{v} = 7$, $\mathbf{u} \times \mathbf{v} = (-5, 3, 1)$
(b) $\mathbf{u} \cdot \mathbf{v} = 2$, $\mathbf{u} \times \mathbf{v} = (14, 14, 0)$
(c) $\mathbf{u} \cdot \mathbf{v} = 1 + a + a^2$, $\mathbf{u} \times \mathbf{v} = (1 - a)(a, -a - 1, 1)$
(d) $\mathbf{u} \cdot \mathbf{v} = 0$, $\mathbf{u} \times \mathbf{v} = \sin \alpha (\sin \alpha \sin \beta, \sin \alpha \cos \beta, -\cos \alpha)$

5.6. (a) 6
(b) $|\sin \alpha|$

5.7. (a) 0
(b) $-2$

5.8. (a) Smoking is a cause of cancer
(b) Smoking does not cause cancer
(c) Smoking can prevent cancer.

5.9. (a) $4\mathbf{w} = (4, 4, 8)$
(b) $2\mathbf{w} = 2(b, a + b, a + b)$