§3. Cartan’s Structural Equations and the Curvature Form

Let $E_1, \ldots, E_n$ be a moving (orthonormal) frame in $\mathbb{R}^n$ and let $\omega^k_j$ its associated connection forms so that:

$$dE_k = \sum_{j=1}^n \omega^j_k E_j.$$  \hspace{1cm} (3.1)

Recall that $\omega^k_j = -\omega^j_k$ and in particular $\omega^k_k = 0$. Let $\theta^j$ be the basic forms associated with this moving frame given by:

$$dx = \sum_{j=1}^n \theta^j E_j,$$  \hspace{1cm} (3.2)

where $x = (x_1, x_2, \ldots, x_n)$. Recall that each $x_j$ is considered as the $j$th coordinate function on $\mathbb{R}^n$ defined by $x_j(p) = p_j$ for each $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$.

Now let us differentiate (3.1) and (3.2): apply operator $d$ which turns vector-valued 1-forms to vector-valued 2-forms. But we must be cautious. There is no problem with the left hand sides: from $d^2 = 0$ we have $d(dE_k) = d^2E_k = 0$ and $d(dx) = d^2x = 0$. For the right hand sides, naturally we apply the product rule. The product rule for 1-forms says $d(f\omega) = d(f) \wedge \omega + f d\omega$, where $\omega$ is any 1-form where $f$ is any smooth function. We can replace $f$ by a vector-valued function, say $F$, and the same identity holds: $d(F\omega) = dF \wedge \omega + F d\omega$.

However, in (3.1) and (3.2), the one–forms $\omega^k_j$ and $\theta^j$ are written in front of our vector-valued functions $E_j$. So we have to figure out a formula for $d(\omega F)$. Here it is:

$$d(\omega F) = d(F\omega) = dF \wedge \omega + F d\omega = (d\omega)F - \omega \wedge dF.$$  \hspace{1cm} (3.3)

The minus sign in front of the last term appears because we have switched the order of the 1-forms $d\omega$ (vector-valued) and $\omega$. Were we careless we would missed this minus sign and would made a big blunder! Applying $d$ on both sides of (3.2), using $d^2 = 0$ and (3.3), changing index, and then substituting $dE_\ell$ by means of (3.1), we get

$$0 = d^2x = \sum_{j=1}^n (d\theta^j E_j - \theta^j \wedge dE_j) = \sum_{j=1}^n d\theta^j E_j - \sum_{k=1}^n \theta^k \wedge dE_k$$

$$= \sum_{j=1}^n d\theta^j E_j - \sum_{\ell=1}^n \theta^\ell \wedge \left( \sum_{j=1}^n \omega^j_\ell E_j \right) = \sum_{j=1}^n \left( d\theta^j - \sum_{\ell=1}^n \theta^\ell \wedge \omega^j_\ell \right) E_j.$$

Since at each point $E_1, \ldots, E_n$ form an orthonormal basis, we have

$$d\theta^j = \sum_{\ell=1}^n \theta^\ell \wedge \omega^j_\ell,$$  \hspace{1cm} (3.4)
which are called Cartan’s first structural equations. Similarly, (3.1) gives

\[ 0 = d^2 E_k = \sum_{j=1}^{n} (d\omega^j_k \ E_j - \omega^j_k \wedge dE_j) = \sum_{j=1}^{n} d\omega^j_k \ E_j - \sum_{\ell=1}^{n} \omega^\ell_k \wedge dE_\ell \]

\[ = \sum_{j=1}^{n} \omega^j_k \ E_j - \sum_{\ell=1}^{n} \omega^\ell_k \wedge (\sum_{j=1}^{n} \omega^j_\ell \wedge E_j) = \sum_{j=1}^{n} \left( d\omega^j_k - \sum_{\ell=1}^{n} \omega^\ell_k \omega^j_\ell \right) E_j. \]

Thus it follows that

\[ d\omega^j_k = \sum_{\ell=1}^{n} \omega^\ell_k \wedge \omega^j_\ell, \quad (3.5) \]

which are called Cartan’s second structural equations. You must learn to appreciate the beauty of these basic structural equations in differential geometry due to Elie Cartan, who is considered by many as the greatest differential geometer.

**Example 3.1.** Verify the structural equations of the forms associated with the Frenet-Serrat frame of a curve in $\mathbb{R}^3$; (see §4.1).

**Solution:** Recall (1.5) from §4.1 with the identification $E_1 = T$, $E_2 = N$, $E_3 = B$, (the unit tangent vector, the normal vector, and the binormal vector respectively), and

\[ \omega^1_1 = \omega^2_2 = \omega^3_3 = \omega^1_3 = \omega^2_3 = 0, \quad \omega^2_1 = -\omega^1_2 = \omega, \quad \omega^3_2 = -\omega^2_3 = \eta. \]

Also $dx = \theta^1 E_1 = \theta^1 T$ (this is because $dx$ is tangential and the tangent of a curve at any point is the one dimensional space spanned by $T$) so that $\theta^2 = \theta^3 = 0$. A curve has only one parameter when it is described by a parametric equation $x = x(t)$. Thus 1-forms such as $\theta^1$, $\omega$ and $\eta$ associated with the curve can be written in the form $f(t)dt$. But

\[ d(f(t)dt) = df(t) \wedge dt = f'(t)dt \wedge dt = 0. \]

So $d\theta^1 = d\omega = d\eta = 0$. Thus it remains to check $\sum_{\ell=1}^{3} \theta^\ell \wedge \omega^j_\ell = 0$ for the first structural equations and $\sum_{\ell=1}^{3} \omega^\ell_j \wedge \omega^k_\ell = 0$ for the second structural equations. Again, since all 1-forms here can be written in the form $f(t)dt$ and since $f_1(t)dt \wedge f_2(t)dt = 0$, these identities are obvious. Thus, both sides of the structure equations for the Frenet-Serrat frame along a curve are vanishing. This tells us that in some sense the local geometry of a curve is not interesting at all. So, let us turn to surfaces.

Let $E_j \ (j = 1, 2, 3)$ be a Darboux frame for a surface $S$ with $E_1$ and $E_2$ tangential and $E_3 \equiv N$ normal. (Here we assume that $S$ is oriented so that the pair $(E_1, E_2)$
has a compatible orientation at each point. We also assume that the Darboux frame given here has the same orientation as the usual one for \( \mathbb{R}^3 \) at each point. These assumptions will be needed at one subtle point in the future, but let us not worry about this now.)

Recall that the basic forms \( \theta^j \) and the connection forms \( \omega^j_k \) are determined by the relations:

\[
\begin{align*}
\text{d}x & = \theta^1 E_1 + \theta^2 E_2 \quad \text{ (that is, } \theta^3 = 0) , \\
\text{d}E_1 & = \omega^2_1 E_2 + \omega^3_1 E_3 , \\
\text{d}E_2 & = \omega^1_2 E_1 + \omega^3_2 E_3 , \\
\text{d}E_3 & = \omega^1_3 E_1 + \omega^2_3 E_2 .
\end{align*}
\]

(3.6)

The structural equations (3.4) and (3.5) become

\[
\begin{align*}
\text{d}\theta^1 & = \theta^2 \wedge \omega^1_2 , \\
\text{d}\theta^2 & = \theta^1 \wedge \omega^2_1 , \\
\text{d}\omega^1_2 & = \omega^3_1 \wedge \omega^2_3 , \\
\text{d}\omega^1_3 & = \omega^2_1 \wedge \omega^3_2 , \\
\text{d}\omega^2_3 & = \omega^1_2 \wedge \omega^3_1 .
\end{align*}
\]

(3.7)

(3.8)

Let \( F \) be a vector field on the surface \( S \) and tangent to the surface. Consider the vector-valued differential \( \text{d}F \). Here we have to be cautious: that \( F \) is tangential does not imply that \( \text{d}F \) is also tangential. In fact, in general we expect that the normal component \( (\text{d}F \cdot N)N \) of \( F \) is nonzero because \( F \) has to turn as \( S \) bends, and the acceleration has a nonzero normal component when the motion is changing its direction, even when the speed remains constant. For each vector (or vector-valued 1-form) \( v \) attached to a point on \( S \), denote by \( v_\parallel \) the tangential component of \( v \), and \( v_\perp \) its normal component. Explicitly,

\[
v = v_\parallel + v_\perp , \quad \text{where} \quad v_\parallel = (v \cdot E_1)E_1 + (v \cdot E_2)E_2 \quad \text{and} \quad v_\perp = (v \cdot E_3)E_3 .
\]

(3.9)

For a vector field (or a vector-valued 1-form) \( F \) tangent to our surface \( S \), let

\[
\text{d}_\parallel F = (\text{d}F)_\parallel , \quad \text{d}_\perp F = (\text{d}F)_\perp ,
\]

which will be called the covariant differential and the normal differential of the vector field \( F \) respectively. The covariant differential is responsible for the intrinsic geometry of our surface, while the normal differential determines its extrinsic shape. To get some idea about the jargon here, take a piece of paper. You can lay it flat on a table, or roll it up into a cylinder or cone. No matter how you change its shape, the intrinsic geometry is still “flat”. Now we state the product rules:

**Rule PC1.** \( \text{d}_\parallel (gF) = \text{d}g \cdot F + g \text{d}_\parallel F \).

**Rule PC2.** \( \text{d}_\parallel (\omega F) = \omega \cdot F - \omega \wedge \text{d}_\parallel F \).
Rule PN1. $d_{\perp}(gF) = gd_{\perp}F$.

Rule PN2. $d_{\perp}(\omega F) = -\omega \wedge d_{\perp}F$.

To verify these identities, write down

$$d(gF) = dg \cdot F + g \, dF \quad \text{and} \quad d(\omega F) = d\omega \cdot F - \omega \wedge dF,$$

and take the tangential components and normal components; (the detail is left to the reader). Notice that $F_T = F$ because $F$ is tangential. In the present section we concentrate on covariant differentials for which only Rules PC1 and PC2 are needed.

Unlike $d^2 = 0$, in general $d^2_T$ is nonvanishing. In fact, this is really what makes the covariant differential $d_T$ interesting. To compute $d^2_T F$ for a tangential vector field $F$, put $F = f^1 E_1 + f^2 E_2$, where $f^1 = F \cdot E_1$ and $f^2 = F \cdot E_2$; (here we use superscript for the components $f^1$ and $f^2$ of $F$ instead of subscripts in order to see a pattern). Now

$$d_T E_1 = (dE_1)_T = (\omega^2_1 E_2 + \omega^2_1 E_3)_T = \omega^2_1 E_2. \quad (3.11)$$

(Notice that $\omega^1_1 = 0$, $E_2$ is tangential and $E_3$ is normal.) Similarly we have $d_T E_2 = \omega^1_2 E_1$, which can be rewritten as $-\omega^2_1 E_1$ if you like. (A systematic way for writing down both identities is $d_T E_k = \sum_{j=1}^2 \omega^j_k E_j$; $k = 1, 2$. Insertion of the extra terms $\omega^1_1 E_1$ and $\omega^2_2 E_2$ is harmless because they vanish. This systematic way is needed if we want to study geometric objects of higher dimensions.) Thus

$$d_T F = d_T (f^1 E_1) + d_T (f^2 E_2)$$

$$= df^1 \cdot E_1 + f^1 d_T E_1 + df^2 \cdot E_2 + f^2 d_T E_2$$

$$= df^1 \cdot E_1 + f^1 \omega^2_1 E_2 + df^2 \cdot E_2 + f^2 \omega^1_2 E_1$$

$$= (df^1 + f^2 \omega^2_1) E_1 + (df^2 + f^1 \omega^1_2) E_2.$$

To compute $d^2_T F \equiv d_T (d_T F)$, we begin with

$$d_T ((df^1 + f^2 \omega^2_1) E_1) = d(df^1 + f^2 \omega^2_1) \cdot E_1 - (df^1 + f^2 \omega^2_1) \wedge d_T E_1$$

$$= (df^2 \wedge \omega^2_1 + f^2 d\omega^1_2) E_1 - (df^1 + f^2 \omega^1_2) \wedge \omega^2_1 E_2$$

$$= (df^2 \wedge \omega^2_1 + f^2 d\omega^1_2) E_1 - (df^1 \wedge \omega^2_1) E_2,$$

because $\omega^1_1 \wedge \omega^2_1 = \omega^2_1 \wedge (-\omega^2_1) = 0$. Similarly, we have

$$d((df^2 + f^1 \omega^1_2) E_1) = (df^1 \wedge \omega^1_1 + f^1 d\omega^2_1) E_2 - (df^2 \wedge \omega^2_1) E_1.$$
Thus
\[ d^2_\nabla F \equiv d^2_\nabla \{ f^1 E_1 + f^2 E_2 \} = f^2 d\omega^1_2 E_1 + f^1 d\omega^2_1 E_2. \] (3.12)

So \( d^2_\nabla \) sends a vector field \( F = f^1 E_1 + f^2 E_2 \) (which can be regarded as a vector-valued 0-form) to a vector-valued 2-form, say \( \varphi^1 E_1 + \varphi^2 E_2 \), such that
\[
\begin{bmatrix}
\varphi^1 \\
\varphi^2
\end{bmatrix} =
\begin{bmatrix}
0 & d\omega^1_2 \\
d\omega^2_1 & 0
\end{bmatrix}
\begin{bmatrix}
f^1 \\
f^2
\end{bmatrix},
\quad \text{where } d\omega^1_2 = \omega^3_2 \wedge \omega^1_3.
\]

We may regard \( d^2_\nabla \) an operator-valued 2-form and its matrix representation with respect to the basis \( E_1, E_2 \) is given by
\[ \Omega = \begin{bmatrix} 0 & d\omega^1_2 \\ d\omega^2_1 & 0 \end{bmatrix}. \] (3.13)

We call this operator-valued 2-form \( d^2_\nabla \), or the matrix-valued 2-form \( \Omega \) given above, the curvature form for the surface \( S \). The following tensor property for the curvature form \( d^2_\nabla \) is important: for all vector fields \( F \) tangent to the surface \( S \) and for all (nice and smooth) function \( g \) on \( S \), we have
\[ d^2_\nabla (gF) = g d^2_\nabla F. \] (3.14)

In contrast, \( d_\nabla (gF) \neq g d_\nabla F \) (in general); instead we have \( d_\nabla (gF) = dg.F + g d_\nabla F. \) (Thus, \( d^2_\nabla \) is a tensor, but \( d_\nabla \) is not.) We are going to give two proofs of (3.14) because of its importance. First proof: start with \( gF = (gf^1)E_1 + (gf^2)E_2 \). Applying (3.12), we have
\[
d^2_\nabla (gF) = (gf^2) d\omega^1_2 E_1 + (gf^1) d\omega^2_1 E_2 = g(f^2 d\omega^1_2 E_1 + f^1 d\omega^2_1 E_2) = g d^2_\nabla F.
\]

Second proof: we have \( d_\nabla (gF) = dg.F + g d_\nabla F \) and hence
\[
d^2_\nabla (gF) = d_\nabla (dg.F) + d_\nabla (g d_\nabla F) = d^2 g.F - dg \wedge d_\nabla F + dg \wedge d_\nabla F + g d^2_\nabla F = g d^2_\nabla F.
\]

(The second proof applies to more general situations.) From \( \omega^2_1 = -\omega^1_2 \) we see that \( \Omega \) is a skew symmetric matrix. The entry \( d\omega^1_2 \) of \( \Omega \) turns out to be independent of the choice of our frame \( E_1, E_2 \). Indeed, suppose that \( F_1, F_2 \) is another frame, having the same orientation; (here we have used the assumption on the orientation of the surface \( S \)). Then \( F_1 = (\cos \theta)E_1 - (\sin \theta)E_2 \) and \( F_2 = (\sin \theta)E_1 + (\cos \theta)E_2 \) for some function \( \theta \) on \( S \). It
follows from (10.12) that \( d^2_+ \mathbf{E}_2 = d \omega^1_2 \mathbf{E}_1 \). To show that \( d \omega^1_2 \) is independent of frames, it is enough to check \( d^2_+ \mathbf{F}_2 = d \omega^1_2 \mathbf{F}_1 \). Indeed, by (3.14),

\[
\begin{align*}
&\quad d^2_+ \mathbf{F}_2 = d^2_+ (\sin \theta \mathbf{E}_1 + \cos \theta \mathbf{E}_2) = \sin \theta \cdot d^2_+ \mathbf{E}_1 + \cos \theta \cdot d^2_+ \mathbf{E}_2 \\
&= \sin \theta \cdot d \omega^1_1 \mathbf{E}_2 + \cos \theta \cdot d \omega^1_2 \mathbf{E}_1 = \sin \theta \cdot d(-\omega^1_2) \mathbf{E}_2 + \cos \theta \cdot d \omega^1_2 \mathbf{E}_1 \\
&= d \omega^1_2 \left( \cos \theta \mathbf{E}_1 - \sin \theta \mathbf{E}_2 \right) = d \omega^1_2 \mathbf{F}_1.
\end{align*}
\]

Done. The 2-from \( d \omega^1_2 \) is called the Pfaffian of \( \Omega \). (The Pfaffian here has nothing to do with Pfaffian equations. It is a numerical invariant associated with a skew symmetric form on an even dimensional linear space. Strictly speaking, it is a topic in the theory of multilinear algebra.) Now a surface is a two dimensional object. The “top forms” defined on it are 2-forms and they must be a multiple of the area form \( \theta^1 \wedge \theta^2 \). In particular,

\[
d \omega^1_2 = K \theta^1 \wedge \theta^2 \tag{3.15}
\]

for some scalar function \( K \) on \( S \), which is called the Gauss curvature for \( S \). The reader is asked to show that \( K \) here is independent of the orientation of \( S \); see Exercise 1 (e).

**Example 3.2.** Find the (Gauss) curvature \( K \) for the unit sphere \( S^2 \) by means of the parametrization given in Example 1.4 in §4.1.

**Solution:** From that example we have \( \theta^1 = \cos \phi \, d\theta, \ \theta^2 = d\phi, \ \omega^2_1 = \sin \phi \, d\theta \). So

\[
d \omega^1_2 = d(-\omega^2_1) = -d(\sin \phi) \wedge d\theta = -\cos \phi \, d\phi \wedge d\theta = \cos \phi \, d\theta \wedge d\phi
\]

and \( \theta^1 \wedge \theta^2 = \cos \phi \, d\theta \wedge d\phi \). Thus \( d \omega^1_2 = \theta^1 \wedge \theta^2 \), telling us that the curvature of \( S^2 \) is a positive constant: \( K = 1 \).

In the present section we mainly study the intrinsic geometry of a surface \( S \), which only depends on the metric tensor of \( S \), but not on the way how \( S \) is embedded in \( \mathbb{R}^3 \). As a consequence, the Gauss curvature here is defined intrinsically. In the next section we study the extrinsic geometry of \( S \), and from that point of view one can define the total curvature for \( S \). Gauss discovered that the total curvature (an extrinsic concept) coincides with what we call the Gauss curvature (an intrinsic concept). He was so pleased with this discovery that he called it theorem aggregium, which means a beautiful theorem. As you will see, using Cartan’s structural equations, the proof is only one line long!
Exercises

1. (a) Check Rules PC1, PC2, PN1 and PN2.

(b) Show that, if $X$ and $Y$ are vector fields tangent to a surface $S$, then

$$d(X \cdot Y) = d\tau X \cdot Y + X \cdot d\tau Y.$$ 

Deduce that, if $F$ is a unit vector field tangent to $S$, then $F \cdot d\tau F = 0$.

(c) Show that the Gauss curvature $K$ is independent of the orientation of $S$. (Hint: there are two simple ways to indicate the change of orientation: first, change the order of $E_1$ and $E_2$; second, replace one of $E_1$ or $E_2$ by its negative. Either way is easy to work with.)

2. In each of the following parts, a parametric surface $r \equiv (x, y, z) = r(u, v)$ and a Darboux frame $E_k$ ($k = 1, 2, 3$) are given. Find (in terms of parameters $u$ and $v$):

1. the fundamental forms $\theta^1$, $\theta^2$ and connection forms $\omega^{12}$, $\omega^{13}$, $\omega^{23}$;
2. the metric $ds^2 = (\theta^1)^2 + (\theta^2)^2$, the area form $\theta^1 \wedge \theta^2$;
3. the covariant differentials $d\tau E_1$, $d\tau E_2$ the normal differentials $d\perp E_1$, $d\perp E_2$; and
4. the Gaussian curvature $K$.

(a) Let $X = X(t)$, $Y = Y(t)$ be a curve in $XY$-plane with an arc-length parametrization: $X'(t)^2 + Y'(t)^2 = 1$. The cylinder $S$ based on this curve is given by $x(u, v) = X(u)$, $y(u, v) = Y(u)$, $z(u, v) = v$. The Darboux frame for $S$ is given by

$$E_1 = \frac{\partial r}{\partial u} = (dX/du, dY/du, 0), \quad E_2 = \frac{\partial r}{\partial v} = (0, 0, 1), \quad E_3 = E_1 \times E_2.$$ 

(b) The unit sphere $S^2$: $r = (\cos u \cos v, \sin u \cos v, \sin v)$ with the Darboux frame $E_1 = (\sin u, \cos u, 0)$, $E_2 = (\cos u \sin v, -\sin u \sin v, \cos v)$, $E_3 = E_1 \times E_2 = r$.

(c) The circular cone with the circle $x^2 + y^2 = 1$ as the “base curve” and the point $A = (0, 0, 1)$ as the apex, together with the Darboux frame given as follows:

$$r(u, v) = ((1 - v) \cos u, (1 - v) \sin u, v), \quad E_1 = \frac{\partial r}{\partial u}, \text{ normalzed} = (- \sin u, \cos u, 0),$$

$$E_2 = \frac{\partial r}{\partial v}, \text{ normalzed} = \frac{1}{\sqrt 2}(- \cos u, - \sin u, 1), \quad E_3 = E_1 \times E_2.$$ 

(d) The unit sphere $S^2$, parametrized by stereographic projection:

$$r(u, v) = (2uh, 2vh, (u^2 + v^2 - 1)h), \quad \text{where} \quad h = 1/(u^2 + v^2 + 1)$$
with the Darboux frame is obtained by normalizing $r_u, r_v, r_u \times r_v$.

(e) The parametric surface $r(u,v) = (u^2/2, uv/\sqrt{2}, v^2/2)$ (see Exercise 6 in §2.8) with the Darboux frame obtained by normalizing $r_u, (r_u \times r_v) \times r_u, r_u \times r_v$. (Patience is needed for completing the computation.)

3. Decomposition (3.9) uses the usual inner product $u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$ for two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{R}^3$. For certain problems (e.g. in theory of relativity) it is more appropriate to use other types of inner products. For example, in deriving the hyperbolic metric (2.8) in §4.2, we consider the hyperboloid $H^2: -x^2 - y^2 + z^2 = 1$ in $\mathbb{R}^3$ with the inner product of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ defined to be $u \cdot v = u_1v_1 + u_2v_2 - u_3v_3$ corresponding to the indefinite metric $dx^2 + dy^2 - dz^2$ chosen. Parametrize the lower sheet of $H^2$ by $r(u,v) = (\cos u \sinh v, \sin u \sinh v, -\cosh v)$. Let $E_k$ be vector fields determined by $r_u = \sinh u \ E_1$, $r_v = E_2$ and $r = E_3$.

(a) Verify that the vector fields $E_1, E_2, E_3$ form a Darboux frame by checking: $E_1 \cdot E_1 = 1, E_2 \cdot E_2 = 1, E_3 \cdot E_3 = -1$ and $E_1 \cdot E_2 = E_1 \cdot E_3 = E_2 \cdot E_3 = 0$,

(b) Compute $dr, dE_k$ to find $\theta^1, \theta^2$ and $\omega^k_j$ ($1 \leq k, j \leq 3$).

(c) Find the metric $(\theta^1)^2 + (\theta^2)^2$, the area form $\theta^1 \wedge \theta^2$, as well as the curvature $K$.

(d) Find the covariant differentials $d_\tau E_1, d_\tau E_2$, normal differentials $d_\perp E_1, d_\perp E_2$, as well as the second covariant differentials $d^2_\tau E_1$ and $d^2_\tau E_2$. 

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