§3. Scalar Fields, Vector Fields and Covector Fields

First we study scalar fields. By a scalar field here we mean a scalar-valued function \( y = f(x) \) with \( x = (x_1, x_2, \ldots, x_n) \) as a point varying in the space \( \mathbb{R}^n \). Imagine yourself traveling in a jeep in a mountainous country with \( f \) as its height function: \( f(x) \) is the altitude at the location \( x \). (Of course \( n = 2 \) in this situation. But let us still write \( n \) instead of 2 for generality.) At time \( t \), your jeep is located at \( x(t) \). As \( t \) changes, \( x(t) \) traces out a curve in the terrain, and you feel the height \( f(x(t)) \) changing as you travel. The rate of change of height you are experiencing is the derivative \( \frac{df}{dt} \). If this derivative is large, you are moving up fast, due to the steep ascend of the road, or the high speed of your jeep, or both. According to the chain rule,

\[
\frac{df}{dt} = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \frac{dx_k}{dt} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dt} + \cdots + \frac{\partial f}{\partial x_n} \frac{dx_n}{dt} .
\] (3.1)

(The last expression reminds you of the meaning of \( \sum_{k=1}^{n} \).) Your velocity is

\[
\mathbf{v} = (v_1, v_2, \ldots, v_n) = \frac{d\mathbf{x}}{dt} = \left( \frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt} \right) .
\] (3.2)

We may rewrite (3.1) succinctly as

\[
\frac{df}{dt} = \nabla f \cdot \mathbf{v},
\]
where \( \nabla f \) is a vector function given by

\[
\nabla f \equiv \text{grad} f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right),
\]
called the gradient of \( f \), and \( \nabla f \cdot \mathbf{v} \) is the dot product of \( \nabla f \) and \( \mathbf{v} \). Recall that the dot product or the inner product of two vectors \( \mathbf{u} = (u_1, u_2, \ldots, u_n) \) and \( \mathbf{v} = (v_1, v_2, \ldots, v_n) \) is defined to be \( \mathbf{u} \cdot \mathbf{v} = \sum_{k=1}^{n} u_k v_k \left( = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \right) \). Also recall that two vectors are perpendicular to each other if their dot product is zero. Notice that \( \nabla f \) is a function of \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \) belonging to the space \( \mathbb{R}^n \), and its values are vectors in this space. Such a function describes a vector field. Thus, by taking the gradient we convert a scalar field to a vector field. If a vector field \( \mathbf{F} \) can be written in the form \( \nabla f \) for some scalar field \( f \), then we call \( \mathbf{F} \) a gradient field, or a conservative field.

The identity \( df/dt = \nabla f \cdot \mathbf{v} \) is about a function in variable \( t \). Thus \( \nabla f \) should be interpreted as \( \nabla f \) evaluated at \( x(t) \) in order to be considered as a function of \( t \). Similarly,
the partial derivatives $\partial f/\partial x_k$ in (3.1) should also be evaluated at $x(t)$. The value of $\nabla f$ at $x(t)$ will be denoted by $\nabla f|_{x(t)}$ instead of $(\nabla f)(x(t))$ for clarity. If we want to be precise, $df/dt = \nabla f \cdot v$ should be written as $f'(t) = \nabla f|_{x(t)} \cdot x'(t)$. But we often sacrifice precision for the sake of convenience and transparency. There is a slightly different way to get this identity. Recall Rule DF5 in §2:

$$df = \sum_{k=1}^{n} (\partial f/\partial x_k) dx_k = \nabla f \cdot dx,$$

where $dx = (dx_1, dx_2, \ldots, dx_n)$. On the other hand, (3.2) gives $dx = v dt$. Thus $df = \nabla f \cdot dx = \nabla f \cdot v dt$, which is essentially the same as $df/dt = \nabla f \cdot v$. More generally, given a vector field $F = (F_1, F_2, \ldots, F_n)$, we consider its associated 1-form

$$\omega_F = F \cdot dx = F_1 dx_1 + F_2 dx_2 + \cdots + F_n dx_n.$$  \hspace{1cm} (3.3)

(Some authors denote $\omega_F$ by $F^\flat$.) Clearly, $F$ is conservative if and only if $\omega_F$ is exact. Using the 1-form $\omega_F$ given in (6.3), we are allowed to consider line integrals of the form $\int_{\gamma} F \cdot dx \equiv \int_{\gamma} \omega_F$, the same kind of line integrals in advanced calculus.

Let $f = f(x)$ be a scalar field. Fix a constant $c$ and consider the level surface (or the level line in case $n = 2$) $L_c : f = c$. More precisely, $L_c$ consists of all those $x$ satisfying $f(x) = c$. Suppose that you are traveling (in a jeep) along $L_c$ and $x(t)$ is your location at time $t$. Then your altitude $f$ remains a constant: $f(x(t)) = c$. Since you feel no change in $f$, $df/dt = 0$. According to the identity $df/dt = \nabla f \cdot v$, we have $\nabla f \cdot v = 0$, that is, your velocity $v$ is perpendicular to the gradient $\nabla f$ all the time, no matter how you are traveling in $L_c$. Now the following assertion is clear: at a point $x$ of the level surface $L_c : f = c$, the gradient $\nabla f|_x$ is perpendicular to $L_c$. This assertion is very handy for finding tangents or normals to curves or surfaces, as illustrated in the following example.

**Example 3.1.** Find the tangent plane and the normal line at the point $p = (1,1,2)$ of the paraboloid $S: z = x^2 + y^2$.

**Solution:** Introduce the scalar field $f$ defined by $f(x,y,z) = x^2 + y^2 - z$. Then $S$ is just the level surface $L_0 : f = 0$. The gradient of $f$ is $\nabla f = (2x, 2y, -1)$, which is $(2,2,-1)$ at $p = (1,1,2)$, giving us the normal direction of the surface at $p$. Thus the required tangent plane is $2(x-1) + 2(y-1) - (z-2) = 0$, or $2x + 2y - z = 2$, and the required normal line in parametric equations is $x = 2(t-1)$, $y = 2(t-1)$, $z = -(t-2)$.

Now we return to $\nabla f \cdot v$. This expression makes sense if $\nabla f$ is evaluated at some point $x$ and $v$ is any vector attached to $x$. It is called the directional derivative of $f$
in the direction of \( \mathbf{v} \) and will be denoted by \( \nabla_{\mathbf{v}} f \), which can be regarded as the rate of change of \( f \) according to a traveler traveling with a constant velocity \( \mathbf{v} \) at the moment he or she goes through \( \mathbf{x} \):

\[
\nabla_{\mathbf{v}} f = \nabla f \cdot \mathbf{v} = \frac{d}{dt} f(\mathbf{x} + tv) \bigg|_{t=0} \equiv \lim_{h \to 0} \frac{f(\mathbf{x} + hv) - f(\mathbf{x})}{h}.
\]

As expected, directional derivatives behave like usual derivatives:

\[
\nabla_{\mathbf{v}} (af + bg) = a\nabla_{\mathbf{v}} f + b\nabla_{\mathbf{v}} g, \quad \nabla_{\mathbf{v}} (fg) = f\nabla_{\mathbf{v}} g + g\nabla_{\mathbf{v}} f,
\]

that is, the linearity and the product rule still hold. We remark that if \( f \) is a scalar field defined on a curve or a surface \( S \) and if \( \mathbf{v} \) is a vector tangential to \( S \) at a point \( \mathbf{a} \) on \( S \), the directional derivative \( \nabla_{\mathbf{v}} f \) still makes sense even though \( \nabla f \) does not. Indeed, we may define \( \nabla_{\mathbf{v}} f \) to be the derivative \((d/dt)f(x(t))\) evaluated at \( t = 0 \) for any curve \( x = x(t) \) lying in \( S \) such that \( x(0) = \mathbf{a} \) and \( x'(0) = \mathbf{v} \).

Scalar fields and vector fields are special cases of tensor fields. Another special case is called **covector fields**. What are covector fields? Well, we see them all the time! All differential 1–forms are covector fields and vice versa. “Covector” is a common word in physics and engineering. But in mathematics it is usually called “linear functional”, a term arising from the notion of duality in linear algebra. We do not intend to give a precise definition of “dual space”. Instead we briefly describe some idea behind this notion.

Basically, a vector \( \mathbf{v} \) and a covector \( \alpha \) can be paired up to produce a number, which will be denoted either by \( \langle \mathbf{v}, \alpha \rangle \) or \( \langle \alpha, \mathbf{v} \rangle \). Take an example in economics to give you some idea: consider a local economy involving \( n \) commodities. A batch of commodities is represented by a “commodity vector” \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \), where \( x_j \) is the quantity (say in tons) of \( j \)th commodity in this batch. Setting prices of these commodities gives us a “price vector” \( \mathbf{p} = (p^1, p^2, \ldots, p^n) \). The pairing \( \langle \mathbf{p}, \mathbf{x} \rangle = \sum_{k=1}^{n} p^k x_k \) gives the total cost of the batch \( \mathbf{x} \). Notice that \( \langle \mathbf{p}, \mathbf{x} \rangle \) is NOT the dot product of \( \mathbf{p} \) and \( \mathbf{x} \), because \( \mathbf{p} \) and \( \mathbf{x} \) are coming from different spaces. They are from different spaces because you cannot add them: \( \mathbf{x} + \mathbf{p} \) does not make sense! Normally we call \( \mathbf{x} \) a vector and \( \mathbf{p} \) a covector.

A vector paired with a covector gives us a scalar. Analogously a vector field paired with a covector field should produce a scalar field. Suppose that \( \mathbf{F} \) is a vector field and \( \omega = g_1 dh_1 + g_2 dh_2 + \cdots + g_r dh_r \) is a covector field. They pair off to give rise to a scalar field denoted by \( \langle \mathbf{F}, \omega \rangle \) according to the following

**Rule VF1.** \( \langle \mathbf{F}, \omega \rangle \equiv \langle \omega, \mathbf{F} \rangle = \sum_{k=1}^{r} g_k \langle \mathbf{F}, dh_k \rangle = \sum_{k=1}^{r} g_k \nabla_{dh_k} \mathbf{F} \).
Again, this rule is straightforward to apply. A consequence of this rule is

\[ \langle F, g_1 \omega_1 + g_2 \omega_2 \rangle = g_1 \langle F, \omega_2 \rangle + g_2 \langle F, \omega_2 \rangle, \quad \langle g_1 F_1 + g_2 F_2, \omega \rangle = g_1 \langle F_1, \omega \rangle + g_2 \langle F_2, \omega \rangle. \]

Another one is: \( \langle df, v \rangle \equiv \langle v, df \rangle = v \cdot \nabla f = \nabla_v f. \) We remind the reader that, to make sense out of the expression \( \nabla_v f \) at a point \( p \), all we need is that \( f \) is defined on a path \( \gamma: x = x(t) \) through a point \( p \) with velocity \( v \) at \( p \), say \( x(a) = p \) and \( x'(a) = v \); in that case, we have \( \nabla_v f = (d/dt)f(x(t)) \bigg|_{t=a}. \)

**Example 3.2.** Find \( \langle F, \omega \rangle \), where \( F = (x, y) \) and \( \omega \) is the angular form in (1.4).

**Solution:** We have \( \nabla_F x = F \cdot \nabla x = (x, y) \cdot (1, 0) = x \) and similarly \( \nabla_F y = y \). So

\[ \langle F, \omega \rangle = \langle F, (x^2 + y^2)^{-1}(xdy - ydx) \rangle = (x^2 + y^2)^{-1}(x \nabla_F y - y \nabla_F x) = 0. \]

This is not surprising. Locally \( \omega = d\theta \) so that \( \langle F, \omega \rangle = \nabla_F \theta \). But each vector in the field \( F \) is pointing in the radial direction and the angle \( \theta \) does not change when a point moves in the radial direction. Hence we should have \( \nabla_F \theta = 0. \)

**Example 3.3.** Let \( X_j \) \( (j = 1, 2, 3) \) be the constant vector fields given by \( X_1 = (1, 0, 0), X_2 = (0, 1, 0) \) and \( X_3 = (0, 0, 1) \). Then we have

\[ \langle X_j, dx_k \rangle = \delta_{jk} \]

where \( \delta_{jk} \) is the Kronecker delta defined by

\[ \delta_{jk} = \begin{cases} 1 & \text{if } j = k; \\ 0 & \text{if } j \neq k. \end{cases} \]

**Example 3.4.** Consider a surface \( S \) defined by the parametric equation

\[ r = r(u, v) = (x(u, v), y(u, v), z(u, v)) \]

where \( (u, v) \) varies in a region \( D \). Let \( p = r(a, b) \) be a point on \( S \), with \( (a, b) \) in \( D \). A vector \( v \) tangent to \( S \) at \( p \) in general can be written as \( v = U r_u + V r_v \), where

\[ r_u = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) \quad \text{and} \quad r_v = \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) \]

are evaluated at \( (a, b) \). Find \( \langle du, v \rangle \) \( (= \langle v, du \rangle) \) and \( \langle dv, v \rangle \).
Solution: Since, by definition, \( \langle v, du \rangle = \nabla_v u \), we must understand the way how \( u \) is regarded as a function on \( S \) before we can do anything! Take a point \( q \) on \( S \). What is the value of \( u \) at \( q \)? Well, since \( q \) is on \( S \), there must be some \((c, d)\) in \( D \) such that \( r(c, d) = q \). The value of \( u \) at \( q = r(c, d) \) should be \( c \). Thus, regarding \( u \) and \( v \) as functions on \( S \), we should have \( r(u(q), v(q)) = q \) for all \( q \) on \( S \). When we put these two functions into a pair as \((u, v)\), then we have a map from \( S \) back to \( D \), which is the inverse map of the one given by the parametric equation \( r = r(u, v) \). To find \( \nabla_v u \), we look for a path \( \gamma \) in \( S \) passing through the point \( p = r(a, b) \) with velocity \( v \) at \( p \). \( \nabla_v u \) is nothing but the rate of change of \( u \) along this path at \( p \). The path \( \gamma \) given by the parametric equation \( x = x(t) \equiv r(u(t), v(t)) \) with \( u(t) = a + Ut \), \( v(t) = b + Vt \) will do the job. Indeed, we have \( x(0) = r(a, b) = p \) and, by the chain rule,

\[
\frac{dx}{dt} = \frac{\partial r}{\partial u} du + \frac{\partial r}{\partial v} dv = ruU + rvV = Uru + Vrv,
\]

which gives \( v \) when it is evaluate at \( t = 0 \). Thus \( u(x(t)) = a + Ut \) gives

\[
\langle du, v \rangle = \frac{d}{dt} \bigg|_{t=0} u(x(t)) = \frac{d}{dt} \bigg|_{t=0} (a + Ut) = U.
\]

Similarly, we have \( \langle dv, v \rangle = V \).

This kind of “path fitting” argument in the above example is important because it helps us to clarify many murky situations in calculus on manifolds, which often involves extremely heavy and complicated notation. The result of the this example is simple but important. It will be needed in Chapter 4. Let us remember it as a rule:

**Rule VF2.** For a vector field \( F = Uru + Vrv \) tangent to a surface given by the parametric equation \( r = r(u, v) \), we have \( \langle du, F \rangle = U \) and \( \langle dv, F \rangle = V \), that is, the identity \( F = \langle F, du \rangle ru + \langle F, dv \rangle rv \) holds for any vector field tangent to \( S \).

In the remaining part of the section, we consider the index of a planar vector field at a singular point. Consider a vector field \( F = (P, Q) \) in the \( xy \)-plane with components \( P \) and \( Q \). A point \( p = (a, b) \) in the domain of \( F \) is called a singular point for \( F \) if \( F(p) = 0 \). Think of \( F \) as a force field. Then a singular point is where you do not feel any force from the field, like the center of a tornado. For a singular point \( p = (a, b) \), we surround it by a tiny circle \( \gamma \) of radius \( r \) with \( x(t) = a + r \cos t \), \( y = b + r \sin t \) \((0 \leq t \leq 2\pi)\) as its parametric equations so that \( p = (a, b) \) is the only singular point inside it. Consider the loop \( \Gamma \equiv F \circ \gamma \) in the \( uv \)-plane given by

\[
u(t) = P(x(t), y(t)), \quad v(t) = Q(x(t), y(t)) \quad (0 \leq t \leq 2\pi).
\]
Notice that the origin of the uv-plane is not on the curve \( \Gamma \). (Otherwise we would have \( P(x(t),y(t)) = 0 \) and \( Q(x(t),y(t)) = 0 \) for some \( t = t_0 \). That is, \( \mathbf{F} = (P,Q) \) would vanish at \( (x(t_0),y(t_0)) \), which is a point on the circle \( \gamma \), contradicting our assumption.) The winding number of \( \Gamma \) about the origin is called the **index** of the vector field at the singularity \( \mathbf{p} = (a,b) \) and will be denoted by \( \text{Ind}(\mathbf{F}; \mathbf{p}) \):

\[
\text{Ind}(\mathbf{F}; \mathbf{p}) = W(\Gamma, \mathbf{0}) = \frac{1}{2\pi} \int_{\Gamma} \frac{udv - vdu}{u^2 + v^2}.
\]

It turns out that \( \text{Ind}(\mathbf{F}, \mathbf{p}) \) is always an integer independent of the radius \( r \) of the small circular \( \gamma \).

**Example 3.5.** Find the index of the vector field \( \mathbf{F} = (-y, x) \) at the origin.

**Solution:** Here we have \( P = -y \) and \( Q = x \). The loop \( \gamma \) is given by \( x = r \cos t \), \( y = r \sin t \). So \( \Gamma \) is given by \( u = -y = -r \sin t \) and \( v = Q = x = r \cos t \). Hence

\[
\text{Ind}(\mathbf{F}, \mathbf{0}) = W(\Gamma, \mathbf{0}) = \frac{1}{2\pi} \int_{0}^{2\pi} (u^2 + v^2)^{-1}(uv' - vu') dt = \frac{1}{2\pi} \int_{0}^{2\pi} r^{-2}((-r \sin t)(-r \sin t) - (r \cos t)(-r \cos t)) dt = 1.
\]

The reader is urged to sketch a simple picture of this vector field and to observe that if one moves around the origin once in the anticlockwise direction \( \mathbf{F} \) turns 360°.

**Example 3.6.** Find the index of \( \mathbf{F} = (x^2 - y^2, 2xy) \) about the origin.

**Solution:** Follow the same routine as the last one: loop \( \Gamma \) is given by \( u = x^2 - y^2 = (r \cos t)^2 - (r \sin t)^2 = r^2 \cos 2t \) and \( v = 2xy = 2(r \cos t)(r \sin t) = r^2 \sin 2t \). So

\[
\text{Ind}(\mathbf{F}, \mathbf{0}) = W(\Gamma, \mathbf{0}) = \frac{1}{2\pi} \int_{0}^{2\pi} (u^2 + v^2)^{-1}(uv' - vu') dt = \frac{1}{2\pi} \int_{0}^{2\pi} r^{-4}((r^2 \cos 2t)(2r^2 \cos 2t) - (r^2 \sin 2t)(-2r^2 \sin t)) dt = 2.
\]

Notice that \( x^2 - y^2 \) and \( 2xy \) are the real and the imaginary parts of the complex function \( f(z) = z^2 \), (again, \( z = x + iy \)). This suggests that there is a connection between the indices of vector fields \( \mathbf{F} = (u, v) \) and line integrals \( (2\pi i)^{-1} \int_{\gamma} \mathbf{f} \ast dz/z \equiv (2\pi i)^{-1} \int_{\gamma} \mathbf{f}'/f dz \) for complex functions \( f = u + iv \) studied in Chapter 3 in future. For detail will be given in Exercise 11 of §3.3.
Let \( f \) be a scalar field and let \( \mathbf{F} = -\nabla f \). If \( \mathbf{p} \) is a critical point of \( f \), then \( \mathbf{F} \equiv \nabla f \) vanishes at \( \mathbf{p} \). We define the index of the critical point \( \mathbf{p} \) for \( f \) to be the index of the vector field \( \mathbf{F} \) at \( \mathbf{p} \). In §3.3 we will prove an “index theorem” which has some interesting topological consequences.

**Exercises**

1. Given the scalar field \( f(x, y, z) = x^2 + 2y^2 - z^2 + 2xy - 2xz + yz \), the point \( \mathbf{p} = (1, 1, 2) \) and the vector \((2, -3, 1)\) attached to \( \mathbf{p} \), find the gradient \( \nabla f \) at \( \mathbf{p} \), the directional derivative \( \nabla_v f \) at \( \mathbf{p} \) and the tangent plane at \( \mathbf{p} \) of the level surface through \( \mathbf{p} \).

2. In each of the following cases, find the index \( \text{Ind}(\mathbf{F}, \mathbf{0}) \) of the given planar vector field at the singularity \( \mathbf{0} \):
   - (a) \( \mathbf{F} = (x, -y) \),
   - (b) \( \mathbf{F} = (y, x) \),
   - (c) \( \mathbf{F} = (x^2 + y^2, 0) \),
   - (d) \( \mathbf{F} = (x^2 - y^2, -2xy) \),
   - (e) \( \mathbf{F} = (x + y, y) \). (Hint: Consider \( \mathbf{F}_\lambda = (x + \lambda y, y) ; 0 \leq \lambda \leq 1. \))

3. Let \( f \) be a scalar field and \( \mathbf{a} \) be a point in the field with \( \nabla f(\mathbf{a}) \neq 0 \). Consider the directional derivatives \( \nabla_v f \) for all unit vectors \( \mathbf{v} \) attached to \( \mathbf{a} \). Show that \( \nabla_v f \) attains the maximum at \( \mathbf{v} = |\nabla f|^{-1} \nabla f \). (Here \( |\mathbf{u}| \) stands for the length of a vector \( \mathbf{u} \).) **Hint**: Cauchy-Schwarz inequality.

4. Verify the identities in (3.5).

5. (a) What can you say about \( \nabla f \) if \( f \) attains a maximum or a minimum at a point?
   - (b) What can you say about \( f \) when \( \nabla f \) is identically zero?

6. (a) If \( \mathbf{F} = (P, Q) \) is a vector field in the \( xy \)-plane, then \( \langle dx, \mathbf{F} \rangle = P \) and \( \langle dy, \mathbf{F} \rangle = Q \). (b) Are the identities \( \nabla_{\mathbf{F}}(fg) = f\nabla_{\mathbf{F}}g + g\nabla_{\mathbf{F}}f \) and \( \langle \mathbf{F}, f\omega \rangle = f\langle \mathbf{F}, \omega \rangle \) contradicting each other?

7. Show that in each of the following cases the given vector field \( \mathbf{F} \) is conservative by finding explicitly a scalar field \( f \) such that \( \nabla f = \mathbf{F} \):
   - (a) \( \mathbf{F} = (x^2 - y^2, -2xy) \),
   - (b) \( \mathbf{F} = (e^x \sin y, e^x \cos y) \),
   - (c) \( \mathbf{F} = (x^2 + 2xy, x^2 + y^3) \)
   - (d) \( \mathbf{F} = (2xyz, x^2z, x^2y) \).

8. Show that each curve from the family \( f(x, y) = c \) is orthogonal to each curve from the family \( g(x, y) = c \) at their point of intersection, if \( *dg = df \). (Recall that \( *(Pdx + Qdy) = Pdy - Qdx \); see (2.1) in §2.)
9. Verify that the tangent plane Π to the hyperboloid $H: x^2 + y^2 - z^2 = 1$ at a point $p = (a, b, c)$ is given by $ax + by - cz = 1$. Show that the intersection $H \cap \Pi$ of $H$ and $\Pi$ is a pair of lines, each of which can be written in parametric form

$$x(t) = a + ut, \quad y(t) = b + vt, \quad z(t) = c + wt,$$

where $q \equiv (u, v, w)$ is lying on the cone $C: x^2 + y^2 - z^2 = 0$ and is orthogonal to $p = (a, b, c)$ in the hyperbolic sense: $(p, q) \equiv au + bv - cw = 0$. Sketch a picture including both $H$ and $C$, showing points $p$ and $q$, lines of intersections in both $H \cap \Pi$ and $C \cap \Pi_0$, where $\Pi_0$ is the plane through the origin parallel to $\Pi$.

10. Consider a point mass $P$ moving in a conservative force field $F = -\nabla V$ with its trajectory $x = x(t)$ parametrized by the time variable $t$. The velocity and the acceleration of $P$ at time $t$ are $v = x'(t)$ and $a = x''(t)$ respectively. Newton’s Law tells us that $F = ma$, where $m$ is the mass of $P$. Verify that

$$E \equiv \frac{1}{2} m |v|^2 + V$$

is constant throughout the motion of $P$. (Hint: differentiate $\frac{1}{2} m x'(t) \cdot x'(t) + V(x(t))$ with respect to $t$. Use the chain rule.) Use this physical idea to describe a method for reducing the following nonlinear second order differential equation to a first order equation: $y'' = F(y)$. (Hint: In 1-dimension all vector fields are conservative!)

11. Describe a way to compute the gradient $\nabla f$, where $f$ is a planar scalar field expressed in polar coordinates: $f = f(r, \theta)$.

12. (a) Verify that if $p$ is a singular point of a planar vector field $F$ and if $f$ is nonzero everywhere, then $\text{Ind}(fF, p) = \text{Ind}(F, p)$. In particular, $\text{Ind}(-F, p) = \text{Ind}(F, p)$; that is, the index remains the same when the field $F$ is reversed. (b) The reflection of a vector field $F = (P, Q)$ in the vertical direction is defined to be $F^* = (P, -Q)$. Verify that $\text{Ind}(F^*, p) = -\text{Ind}(F, p)$. 

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