§4. The Second Fundamental Form

In the last section we developed the theory of intrinsic geometry of surfaces by considering the covariant differential \( d\gamma F \), that is, the tangential component of \( dF \) for a vector field \( F \) tangent to the given surface \( S \). In the present section we take up the study of the shapes of surfaces by considering the normal component \( d\perp F \) of \( dF \), which mainly tells us how \( S \) is embedded in the ambient space \( \mathbb{R}^3 \). For example, a cylinder is intrinsically flat like a plane but its shape is different from a plane: for a plane, the normal differential \( d\perp F \) is always zero, but this is not the case for a cylinder. Notice that, if \( \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3 \) for a Darboux frame for a surface \( S \) with \( \mathbf{E}_3 \) normal to \( S \), then

\[
  d\perp F = (dF \cdot \mathbf{E}_3) \mathbf{E}_3. \tag{4.1}
\]

Since \( dF \cdot \mathbf{E}_3 \) is a 1-form, we may consider \( \langle dF \cdot \mathbf{E}_3, \mathbf{G} \rangle \) for given vector fields \( F \) and \( G \), which is a scalar function as the result of pairing a covector field with a vector field.

**Example 4.1.** Consider the cylinder \( C \): \( x^2 + y^2 = 1 \) with \( -\infty < z < \infty \), and the Darboux frame \( \mathbf{E}_1 = (-y, x, 0) \), \( \mathbf{E}_2 = (0, 0, 1) \) and \( \mathbf{E}_3 = (x, y, 0) \) (at the point \( (x, y, z) \) on \( C \)). A vector field \( F \) tangent to \( C \) can be expressed as \( F = f_1 \mathbf{E}_1 + f_2 \mathbf{E}_2 \), where \( f_1 \) and \( f_2 \) are scalars depending on the point at which \( F \) is evaluated. Find an expression for the 1-form \( dF \cdot \mathbf{E}_3 \). Also, given another tangential vector field \( G = g_1 \mathbf{E}_1 + g_2 \mathbf{E}_2 \), find an expression for the pairing \( \langle dF \cdot \mathbf{E}_3, \mathbf{G} \rangle \).

**Solution:** We have \( d\mathbf{E}_1 = (-dy, dx, 0) \), \( d\mathbf{E}_2 = (0, 0, dz) \) and \( \mathbf{E}_3 = (x, y, 0) \). Hence

\[
  d\mathbf{E}_1 \cdot \mathbf{E}_3 = -xdy + ydx \quad \text{and} \quad d\mathbf{E}_2 \cdot \mathbf{E}_3 = 0.
\]

Thus

\[
  dF \cdot \mathbf{E}_3 = (df_1 \mathbf{E}_1 + f_1 d\mathbf{E}_1 + df_2 \mathbf{E}_2 + f_2 d\mathbf{E}_2) \cdot \mathbf{E}_3 = f_1 (-xdy + ydx).
\]

Also, \( \langle dF \cdot \mathbf{E}_3, \mathbf{G} \rangle = \langle f_1 (-xdy + ydx), \mathbf{G} \rangle = f_1 (-x\langle dy, \mathbf{G} \rangle + y\langle dx, \mathbf{G} \rangle) \). For convenience, introduce the standard basis \( \mathbf{e}_1 = (1, 0, 0) \), \( \mathbf{e}_2 = (0, 1, 0) \) and \( \mathbf{e}_3 = (0, 0, 1) \) of \( \mathbb{R}^3 \). Then \( \langle dy, \mathbf{G} \rangle = \nabla y \cdot \mathbf{G} = \mathbf{e}_2 \cdot \mathbf{G} = g_1 \mathbf{e}_2 \cdot \mathbf{E}_1 + g_2 \mathbf{e}_2 \cdot \mathbf{E}_2 = g_1 x \). Similarly, \( \langle dx, \mathbf{G} \rangle = -g_1 y \). Thus

\[
  \langle dF \cdot \mathbf{E}_3, \mathbf{G} \rangle = f_1 (-x(g_1 x) + y(-g_1 y)) = -f_1 g_1 (x^2 + y^2) = -f_1 g_1,
\]

in view of the fact that \( (x, y, z) \) is a point on \( C \) and hence \( x^2 + y^2 = 1 \).

**Example 4.2.** Take a parametric curve \( \gamma: Y = Y(s), Z = Z(s) \) in \( YZ \)-plane with \( Y'(s) > 0 \), assuming the arc-length parametrization: \( Y''(s)^2 + Z'(s)^2 = 1 \). The surface of revolution obtained by revolving \( \gamma \) about the \( z \)-axis is, in parametric equations,

\[
  x = x(u, v) = Y(v) \cos u, \quad y = y(u, v) = Y(v) \sin u, \quad z = z(u, v) = Z(v).
\]
The tangent fields \( r_u \) and \( r_v \) along “\( u \)-curves” and “\( v \)-curves” are respectively given by \( r_u = (-Y(v) \sin u, Y(v) \cos u, 0) \) and \( r_v = (Y'(v) \cos u, Y'(v) \sin u, Z'(v)) \). An easy computation shows \( r_u \cdot r_v = 0 \). Normalizing, we have unit tangent fields \( \mathbf{E}_1 \) and \( \mathbf{E}_2 \). Taking their cross product, we get a unit normal field \( \mathbf{E}_3 \):

\[
\mathbf{E}_1 = |\mathbf{r}_u|^{-1} \mathbf{r}_u = (-\sin u, \cos u, 0), \\
\mathbf{E}_2 = |\mathbf{r}_v|^{-1} \mathbf{r}_v = (Y'(v) \cos u, Y'(v) \sin u, Z'(v)) \\
\mathbf{E}_3 = \mathbf{E}_1 \times \mathbf{E}_2 = (Z'(v) \cos u, Z'(v) \sin u, -Y'(v)).
\]

Find \( \langle \mathbf{dF} \cdot \mathbf{E}_3, \mathbf{G} \rangle \) for given tangent fields \( \mathbf{F} = f_1 \mathbf{E}_1 + f_2 \mathbf{E}_2 \) and \( \mathbf{G} = g_1 \mathbf{E}_1 + g_2 \mathbf{E}_2 \).

**Solution:** \( \mathbf{dF} \cdot \mathbf{E}_3 = (df_1 \mathbf{E}_1 + df_1 \mathbf{E}_1 + df_2 \mathbf{E}_2 + f_2 d \mathbf{E}_2) \cdot \mathbf{E}_3 = f_1 d \mathbf{E}_1 \cdot \mathbf{E}_3 + f_2 d \mathbf{E}_2 \cdot \mathbf{E}_3. \)

Note that \( d \mathbf{E}_1 = (-\cos u \, du, -\sin u \, du, 0) \) and hence

\[
d \mathbf{E}_1 \cdot \mathbf{E}_3 = (-\cos u \, du)(Z'(v) \cos u) + (-\sin u \, du)(Z'(v) \sin u) = -Z'(v) \, du.
\]

Also, \( d \mathbf{E}_2 = (-Y'(v) \sin u \, du + Y''(v) \cos u \, dv, Y'(v) \cos u \, du + Y''(v) \sin u \, dv, Z''(v) \, dv). \)

A short computation shows \( d \mathbf{E}_2 \cdot \mathbf{E}_3 = \{Z'(v)Y''(v) - Y'(v)Z''(v)\} \, dv. \) Thus

\[
d \mathbf{F} \cdot \mathbf{E}_3 = -f_1 Z'(v) \, du - f_2 (Y'(v)Z''(v) - Z'(v)Y''(v)) \, dv.
\]

To find \( \langle d \mathbf{F} \cdot \mathbf{E}_3, \mathbf{G} \rangle \), we need to know \( \langle du, \mathbf{G} \rangle \) and \( \langle dv, \mathbf{G} \rangle \). Recall Rule VF2 from §1.3 that \( \langle du, \mathbf{U} \mathbf{r}_u + \mathbf{V} \mathbf{r}_v \rangle = \mathbf{U} \) and \( \langle dv, \mathbf{U} \mathbf{r}_u + \mathbf{V} \mathbf{r}_v \rangle = \mathbf{V}. \) On the other hand, we have

\[
\mathbf{G} = g_1 \mathbf{E}_1 + g_2 \mathbf{E}_2 = g_1 |\mathbf{r}_u|^{-1} \mathbf{r}_u + g_2 |\mathbf{r}_v|^{-1} \mathbf{r}_v = g_1 Y(v)^{-1} \mathbf{r}_u + g_2 \mathbf{r}_v.
\]

Thus \( \langle du, \mathbf{G} \rangle = g_1 Y(v)^{-1} \) and \( \langle dv, \mathbf{G} \rangle = g_2. \) So we have

\[
\langle d \mathbf{F} \cdot \mathbf{E}_3, \mathbf{G} \rangle = -\{Y(v)^{-1} Z'(v)\} f_1 g_1 - \{Y'(v)Z''(v) - Z'(v)Y''(v)\} f_2 g_2. \tag{4.2}
\]

Exercise 1 in §4.1 tells us that \( Y'Z'' - Z'Y'' \) is the curvature \( \kappa \) of the generating curve \( \gamma \) of \( S \). Note that the last example is the special case with \( Y(v) = 1 \) and \( Z(v) = v \). Identity (4.2) indicates that \( \langle d \mathbf{F} \cdot \mathbf{E}_3, \mathbf{G} \rangle \) is a symmetric bilinear form in \( \mathbf{F} \) and \( \mathbf{G} \). As we shall see, this is not accidental!

A remarkable thing about the expression \( d \mathbf{F} \cdot \mathbf{E}_3 \) is the identity

\[
d(g \mathbf{F}) \cdot \mathbf{E}_3 = g(d \mathbf{F} \cdot \mathbf{E}_3), \tag{4.3}
\]
that is, the function \( g \) can be extracted. This follows immediately from Rule PN1 in the previous section and the obvious identity \( d\mathbf{F} \cdot \mathbf{E}_3 = d_\mathbf{F} \mathbf{E}_3 \) obtained from (4.1). There is a better way to see (4.3): differentiate the identity \( \mathbf{F} \cdot \mathbf{E}_3 = 0 \) (which is a consequence of assumption that \( \mathbf{F} \) is tangential and \( \mathbf{E}_3 \) is normal) to get \( d\mathbf{F} \cdot \mathbf{E}_3 + \mathbf{F} \cdot d\mathbf{E}_3 = 0 \), or

\[
d\mathbf{F} \cdot \mathbf{E}_3 = -\mathbf{F} \cdot d\mathbf{E}_3.
\] (4.4)

Thus (4.3) follows from the obvious identity \((g\mathbf{F}) \cdot d\mathbf{E}_3 = g(\mathbf{F} \cdot d\mathbf{E}_3)\). Identity (4.3) is remarkable because it shows that the 1-form \( d\mathbf{F} \cdot \mathbf{E}_3 \) at a point \( p \) on the surface \( S \) only depends on \( \mathbf{F}_p \), the vector of \( \mathbf{F} \) at the point \( p \), although \( d\mathbf{F} \) at \( p \) also depends on \( \mathbf{F} \) at points near \( p \). Indeed, suppose that we have another vector field \( \mathbf{G} \) which coincides with \( \mathbf{F} \) at the point \( p \): \( \mathbf{F}_p = \mathbf{G}_p \). Then \( \mathbf{F} - \mathbf{G} \) is zero at \( p \) and hence we can write \( \mathbf{F} - \mathbf{G} = f\mathbf{H} \) for some smooth scalar function \( f \) with \( f(p) = 0 \) and some vector field \( \mathbf{H} \). Thus

\[
d\mathbf{F} \cdot \mathbf{E}_3 - d\mathbf{G} \cdot \mathbf{E}_3 = d(\mathbf{F} - \mathbf{G}) \cdot \mathbf{E}_3 = d(f\mathbf{H}) \cdot \mathbf{E}_3 = f d\mathbf{H} \cdot \mathbf{E}_3,
\]

which vanishes at \( p \). The reader should find out how this argument breaks down when \( d\mathbf{F} \cdot \mathbf{E}_3 \) is replaced by \( d\mathbf{F} \) in order to enhance his or her understanding.

The vector-valued differential form \( d\mathbf{E}_3 \) on \( S \) will be called the **Weingarten form** for \( S \). One can see that \( d\mathbf{E}_3 \) is tangential by differentiating \( \mathbf{E}_3 \cdot \mathbf{E}_3 = 1 \), giving us \( \mathbf{E}_3 \cdot d\mathbf{E}_3 = 0 \). Actually, we know this already: recall \( d\mathbf{E}_3 = \omega_1^3 \mathbf{E}_1 + \omega_2^3 \mathbf{E}_2 \) from (3.6) in §4.3, from which we see that \( d\mathbf{E}_3 \) is tangential because both \( \mathbf{E}_1 \) and \( \mathbf{E}_2 \) are. On the LHS of (11.4), \( \mathbf{E}_3 \) is normal and \( d\mathbf{F} \) in general is neither normal nor tangential. However, both \( \mathbf{F} \) and \( d\mathbf{E}_3 \) on the RHS are tangential.

Before going any further, let us recall the basic setting from the last section: given a Darboux frame \( \mathbf{E}_j \) \((j = 1, 2, 3)\) for the surface \( S \), where \( \mathbf{E}_1, \mathbf{E}_2 \) tangential and \( \mathbf{E}_3 \) is normal. The fundamental forms \( \theta^j \) are determined by \( d\mathbf{r} = \theta^1 \mathbf{E}_1 + \theta^2 \mathbf{E}_2 (\theta^3 = 0) \), and their differentials satisfy the first structural equations

\[
d\theta^1 = \theta^2 \wedge \omega_2^1, \quad d\theta^2 = \theta^1 \wedge \omega_1^2, \quad (d\theta^3 = 0) \quad 0 = \theta^1 \wedge \omega_1^2 + \theta^2 \wedge \omega_3. \] (4.5)

The connection forms \( \omega^j_k \) are determined by \( d\mathbf{E}_j = \sum_{k=1}^3 \omega^j_k \mathbf{E}_k \), with \( \omega^j_j + \omega^j_k = 0 \) and their differentials satisfy the second structural equations

\[
d\omega_1^2 = \omega_1^3 \wedge \omega_2^1, \quad d\omega_2^3 = \omega_2^1 \wedge \omega_3^3, \quad d\omega_3^2 = \omega_2^3 \wedge \omega_1^2. \] (4.6)

Now we claim:
**Rule VF3.** \( \theta^1 \) and \( \theta^2 \) form a **coframe** of \( S \) dual to the tangent frame \( \mathbf{E}_1, \mathbf{E}_2 \) in the sense that \( \langle \theta^j, \mathbf{E}_k \rangle = \delta^j_k \).

Here \( \delta^j_k \) is Kronecker’s delta; in detail, \( \langle \theta^1, \mathbf{E}_1 \rangle = \langle \theta^2, \mathbf{E}_2 \rangle = 1 \) and \( \langle \theta^1, \mathbf{E}_2 \rangle = \langle \theta^2, \mathbf{E}_1 \rangle = 0 \). That \( \theta^j \) \( (j = 1, 2) \) is a coframe means that every 1-form \( \omega \) defined on \( S \) can be written as \( \omega = g_1 \theta^1 + g_2 \theta^2 \) for some scalar functions \( g_1 \) and \( g_2 \) on \( S \). This rule, a companion to Rule VF2 in §1.3, is a bit hard to swallow. Its justification needs some elaborate theoretical background and hence is postponed to the end of the section.

The key for studying the shape of a surface is the last identity in (4.5), namely

\[ \theta^1 \wedge \omega^1 + \theta^2 \wedge \omega^2 = 0. \]

Since \( \theta^1, \theta^2 \) form a coframe for \( S \), we may put

\[ \omega^1_1 = h_{11} \theta^1 + h_{12} \theta^2 \quad \text{and} \quad \omega^1_2 = h_{21} \theta^1 + h_{22} \theta^2. \]  \( (4.7) \)

Thus \( 0 = \theta^1 \wedge \omega^1_1 + \theta^2 \wedge \omega^1_2 = h_{12} \theta^1 \wedge \theta^2 + h_{21} \theta^2 \wedge \theta^1 = (h_{12} - h_{21}) \theta^1 \wedge \theta^2. \) Since \( \theta^1 \wedge \theta^2 \) is the area form, it is non-vanishing everywhere. So we must have \( h_{12} - h_{21} = 0 \), or

\[ h_{12} = h_{21}. \]  \( (4.8) \)

Let \( \mathbf{F} = f^1 \mathbf{E}_1 + f^2 \mathbf{E}_2 \) and \( \mathbf{G} = g^1 \mathbf{E}_1 + g^2 \mathbf{E}_2 \) be vector fields tangent to \( S \), where \( f^j = \mathbf{F} \cdot \mathbf{E}_j \) and \( g^j = \mathbf{G} \cdot \mathbf{E}_j \). We compute \( \langle d\mathbf{F} \cdot \mathbf{E}_3, \mathbf{G} \rangle \) as follows. From (4.4), we have

\[ d\mathbf{F} \cdot \mathbf{E}_3 = -\mathbf{F} \cdot d\mathbf{E}_3 = -\mathbf{F} \cdot (\omega^1_3 \mathbf{E}_1 + \omega^2_3 \mathbf{E}_2) = -\omega^1_3 \mathbf{F} \cdot \mathbf{E}_1 - \omega^2_3 \mathbf{F} \cdot \mathbf{E}_2 \]

\[ = -f_1 \omega^1_3 - f_2 \omega^2_3 = f_1 \omega^3_1 + f_2 \omega^3_2. \]

Thus \( \langle d\mathbf{F} \cdot \mathbf{E}_3, \mathbf{G} \rangle = f_1 \langle \omega^3_1, \mathbf{G} \rangle + f_2 \langle \omega^3_2, \mathbf{G} \rangle \). Now

\[ \langle \omega^3_1, \mathbf{G} \rangle = \langle h_{11} \theta^1 + h_{12} \theta^2, g^1 \mathbf{E}_1 + g^2 \mathbf{E}_2 \rangle = h_{11} g^1 + h_{12} g^2, \]

in view of Rule VF3 above. Similarly, we have \( \langle \omega^3_2, \mathbf{G} \rangle = h_{21} g^1 + h_{22} g^2 \). Finally,

\[ \langle d\mathbf{F} \cdot \mathbf{E}_3, \mathbf{G} \rangle = f_1 (h_{11} g^1 + h_{12} g^2) + f_2 (h_{21} g^1 + h_{22} g^2) = \sum_{j,k=1}^{2} h_{jk} f^j g^k. \]  \( (4.10) \)

Due to (11.8), the last expression is a symmetric bilinear form in \( (f^1, f^2) \) and \( (g^1, g^2) \), and, as a consequence, \( \langle d\mathbf{F} \cdot \mathbf{E}_3, \mathbf{G} \rangle = \langle d\mathbf{G} \cdot \mathbf{E}_3, \mathbf{F} \rangle \). We call

\[ II(\mathbf{F}, \mathbf{G}) \equiv \langle d\mathbf{F} \cdot \mathbf{E}_3, \mathbf{G} \rangle \equiv \langle d\mathbf{E}_3 \cdot \mathbf{F}, \mathbf{G} \rangle \]

the **second fundamental form** for the surface. By the way, the first fundamental form is just the inner product: \( I(\mathbf{F}, \mathbf{G}) = \mathbf{F} \cdot \mathbf{G} = f^1 g^1 + f^2 g^2 \). Both fundamental forms are symmetric in the sense that \( I(\mathbf{F}, \mathbf{G}) = I(\mathbf{G}, \mathbf{F}) \) and \( II(\mathbf{F}, \mathbf{G}) = II(\mathbf{G}, \mathbf{F}) \).
The trace and the determinant of the symmetric matrix \[
\begin{bmatrix}
  h_{11} & h_{12} \\
  h_{21} & h_{22}
\end{bmatrix}
\] associated with the second fundamental form, namely
\[
H = h_{11} + h_{22}, \quad K = h_{11}h_{22} - h_{12}^2,
\]
are called the \textbf{mean curvature} and the \textbf{total curvature} of \( S \) respectively; (some books define the mean curvature \( H \) as \(-h_{11} - h_{22}, \) or \((-h_{11} - h_{22})/2 \)).

**Example 4.3.** Find the total curvature of the surface of revolution in Example 4.2.

**Solution:** Comparing the answer to Example 4.2 with (4.10), we have
\[
h_{11} = -Z'/Y, \quad h_{22} = -\{Y'Z'' - Z'Y''\} \quad \text{and} \quad h_{12} = h_{21} = 0;
\]
\((Y \text{ and } Z \text{ are functions of } v). \) So the total curvature \( K \) is \( \{Y'Z'' - (Z')^2Y''\}/Y. \)
Differentiating \((Y')^2 + (Z')^2 = 1, \) we have \( Y'' + Z'Z'' = 0. \) Thus
\[
K = \{Y'(-Y''Y') - (Z')^2Y''\}/Y = -\{(Y')^2 + (Z')^2\}Y''/Y = -Y''/Y.
\]
(In the next section we give another way to obtain this answer.)

A pleasant surprise is: the total curvature \( K \) is the same as the Gaussian curvature \( K \) described in the last section; (the same letter \( K \) used here for the total curvature is not a mistake!) This is surprising because \( K = h_{11}h_{22} - h_{12}^2 \) is an expression involving extrinsic quantities \( h_{jk}, \) while the Gaussian curvature given in the last section, determined by \( d\omega^1 = K \theta^1 \wedge \theta^2, \) is intrinsic. This highly nontrivial fact, known as \textbf{theorem egregium of Gauss}, is one of the most important discoveries in classical differential geometry. Now Cartan’s method of moving frame makes it almost transparent. If you play with Cartan’s structural equations long enough, you’ll get it!

**Example 4.4.** Prove the theorem egregium of Gauss.

**Solution:** The first identity in (4.6) gives
\[
d\omega^2 = \omega^3 \wedge \omega^3 = \omega_2^3 \wedge (-\omega^3) \]
\[
= (h_{21}\theta^1 + h_{22}\theta^2) \wedge (-h_{11}\theta^1 - h_{12}\theta^2) = (h_{11}h_{22} - h_{12}^2) \theta^1 \wedge \theta^2.
\]
Comparing with \( d\omega^1 = K \theta^1 \wedge \theta^2, \) we immediately get \( K = h_{11}h_{22} - h_{12}^2. \) Done!

Finally we explain why we should believe in Rule VF3. Let us first consider the situation that \( E_j \ (j = 1, 2, 3) \) is a frame filled a spatial region \( U \) in \( \mathbb{R}^3; \) (technically, \( U \) is
an open set in \( \mathbb{R}^3 \). Again \( \theta_j \) is determined by \( dx = \sum_{j=1}^{3} \theta^j \mathbf{E}_j \), where \( x = (x_1, x_2, x_3) \).

We claim: \( \langle \theta^j, \mathbf{E}_k \rangle = \delta^j_k \). To prove this, we introduce the standard basis vectors for \( \mathbb{R}^3 \):
\[
\mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0) \quad \text{and} \quad \mathbf{e}_3 = (0, 0, 1).
\]

Write \( \mathbf{E}_k = (E_{k1}, E_{k2}, E_{k3}) \) \((k = 1, 2, 3)\). Then \( E_{k\ell} = \mathbf{E}_k \cdot \mathbf{e}_\ell \) \((k, \ell = 1, 2, 3)\). Notice that
\[
\nabla x_\ell = \left( \frac{\partial x_1}{\partial x_\ell}, \frac{\partial x_2}{\partial x_\ell}, \frac{\partial x_3}{\partial x_\ell} \right) = \mathbf{e}_\ell \; \text{ for } \ell = 1, 2, 3.
\]

Hence \( \langle dx_\ell, \mathbf{E}_k \rangle = \nabla x_\ell \cdot \mathbf{E}_k = \mathbf{e}_\ell \cdot \mathbf{E}_k = E_{k\ell} \). On the other hand, from the identity \( \sum_{k=1}^{3} \theta^k \mathbf{E}_k = dx \) and \( \mathbf{E}_j \cdot \mathbf{E}_k = \delta_{jk} \), we have
\[
\theta^j = \sum_{k=1}^{3} \theta^k \mathbf{E}_j \cdot \mathbf{E}_k = (E_{j1}, E_{j2}, E_{j3}) \cdot (dx_1, dx_2, dx_3) = \sum_{\ell=1}^{3} E_{j\ell} dx_\ell.
\]

Thus \( \langle \theta^j, \mathbf{E}_k \rangle = \sum_{\ell=1}^{3} E_{j\ell} \langle dx_\ell, \mathbf{E}_k \rangle = \sum_{\ell=1}^{3} E_{j\ell} E_{k\ell} = \mathbf{E}_j \cdot \mathbf{E}_k = \delta_{jk} \). Done.

Now we consider a Darboux frame \( \mathbf{E}_j \) \((j = 1, 2, 3)\) for a surface \( S \). This frame is defined only at points on \( S \). We extend each \( \mathbf{E}_j \) to some \( \tilde{\mathbf{E}}_j \) \((j = 1, 2, 3)\) to some spatial region \( U \), which is a neighborhood of \( S \). It follows from our previous discussion that \( \langle \tilde{\theta}^j, \tilde{\mathbf{E}}_k \rangle = \delta^j_k \) \((j, k = 1, 2, 3)\), where \( \tilde{\theta}^j \) are 1-forms determined by
\[
dx = \sum_{j=1}^{3} \tilde{\theta}^j \tilde{\mathbf{E}}_j. \tag{4.12}
\]

Clearly, the restriction of each \( \tilde{\mathbf{E}}_j \) to \( S \) is \( \mathbf{E}_j \) for each \( j \). Along the surface \( S \), the tangential component \( (dx)_\uparrow \) of \( dx \) is, in view of (4.12) above and the fact that the restriction of \( \tilde{\mathbf{E}}_j \) to \( S \) is \( \mathbf{E}_j \), given by \( (dx)_\uparrow = \tilde{\theta}^1 \mathbf{E}_1 + \tilde{\theta}^2 \mathbf{E}_2 \). So the restriction of \( \tilde{\theta}^1 \) and \( \tilde{\theta}^2 \) to \( S \) are \( \theta^1 \) and \( \theta^2 \) respectively. Thus \( \langle \theta^j, \mathbf{E}_k \rangle = \langle \tilde{\theta}^j, \tilde{\mathbf{E}}_k \rangle = \delta^j_k \).

There is one question remains: why can we extend \( \mathbf{E}^j \) to \( \tilde{\mathbf{E}}^j \)? This is a very technical question and we answer briefly as follows. Anyone with proper background in modern mathematical analysis should have no problem to fill in the details of the proof outlined below. The uninterested reader may skip this technical Mumbo Jumbo.

To begin with, we assert that every (smooth) function \( f \) on \( S \) can be extended to a function \( \tilde{f} \) defined on a (spatial) neighborhood of \( S \). This can at least be done in a local level, mainly because at any point \( p \) of the surface we can choose a local coordinate \((u, v, w)\) so that in a neighborhood of \( p \), \( S \) is described by the equation \( w = 0 \). To extend \( f \) globally, we may use a partition of unity to piece local extensions together.

Once we know that a (smooth) function can be extended, we can extend every (smooth) vector field by extending its components. Now let us agree that each vector
field \( \mathbf{E}^j \) of the given moving frame can be extended to a vector field \( \mathbf{\bar{E}}^j \). The problem here is that \( \{ \mathbf{\bar{E}}^j \ (j = 1, 2, 3) \} \) is not necessarily an orthonormal frame. All we can say is that, at a point close enough to \( S \), \( \{ \mathbf{\bar{E}}^j \} \) is almost orthonormal and hence is linearly independent. We may allow the frame \( \mathbf{\bar{E}}^j \ (j = 1, 2, 3) \) to be linearly independent at each point by shrinking its domain appropriately. Now we can use the Gram-Schmidt process to obtain an orthonormal frame out of \( \mathbf{\bar{E}}^j \ (j = 1, 2, 3) \), which is our required extension of the Darboux frame \( \mathbf{E}^j \ (j = 1, 2, 3) \).

**Exercises**

1. In each of the following parts, find the second fundamental form, the mean curvature and the total curvature of the given surface of revolution:
   (a) Cone: \( x^2 + y^2 = z^2 \) \((z > 0)\).  
   (b) Sphere: \( x^2 + y^2 + z^2 = R^2 \) \((R > 0)\).
   (c) Paraboloid: \( x^2 + y^2 = z \).
   (d) Hyperboloid: \( x^2 + y^2 = 1 + z^2 \).
   (e) Catenoid: \( x = \cosh u \cos v, y = \cosh u \sin v, z = u \).
   (f) Pseudosphere: \( x = \sech u \cos v, y = \sech u \sin v, z = \tanh u - u \).
   (g) Torus: \( x = (R + r \cos u) \cos v, y = (R + r \cos u) \sin v, z = r \sin u \) \((0 < r < R)\).
   **Hint:** Apply the solution to Example 4.2 with caution: the generating curve \( Y = Y(v), Z = Z(v) \) there has the arc-length parametrization: \((Y')^2 + (Z')^2 = 1\).

2. (a) What happens to the mean curvature \( H \) and the total curvature \( K \) of a surface if the unit normal field \( \mathbf{E}_3 \) on \( S \) is reversed, that is, replaced by \( -\mathbf{E}_3 \)? (b) Same question if \( S \) is dilated. In detail, for a constant \( a > 0 \), let \( S_a \) be the surface such that a point \( p \) is on \( S \) if and only if \( a \mathbf{p} \) is on \( S_a \). Notice that \( S_1 = S \). Denote by \( H_a \) and \( K_a \) the mean curvature and the total curvature of \( S_a \) respectively. What are the relations between \( H_a \) and \( H_1 \), and between \( K_a \) and \( K_1 \)?

3. Show that if the mean curvature \( H \) and the total curvature \( K \) of a surface \( S \) are vanishing everywhere, then \( S \) is a plane. **Hint:** Since \( [h_{ij}] \) is a symmetric matrix with vanishing eigenvalues \((H, K = \text{the sum and the product of eigenvalues})\), we must have \( h_{ij} \equiv 0 \). By (11.8), \( \omega_1^2 = \omega_2^2 = 0 \). So the Weingarten form \( d\mathbf{E}_3 \) also vanishes.)

4. We say that a point \( \mathbf{p} \) on a surface \( S \) is **umbilic**, if \( h_{11} = h_{22} \) and \( h_{12} = h_{21} = 0 \) at that point, in other words, there is a scalar \( \lambda \) such that \( II(\mathbf{F}, \mathbf{G})_p = \lambda I(\mathbf{F}, \mathbf{G})_p \) for all tangential vector fields \( \mathbf{F} \) and \( \mathbf{G} \). Prove that, if every point of a surface \( S \) is umbilic, then \( S \) is either planar or spherical.
6. In each of the following parts, use the result of the last exercise to compute the mean curvature and the total curvature of the given parametric surface:

(a) the unit sphere, using the stereographic parametrization.

(b) Verify that \( d\mathbf{N} = \lambda d\mathbf{x} \) gives \( \mathbf{N}_u = \lambda \mathbf{x}_u \) and \( \mathbf{N}_v = \lambda \mathbf{x}_v \). So \( \mathbf{N}_{uv} = \lambda \mathbf{x}_u + \lambda \mathbf{x}_{uv} \) and \( \mathbf{N}_{vu} = \lambda \mathbf{x}_v + \lambda \mathbf{x}_{vu} \).

Hence \( \mathbf{N} \) is normal to \( \mathbf{x} \) and \( \mathbf{x}_u \times \mathbf{x}_v \neq 0 \) we can derive \( \lambda_u = \lambda_v = 0 \). Thus \( \lambda \) is a constant. Now \( d\mathbf{N} = \lambda d\mathbf{x} \) can be rewritten as \( d(\mathbf{N} - \lambda \mathbf{x}) = 0 \).

5. Let \( S \) be a surface given by parametric equation \( \mathbf{r} = \mathbf{r}(u, v) \). Recall that the metric tensor for \( S \) is \( ds^2 = g_{uu} du^2 + 2g_{uv} dudv + g_{vv} dv^2 \), where \( g_{uu} = \mathbf{r}_u \cdot \mathbf{r}_u \), \( g_{uv} = \mathbf{r}_u \cdot \mathbf{r}_v \) and \( g_{vv} = \mathbf{r}_v \cdot \mathbf{r}_v \). Also, \( \mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \) is normal to \( S \) with \( |\mathbf{N}| = (g_{uu}g_{vv} - g_{uv}^2)^{1/2} \). We may take \( \mathbf{E}_3 = \mathbf{N}/|\mathbf{N}|. \)

(a) Show that \( \langle d\mathbf{F} \cdot \mathbf{E}_3, \mathbf{G} \rangle = -|\mathbf{N}|^{-1} \langle d\mathbf{N} \cdot \mathbf{F}, \mathbf{G} \rangle \). Hint: Again, from \( \mathbf{N} \cdot \mathbf{F} = 0 \) we get \( d\mathbf{N} \cdot \mathbf{F} = -\mathbf{N} \cdot d\mathbf{F} \). Recall that \( \mathbf{E}_3 = \mathbf{N}/|\mathbf{N}|. \)

(b) Verify that \( \langle d\mathbf{N} \cdot \mathbf{F}, \mathbf{G} \rangle = L_{uu} F^u G^u + L_{uv} F^u G^v + L_{vv} F^v G^v \), where \( F^u = \langle \mathbf{F}, du \rangle, F^v = \langle \mathbf{F}, dv \rangle, G^u = \langle \mathbf{G}, du \rangle \) and \( G^v = \langle \mathbf{G}, dv \rangle \). Hint: By Rule VF2 in §1.3, we have \( \mathbf{F} = \langle \mathbf{F}, du \rangle \mathbf{r}_u + \langle \mathbf{F}, dv \rangle \mathbf{r}_v. \)

(c) Show that the mean curvature and the total curvature of \( S \) are given by

\[
H = \frac{L_{uu} g_{vv} - 2L_{uv} g_{uv} + L_{vv} g_{uu}}{g_{uu} g_{vv} - g_{uv}^2}, \quad K = \frac{L_{uu} L_{vv} - L_{uv}^2}{g_{uu} g_{vv} - g_{uv}^2}.
\]

6. In each of the following parts, use the result of the last exercise to compute the mean curvature and the total curvature of the given parametric surface:

(a) the unit sphere, using the stereographic parametrization.

(b) the torus given in part (g) of Exercise 1 above.

(c) the helicoid \( \mathbf{r}(u, v) = (u \cos v, u \sin v, bv) \) \( (b > 0) \); (note that, for fixed \( v \), equation \( \mathbf{r} = \mathbf{r}(u, v) \) with \( u \) as a parameter describes a line).

7. Suppose that a surface \( S \) is given by an equation of the form \( z = h(x, y) \). Show that the mean curvature and the total curvature of \( S \) are given by

\[
H = \frac{(1 + h_y^2) h_{yy} - 2h_x h_y h_{xy} + (1 + h_x^2) h_{xx}}{(1 + h_x^2 + h_y^2)^{3/2}}, \quad K = \frac{h_{xx} h_{yy} - h_{xy}^2}{(1 + h_x^2 + h_y^2)^2}.
\]