The differential of a vector field in a Euclidean space is straightforward to define. If $F$ is a vector field in $\mathbb{R}^n$ with components $F_1, F_2, \ldots, F_n$, that is, $F = (F_1, F_2, \ldots, F_n)$, then $dF$ is just the vector-valued differential form with components $dF_1, dF_2, \ldots, dF_n$, that is, $dF = (dF_1, dF_2, \ldots, dF_n)$.

For example, writing $r = (x, y, z)$, we have $dr = (dx, dy, dz)$. Another example: let $F = (x^2 - y^2, 2xy)$. The components of this vector field are $F_1 = x^2 - y^2$ and $F_2 = 2xy$ with $dF_1 = 2xdx - 2ydy$ and $dF_2 = 2xdy + 2ydx$ and hence $dF = 2(xdx - ydy, xdy + ydx)$, or $dF = 2(xdx, ydx) + 2(-ydy, xdy) = G_1 dx + G_2 dy$, where the “coefficients” $G_1$ and $G_2$ of $dF$ are themselves vector fields given by $G_1 = 2(x, y)$ and $G_2 = 2(-y, x)$.

In general, a vector-valued differential in the Euclidean space $\mathbb{R}^n$ can be written as $\sum_{k=1}^n V_k dx_k$, where $V_k$ are vector fields. Another description is that a vector-valued differential is an expression like a vector with differential forms as its components. In actual computation of differentials of vector fields, we often use various versions of the product rule, depending on what kind of products we are dealing with. In what follows, $F$ and $G$ are vector fields and $f$ is a scalar field in $\mathbb{R}^n$:

**Rule VD1** (Product Rule). $d(fG) = df \cdot G + fdG$, $d(F \cdot G) = dF \cdot G + F \cdot dG$, and when $n = 3$, $d(F \times G) = dF \times G + F \times dG$.

($F \cdot G$ is the dot product and $F \times G$ is the cross product of $F$ and $G$.) Often a vector field or a vector-valued differential spreads over a smaller set called a submanifold, such as a curve or a surface, so that it depends on a smaller number of parameters, as shown in the following example.

**Example 1.1.** Find the differentials $dT$ and $dN$ of the unit tangent field $T$ and the unit normal field $N$ respectively along the parabola $y = x^2/2$.

**Solution:** Put the parabola $y = x^2/2$ into parametric equations as $x = t$, $y = t^2/2$. Then the velocity vector is given by $v = (dx/dt, dy/dt) = (1, t)$. The unit tangent field $T$, which is a field of unit vectors tangent to the parabola, can be obtained by normalizing $v$:

$$T = |v|^{-1}v = (1 + t^2)^{-1/2}(1, t) \equiv (1 + x^2)^{-1/2}(1, x).$$
To find the normal direction at each point on the parabola, we treat the parabola as the level curve \( f = 0 \) of the scalar field \( f(x, y) = y - x^2/2 \) and compute the gradient of \( f \):
\[
\nabla f = (\partial f/\partial x, \partial f/\partial y) = (-x, 1).
\]
Normalizing \( \nabla f \), we get the unit normal field
\[
\mathbf{N} = |\nabla f|^{-1} \nabla f = (1 + x^2)^{-1/2}(-x, 1).
\]
Alternatively, we can get \( \mathbf{N} \) by turning \( \mathbf{T} \) 90°: turning a vector \( \mathbf{v} = (a, b) \) in 90° anticlockwise, we get \( \mathbf{v}^\perp = (-b, a) \). By the product rule, we have
\[
d\mathbf{T} = d((1 + x^2)^{-1/2}). (1, x) + (1 + x^2)^{-1/2} d(1, x)
\]
\[
= (-x(1 + x^2)^{-3/2}). (1, x) dx + (1 + x^2)^{-1/2} (0, 1) dx
\]
\[
= (1 + x^2)^{-3/2} \{ -x(1, x) + (1 + x^2)(0, 1) \} dx = (1 + x^2)^{-3/2} (-x, 1) dx.
\]
Similarly, we have \( d\mathbf{N} = (1 + x^2)^{-3/2} (-1, -x) dx \). Putting \( \omega = (1 + x^2)^{-1} dx \), we can rewrite the answer as \( d\mathbf{T} = \omega \mathbf{N}, \ d\mathbf{N} = -\omega \mathbf{T} \).

In the above example, the pair of vector fields \( \mathbf{T}, \mathbf{N} \) form an orthonormal frame field, or a moving frame, along the parabola. By an orthonormal frame in the Euclidean space \( \mathbb{R}^n \) we mean an orthonormal basis in this linear space, which consists of \( n \) unit vectors orthogonal to each other, say \( \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \), with \( \mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk} \) (the Kronecker delta, which is 1 when \( j = k \) and is 0 when \( j \neq k \)), that is, \( \mathbf{e}_j \cdot \mathbf{e}_j = 1 \) and \( \mathbf{e}_j \cdot \mathbf{e}_k = 0 \) for \( j \neq k \), \( 1 \leq j, k \leq n \). By a moving frame along a manifold \( M \) (such as a curve, or a surface, or a hypersurface) in \( \mathbb{R}^n \) we mean at each point \( p \) of the manifold \( M \), a frame
\[
\mathbf{E}_1(p), \mathbf{E}_2(p), \ldots, \mathbf{E}_n(p)
\]
attaching to \( p \) is assigned. Thus each \( \mathbf{E}_k \) \( (1 \leq k \leq n) \) individually is a vector field along \( M \), and together they form an orthonormal basis at each point of \( M \). The elegant method of moving frames is credited to Elie Cartan for studying differential geometry, although Darboux used them before him.

Suppose that a moving frame \( \mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_n \) is given. Then, for each \( k \) \( (1 \leq k \leq n) \), the differential \( d\mathbf{E}_k \) can be written as a linear combination of \( \mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_n \) with some differential forms as coefficients, say
\[
d\mathbf{E}_k = \sum_{j=1}^{n} \omega^j_k \mathbf{E}_j; \tag{1.1}
\]
The differential forms \( \omega^j_k \) are called the connection forms. We can write out \( \omega^j_k \) explicitly by using the orthonormality of the frame \( \{ \mathbf{E}_k \}_{k=1}^{n} \), that is, \( \mathbf{E}_j \cdot \mathbf{E}_k = \delta_{jk} \). Indeed,
\[
d\mathbf{E}_k \cdot \mathbf{E}_j = \sum_{\ell=1}^{n} \omega^\ell_k \mathbf{E}_\ell \cdot \mathbf{E}_j = \sum_{\ell=1}^{n} \omega^\ell_k \delta_{\ell j} = \omega^j_k.
\]
and hence we have $\omega_j^k = dE_k \cdot E_j$. Applying “$d$” to both sides of the orthonormal relation $E_j \cdot E_k = \delta_{jk}$ and using the product rule, we have $dE_j \cdot E_k + E_j \cdot dE_k = 0$, that is, $\omega_j^k + \omega_k^j = 0$. In particular, when $j = k$, we have $\omega_k^k + \omega_k^k = 0$ and hence $\omega_k^k = 0$. Thus we conclude

$$\omega_k^k = 0 \quad \text{and} \quad \omega_k^j = -\omega_j^k \quad \text{for all} \quad j, k \quad \text{with} \quad 1 \leq j, k \leq n. \quad (1.2)$$

In Example 1.1, we may take $E_1 = T$ and $E_2 = N$. Relation (1.2) reduces four connection forms $\omega_j^k$ ($1 \leq j, k \leq 2$) to one. In that example, (1.1) becomes $dE_1 = \omega_1^2 E_2 = \omega E_2$ and $dE_2 = \omega_1^1 E_1 = -\omega E_1$, where $\omega = (1 + x^2)^{-1}dx$.

As usual, we denote the Cartesian coordinate functions by $x_1, x_2, \ldots, x_n$. (For any point $p = (p_1, p_2, \ldots, p_n)$ in $\mathbb{R}^n$, $x_k(p)$ is the $k$th component of $p$, namely, $x_k(p) = p_k$. So $x_k$ is indeed a function defined on $\mathbb{R}^n$.) Put $x = (x_1, x_2, \ldots, x_n)$, which is considered as a vector field. (When $n = 2$ or $3$, we often use the letter $r$ instead. Thus $r = (x, y)$ or $r = (x, y, z)$.) The vector-valued differential form $dx = (dx_1, dx_2, \ldots, dx_n)$ is a linear combination of $E_1, \ldots, E_n$ with some differential forms as coefficients, say

$$dx = \sum_{k=1}^n \theta^k E_k. \quad (1.3)$$

As before, we can write out $\theta^k$ explicitly as: $\theta^k = dx \cdot E_k$. (In the next chapter we shall see that the “metric tensor” of a submanifold is given by $\sum_{k=1}^n (\theta^k)^2$.)

**Example 1.2.** Find the covector fields $\theta^1$ and $\theta^2$ associated with the frame $E_1 = T$ and $E_2 = N$ in Example 1.1.

**Solution:** We have $r = (x, y)$ and $dr = (dx, dy) = (dx, d(x^2/2)) = (dx, xdx)$. Since $E_1 = T = (1 + x^2)^{-1/2}(1, x)$ and $E_2 = N = (1 + x^2)^{-1/2}(-x, 1)$,

$$\theta^1 = dr \cdot E_1 = (1 + x^2)^{-1/2}\{(dx, xdx) \cdot (1, x)\} = (1 + x^2)^{1/2}dx,$$

and similarly $\theta^2 = (1 + x^2)^{-1/2}\{(dx, xdx) \cdot (-x, 1)\} = 0$. (As we shall see, $\theta^2 = 0$ is expectable because $E_1$ and $E_2$ form a so-called Darboux frame for the parabola.)

A moving frame $E_1, E_2, \ldots, E_n$ along a submanifold $M$, such as a curve or a surface, is called a **Darboux frame** for $M$ if each $E_k$ is either tangent to $M$ or normal to $M$. The moving frame $T, N$ in Example 1.1 is a Darboux frame along the parabola $y = x^2/2$.

For a general curve in $\mathbb{R}^2$ described by parametric equations $x = x(t), y = y(t)$, we can get a Darboux frame as follows. First compute the velocity $v \equiv (dx/dt, dy/dt)$. Let

$$v = |v| \equiv \sqrt{(dx/dt)^2 + (dy/dt)^2} \quad \text{and} \quad ds = v \, dt; \quad (1.4)$$

3
Now equations (1.1) become
\[ d \mathbf{v} \]
Let us assume that \( f \ dt \) depends only on one variable, namely the parameter \( t \). This way we obtain a Darboux frame along the curve consisting of \( \mathbf{E}_1 = \mathbf{T} \) and \( \mathbf{E}_2 = \mathbf{N} \). Now equations (1.1) become \( d \mathbf{T} = \omega \mathbf{N} \) and \( d \mathbf{N} = -\omega \mathbf{T} \). Here the connection form \( \omega \) depends only on one variable, namely the parameter \( t \). Hence it can be written in the form \( f \ dt \), where \( f \) is a function with a single variable \( t \). Recall the relation \( ds = v dt \) from (1.4).

Let us assume that \( v \neq 0 \) so that we may rewrite this relation as \( dt = v^{-1} ds \). Thus we have \( \omega = f \ dt = \kappa ds \), where \( \kappa = v^{-1} f \). We call \( \kappa = \kappa(t) \) the curvature of the parametric curve at point \((x(t), y(t))\). In Example 1.1, we have \( \mathbf{v} = (1, t) \) and \( \omega = (1 + t^2)^{-1} dt \) (notice that \( x = t \)). Hence \( v = |\mathbf{v}| = (1 + t^2)^{1/2} \), which gives \( ds = v dt = (1 + t^2)^{1/2} dt \). Consequently \( \omega = (1 + t^2)^{-3/2}(1 + t^2)^{1/2} dt = (1 + t^2)^{-3/2} ds \). Therefore the curvature of the parabola in that example is given by \( \kappa(t) = (1 + t^2)^{-3/2} \) at the point \((t, t^2/2)\). For a general formula for computing curvature of a planar parametric curve, see Exercise 1 of the present section. The identity \( d \mathbf{T} = (\kappa ds) \mathbf{N} \) tells us that the curvature \( \kappa \) is negative if the curve bends in the opposite direction of \( \mathbf{N} \).

Next, consider a spatial curve in parametric equations \( x = x(t), y = y(t), z = z(t) \). We construct a Darboux frame along this curve as follows. First, we take the velocity vector \( \mathbf{v} \equiv (dx/dt, dy/dt, dz/dt) \) and normalize it to get a unit tangent field \( \mathbf{T} = |\mathbf{v}|^{-1} \mathbf{v} \). Then take the derivative \( \mathbf{T}' \equiv d\mathbf{T}/dt \) and normalize it to obtain a unit vector field

\[ \mathbf{N} = |\mathbf{T}'|^{-1} \mathbf{T}' , \]
called the principal normal of the curve; (we tacitly assume that \( \mathbf{v} \neq 0 \) and \( \mathbf{T}' \neq 0 \)). Differentiating both sides of \( \mathbf{T} \cdot \mathbf{T} = |\mathbf{T}|^2 = 1 \) and applying the product rule, we obtain \( 2 \mathbf{T} \cdot \mathbf{T}' = 0 \). Thus \( \mathbf{T} \cdot \mathbf{N} = 0 \), telling us that \( \mathbf{N} \perp \mathbf{T} \). Finally, let

\[ \mathbf{B} = \mathbf{T} \times \mathbf{N} , \]
which is called the binormal to the curve. The vector fields \( \mathbf{E}_1 = \mathbf{T}, \mathbf{E}_2 = \mathbf{N} \) and \( \mathbf{E}_3 = \mathbf{B} \) along the curve form a Darboux frame, called the Frenet-Serrat frame. From \( \mathbf{N} = |\mathbf{T}'|^{-1} \mathbf{T}' \) we have

\[ d\mathbf{E}_1/dt = \mathbf{T}' = |\mathbf{T}'|\mathbf{N} = |\mathbf{T}'|\mathbf{E}_2 . \]

So \( \omega_1^2 = |\mathbf{T}'| dt, \omega_1^3 = 0 \), and \( \omega_3^1 = -\omega_1^3 = 0 \). Thus, the nine connection forms \( \omega^k_j \) are
reduced to two, namely $\omega_1^2$ and $\omega_3^2$. Putting $\omega = \omega_1^2$ and $\eta = \omega_3^2$, (1.1) gives:

\[
\begin{align*}
   d\mathbf{T} &= \omega \mathbf{N}, \\
   d\mathbf{N} &= -\omega \mathbf{T} + \eta \mathbf{B}, \\
   d\mathbf{B} &= -\eta \mathbf{N},
\end{align*}
\]

which are called Frenet-Serrat’s equations. Again, using the relation $ds = vdt$, where $v = |v| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$, the connection forms $\omega$ and $\eta$ can be written as $\omega = \kappa ds$ and $\eta = \tau ds$. The functions $\kappa = \kappa(t)$ and $\tau = \tau(t)$ are called the curvature and the torsion respectively of the curve at $(x(t), y(t), z(t))$. Recall that $\omega = \omega_1^2 = |\mathbf{T}'| dt$

Thus we have $\kappa v dt = \kappa ds = \omega = |\mathbf{T}'| dt$ and hence $\kappa = v^{-1}|\mathbf{T}'| \geq 0$. Thus, unlike the planar curve, the curvature of a spatial curve does not take negative values.

**Example 1.3.** Find the Frenet-Serrat frame and the associated connection forms, as well as the curvature $\kappa$ and the torsion $\tau$, for the helix described by the parametric equations $x = \cos t$, $y = \sin t$ and $z = t$.

**Solution:** A direct computation shows $\mathbf{v} = (dx/dt, dy/dt, dz/dt) = (-\sin t, \cos t, 1)$, giving $v \equiv |\mathbf{v}| = ((-\sin t)^2 + (\cos t)^2 + 1)^{1/2} = \sqrt{2}$ and $\mathbf{T} = \mathbf{v}/|\mathbf{v}| = (-\sin t, \cos t, 1)/\sqrt{2}$. Next, $\mathbf{T}' = (-\cos t, -\sin t, 0)/\sqrt{2}$, which gives $\mathbf{N} = |\mathbf{T}'|^{-1}\mathbf{T}' = (-\cos t, -\sin t, 0)$. The binormal $\mathbf{B}$ can be obtained by forming $\mathbf{T} \times \mathbf{N}$:

\[
\mathbf{B} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 & -\sin t \\ -\sin t & 0 & -\cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = \frac{1}{\sqrt{2}}(\sin t, -\cos t, 1).
\]

So we have $d\mathbf{T} = d(-\sin t, \cos t, 1)/\sqrt{2} = (-\cos t, -\sin t, 0)dt/\sqrt{2} = \mathbf{N} dt/\sqrt{2}$ and similarly, $d\mathbf{B} = (\cos t, \sin t, 0) dt/\sqrt{2} = -\mathbf{N} dt/\sqrt{2}$. Thus, according to the notation in (9.5), the required connection forms are $\omega = dt/\sqrt{2}$ and $\eta = dt/\sqrt{2}$. On the other hand, since $v = \sqrt{2}$, we have $\omega = \kappa ds = \kappa. \sqrt{2} dt$ and consequently $\kappa = 1/2$. Similarly, $\tau = 1/2$.

A Darboux frame for a surface $S$ in $\mathbb{R}^3$ has two tangential fields, say $\mathbf{T}_1$ and $\mathbf{T}_2$, and one normal field, say $\mathbf{N}$. When $S$ is given by parametric equations $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, we can obtain a Darboux frame $\mathbf{T}_1, \mathbf{T}_2, \mathbf{N}$ for $S$ as follows. Write

\[
\mathbf{r} = \mathbf{r}(u, v) \equiv (x(u, v), y(u, v), z(u, v)).
\]

Compute the tangent fields

\[
r_u \equiv \partial\mathbf{r}/\partial u \quad \text{and} \quad r_v \equiv \partial\mathbf{r}/\partial v,
\]
which are tangent to “u”-curves and “v”-curves in $S$ respectively. Let $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$, which is normal to $S$. Orthonormalize $\mathbf{r}_u$, $\mathbf{r}_v$ to obtain unit tangent fields $T_1$ and $T_2$ and normalize $\mathbf{n}$ to obtain a unit normal field $N (= |\mathbf{n}|^{-1} \mathbf{n})$.

Example 1.4. Consider the unit sphere $S^2$ parametrized by the latitude $\phi$ and longitude $\theta (-\pi < \theta < \pi, -\pi/2 < \phi < \pi/2)$: $x = \cos \theta \cos \phi$, $y = \sin \theta \cos \phi$, $z = \sin \phi$. Find a Darboux frame and the associated connection forms $\omega^j_k$ and forms $\theta^i$.

Solution: Let us write $\mathbf{r} = (x, y, z) = (\cos \theta \cos \phi, \sin \theta \cos \phi, \sin \phi)$. Then we have $t_1 = r_\theta = (-\sin \theta \cos \phi, \cos \theta \cos \phi, 0)$, $t_2 = r_\phi = (-\cos \theta \sin \phi, -\sin \theta \sin \phi, \cos \phi)$ and $n = t_1 \times t_2 = (\cos \theta \cos^2 \phi, \sin \theta \cos \phi, \cos \phi \sin \phi) = \cos \phi \mathbf{r}$. It is easy to check $t_1 \cdot t_2 = 0$. Normalizing $t_1$, $t_2$ and $n$, we get a Darboux frame:

$$T_1 = t_1/|t_1| = (-\sin \theta, \cos \theta, 0), \quad T_2 = t_2/|t_2| = (-\cos \theta \sin \phi, -\sin \theta \sin \phi, \cos \phi),$$

and $N = n/|n| = r = (\cos \theta \cos \phi, \sin \theta \cos \phi, \sin \phi)$. (Note that $|t_2| = 1$ and $|t_1| = |\cos \phi| = \cos \phi$ because $-\pi/2 < \phi < \pi/2$.) Comparing $d\mathbf{r} = \theta^1 T_1 + \theta^2 T_2$ with

$$d\mathbf{r} = r_\theta d\theta + r_\phi d\phi = t_1 d\theta + t_2 d\phi = T_1. |t_1| d\theta + T_2. |t_2| d\phi = T_1. \cos \phi d\theta + T_2. d\phi$$

we have

$$\theta^1 = \cos \phi \, d\theta, \quad \theta^2 = \cos \phi \, d\phi. \quad (1.6)$$

(This will give us the metric tensor $(\theta^1)^2 + (\theta^2)^2 = \cos^2 \phi \, d\theta^2 + d\phi^2$ and the area form $\theta^1 \wedge \theta^2 = \cos \phi \, d\theta \wedge d\phi$.) Also,

$$dT_1 = (-\cos \theta, -\sin \theta, 0) \, d\theta$$
$$dT_2 = (\sin \theta \sin \phi, -\cos \theta \sin \phi, 0) \, d\theta - (\cos \theta \cos \phi, \sin \theta \cos \phi, \sin \phi) \, d\phi$$
$$dN = (-\sin \theta \cos \phi, \cos \theta \cos \phi, 0) \, d\theta + (-\cos \theta \sin \phi, -\sin \theta \cos \phi, \cos \phi) \, d\phi.$$

The connection forms $\omega^2_1$, $\omega^3_1$ and $\omega^3_2$ in

$$dT_1 = \omega^2_1 T_2 + \omega^3_1 N, \quad dT_2 = -\omega^2_1 T_1 + \omega^3_2 N, \quad dN = -\omega^3_1 T_1 - \omega^3_2 T_2$$

can be computed as follows:

$$\omega^2_1 = dT_1 \cdot T_2 = (\cos^2 \theta \sin \phi + \sin^2 \theta \sin \phi) \, d\theta = \sin \phi \, d\theta$$
$$\omega^3_1 = dT_1 \cdot N = (-\cos^2 \theta \cos \phi - \sin^2 \theta \cos \phi) \, d\theta = -\cos \phi \, d\theta$$
$$\omega^3_2 = dT_2 \cdot N = (\sin \theta \sin \phi \cos \theta \cos \phi - \cos \theta \sin \phi \sin \theta \cos \phi) \, d\theta +$$
$$(-\cos^2 \theta \cos^2 \phi - \sin^2 \theta \cos^2 \phi - \sin^2 \phi) \, d\phi = -d\phi. \quad (1.7)$$
As you can see, even for a simple example in differential geometry, the amount of work could be quite formidable. Naturally we need to find ideas for “seeing things” (such as reduction by symmetries) in order to avoid heavy and tedious computation. In fact, many general formulas in differential geometry are good for theoretical purposes rather than practical calculation!

Exercises

1. Let $\gamma$ be a planar curve given by parametric equations $x = x(t)$ and $y = y(t)$. Assume $v = \sqrt{x'(t)^2 + y'(t)^2} \neq 0$ for all $t$. Verify that $T = \frac{1}{v}(x', y')$ and $N = \frac{1}{v}(-y', x')$ form a Darboux frame along $\gamma$. Verify $dT = \omega N$ and $dN = -\omega T$, where

$$\omega = \frac{x'y'' - y'x''}{v^2} dt.$$  \hspace{1cm} (1.8)

Thus we obtain the formula $\kappa(t) = (x'y'' - y'x'')/v^3$ for computing curvature. Also, employ (1.8) to find the connection 1-form $\omega$ for the parabola $y = \frac{1}{2}x^2$ by using the parametrization $x = 2t, y = 2t^2$. Show that the curvature at the point $(x, x^2/2)$ is $\kappa = (1 + x^2)^{-3/2}$.

2. In each of the following cases, find the Darboux frame, $T$, $N$, its associated connection form $\omega$ and the curvature function $\kappa$ for the given planar parametric curves.

(a) Archimedian spiral: $x = e^{at} \cos t, y = e^{at} \sin t$; ($a$ is any constant).

(b) catenary: $x = t, y = \cosh t$.

(c) cubic curve $x = t^2, y = t - \frac{1}{3}t^3$.

Partial answers: (a) $\kappa = 1/As$, where $s = \int_0^t v$; (b) $\kappa = 1/\cosh^2 t$.

3. Given a planar curve $\gamma$ with parametric equations $x = x(t), y = y(t)$. The involute of $\gamma$ is defined to be the parametric curve $\gamma_e : X = X(t), Y = Y(t)$ given by $(X(t), Y(t)) = (x(t), y(t)) + \kappa(t)^{-1}N$, where $N$ and $\kappa$ are respectively the unit normal vector field and the curvature of $\gamma$. (a) Show that the tangents of $\gamma_e$ are normal to $\gamma$.

(b) Find the evolute of the catenary $x(t) = t, y(t) = \cosh t$.

Note: We call $\kappa(t)^{-1}$ the radius of curvature. The circle with radius $\kappa(t)^{-1}$ centered at $(x(t), y(t)) + \kappa(t)^{-1}N$ is “kissing” the curve $\gamma$ at the point $P \equiv (x(t), y(t))$. It is called the osculating circle of $\gamma$ at $P$. So the involute of $\gamma$ is nothing but the trajectory of the center of the osculating circle.
4. In this exercise we describe a way to construct a planar curve with a given curvature function \( \kappa(t) \), \( a \leq t \leq b \). Fix any \( c \) between \( a \) and \( b \). Let \( \phi(t) = \int_c^t \kappa(u) \, du \). Show that the parametric curve \( x = x(t), y = y(t) \) \( (a < t < b) \) given by

\[
x(t) = \int_c^t \cos \phi(t) \, dt, \quad y(t) = \int_c^t \sin \phi(t) \, dt
\]

has the prescribed curvature \( \kappa(t) \).

5. Consider a spatial parametric curve \( \mathbf{r}(t) = (x(t), y(t), z(t)) \). As usual, we write \( \mathbf{r}'(t) = (x'(t), y'(t), z'(t)) \) and \( \mathbf{r}''(t) = (x''(t), y''(t), z''(t)) \). Verify that Frenet-Serrat’s equations (1.5) hold with \( \omega = \kappa \cdot v dt \) and \( \eta = \tau \cdot v dt \) \( (v = |\mathbf{r}'|) \), where

\[
\kappa = \kappa(t) = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}, \quad \text{(curvature)} \quad \tau = \tau(t) = -\frac{[\mathbf{r}, \mathbf{r}', \mathbf{r}'']}{|\mathbf{r}' \times \mathbf{r}''|^2}, \quad \text{(torsion)} \quad (1.9)
\]

\([\mathbf{a}, \mathbf{b}, \mathbf{c}] \) designates the triple product \( (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \).

6. (a) Use (1.9) to compute the curvature \( \kappa \) and the torsion \( \tau \) for the helix \( x(t) = \cos t, y(t) = \sin t \) and \( z(t) = t \). Verify that your answer agrees with the one in Example 1.3 above. (b) Find Frenet-Serrat frame, the curvature and the torsion for the twisted cubic given by the parametric equations \( x(t) = t, y(t) = t^2, y = \frac{2}{3}t^3 \).

7. Let \( \gamma \) be a parametric curve \( \gamma \) in the \( XY \)-plane given by \( X = X(t) \) and \( Y = Y(t) \). Assume that \( X'(t)^2 + Y'(t)^2 = 1 \); that is, \( t \) is an arc length parameter. Consider the cylinder given by \( x(u, v) = X(u), y(u, v) = Y(u), z(u, v) = v \). Show that

\[
\mathbf{T}_1(u, v) = (X'(u), Y'(u), 0), \quad \mathbf{T}_2 = (0, 0, 1), \quad \mathbf{N}(u, v) = (-Y'(u), X'(u), 0)
\]

form a Darboux frame for the cylinder. Verify \( d\mathbf{T}_2 = 0, d\mathbf{T}_1 = \eta \mathbf{N} \) and \( d\mathbf{N} = -\eta \mathbf{T}_1 \), where \( \eta = (X'(u)Y''(u) - Y'(u)X''(u)) \, du \).

8. Let \( Y = Y(t), Z = Z(t) \) be a parametric curve in the \( yz \)-plane with \( Y(t) > 0 \) for all \( t \). Let \( \mathbf{r}(u, v) = (Y(u) \cos v, Y(u) \sin v, Z(u)) \) be the surface of rotation it generates. Compute \( \mathbf{r}_u, \mathbf{r}_v \) and \( \mathbf{r}_u \times \mathbf{r}_v \) and verify that, after normalization, we obtain a Darboux frame

\[
\mathbf{T}_1 = h(u) \, (Y'(u) \cos v, Y'(u) \sin v, Z'(u)), \quad \mathbf{T}_2 = (-\sin v, \cos v, 0), \quad \mathbf{N} = h(u) \, (-Z'(u) \cos v, -Z'(u) \sin v, Y'(u)),
\]

where \( h(u) = (Y'(u))^2 + Z'(u)^2)^{-1/2} \). Find the connection forms \( \omega^j_i \) and the fundamental forms \( \theta^j_i \) associated with this Darboux frame in terms of the functions \( Y(u), Z(u) \) and their derivatives.