Chapter 12. Complex Differential Forms

In the present chapter we introduce a powerful method of differential forms to handle many calculus problems. We will state some rules of thumb, explained briefly but not proved with mathematical rigor, and a few examples to illustrate the ease of working with them. Let us begin with a complex function \( f(z) \) with variable \( z = x + iy \). We may consider it as a function of two real variables \( x \) and \( y \) so that the partial derivatives \( \partial f/\partial x \) and \( \partial f/\partial y \). The differential \( df \) of \( f \) is related to these partial derivatives as follows:

\[
df = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy.
\]

But normally we do use this identity to compute \( df \); instead, we use the following rules (the product rule, the quotient rule and the chain rule:

\[d(\frac{uv}{v^2}) = \frac{vdu - u dv}{v^2},\]

\[d f = \frac{df}{du} \, du \text{ for } f = f(u)\]

**Example 12.1.** To compute \( d|z| \), we write \( v = |z| \). Then \( v^2 = |z|^2 = x^2 + y^2 \). So \( d(v^2) = 2x \, dx + 2y \, dy \). On the other hand \( d(v^2) = 2vdv \). Thus \( 2vdv = 2x \, dx + 2y \, dy \), giving us \( dv = (x \, dx + y \, dy)/v \), or

\[d|z| = \frac{x \, dx + y \, dy}{|z|}.
\]

As usual, we write \( r = |z| \). Then

\[
\frac{dr}{r} = \frac{x \, dx + y \, dy}{|z|^2} = \frac{x \, dx + y \, dy}{x^2 + y^2}.
\]

(12.2)

If we start with the polar form \( z = re^{i\theta} \), then, using the product rule, we have

\[dz = d(re^{i\theta}) = e^{i\theta} \, dr + re^{i\theta} \, d\theta = e^{i\theta} \, dr + i \, re^{i\theta} \, d\theta = re^{i\theta} \left( \frac{dr}{r} + id\theta \right).
\]

Consequently we have

\[
\frac{dz}{z} = \frac{dr}{r} + i d\theta.
\]

(12.3)

This identity should be read with care. First, we require \( z \neq 0 \); otherwise the left hand side does not make sense. In this case we have \( r > 0 \) and we can rewrite \( dr/r \) as \( d\ln r \). However, writing \( dz/z \) as \( d\ln z \) is problematic! Second, there is some ambiguity about the
way to define $\theta$. If we do this carefully, then we have to talk about “branches of $\theta$”. To avoid the potential trouble caused by $\theta$ in $d\theta$, in the future we often write $\omega$ instead of $d\theta$.

**Example 12.2.** We can use $z = x + iy$ to put $dz/z$ in terms of $x$ and $y$:

$$
\frac{dz}{z} = \frac{\overline{z} dz}{zz} = \frac{(x - iy)d(x + iy)}{x^2 + y^2} = \frac{xdx + ydy}{x^2 + y^2} + i \frac{xdy - ydx}{x^2 + y^2} \tag{12.4}
$$

Comparing the imaginary parts of the last expression with (12.3), we obtain

$$
\omega = \frac{xdy - ydx}{x^2 + y^2}
$$

which will be called the **angular form**. As we have mentioned above, the expression $d\theta$ on the left hand side is problematic. But the right hand side is fine, as long as $x + iy \neq 0$. Comparing the real parts of (12.4) and (12.3), we have $dr/r = (xdx + ydy)/(x^2 + y^2)$. But this is not new. It is just (12.2) above.

**Exercise 12.1.** Check that the angular form is homogenous in the sense that, if $f = f(x, y)$ is any no-where vanishing function, and if $u = fx$ and $v = fy$, then $(udv - vdu)/(u^2 + v^2) = (xdy - ydx)/(x^2 + y^2)$.

The complex conjugate of $z = x + iy$ is $\overline{z} = x - iy$. Now we have two pairs of variables: $\{x, y\}$ and $\{z, \overline{z}\}$, which are related by

$$
z = x + iy, \quad \overline{z} = x - iy, \quad \text{and} \quad x = \frac{z + \overline{z}}{2}, \quad y = \frac{z - \overline{z}}{2i}.
$$

These identities can be formally carried over to differentials:

$$
dz = dx + idy, \quad d\overline{z} = dx - idy, \quad dx = \frac{dz + d\overline{z}}{2}, \quad dy = \frac{dz - d\overline{z}}{2i}. \tag{12.5}
$$

As we know, the real variables $x$ and $y$ are independent of each other and hence so are their differentials. But the complex variables $z$ and $\overline{z}$ are related. They cannot be regarded as independent variables. However, their differentials $dz$ and $d\overline{z}$ are *linearly independent* in the sense of the following:

**Fact** If $g dz + h d\overline{z} = 0$, then $g = 0$ and $h = 0$.

Here $g$ and $h$ are arbitrary (complex-valued) functions in $z$. We explain why this is true. Indeed, substituting $dz = dx + idy$ and $d\overline{z} = dx - idy$ in $g dz + hd\overline{z} = 0$, we have
\[ g(dx + idy) + h(dx - idy) = 0, \] which is equivalent to \( (g + h)dx + (ig - ih)dy = 0. \) Since \( dx \) and \( dy \) are independent, we have \( g + h = 0 \) and \( ig - ih = 0 \) (by Rule DF6 in §1.2), from which it follows that \( g = 0 \) and \( h = 0. \)

In (12.1), the differential \( df \) of \( f \) is expressed in terms of \( dx \) and \( dy \). Using (12.5), we can express \( df \) in terms of \( dz \) and \( \overline{dz} \). Indeed,

\[
\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} d\overline{z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) dz + \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) d\overline{z}. \tag{12.6}
\]

Now we define \( \frac{\partial f}{\partial z} \) and \( \frac{\partial f}{\partial \overline{z}} \) in such a way that the identity

\[
df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} d\overline{z} \tag{12.7}
\]

holds. Comparing (12.7) with the previous identity (12.6), naturally we define:

\[
\frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right). \tag{12.8}
\]

The second identity gives the definition of the so-called \( \overline{\mathcal{D}} \) - operator \( \partial / \partial \overline{z} \); (here \( \mathcal{D} \) is pronounced as “dee bar”). We interpret these operators as rates of change. Consider the change \( \Delta f = f(z + \Delta z) - f(z) \) in \( f \) caused by an increment \( \Delta z \equiv \Delta x + i \Delta y \) in \( z \equiv x + iy \).

Under what condition does the limit \( \lim_{|\Delta z| \to 0} \Delta f / \Delta z \) exist, which gives the complex derivative \( f'(z) \) of \( f \) at \( z \)? To answer this question, let us begin with identity (12.1). This identity can be viewed as a “infinitesimal version” of the following more precise one:

\[
\Delta f = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + o \left\{ \sqrt{(\Delta x)^2 + (\Delta y)^2} \right\}. \tag{The little “o” notation \( o\{ \ldots \} \) will be explained soon.}
\]

Substitute \( \Delta x \) by \( \frac{1}{2} (\Delta z + \overline{\Delta z}) \), and \( \Delta y \) by \( \frac{1}{2i} (\Delta z - \overline{\Delta z}) \). Replace \( \sqrt{(\Delta x)^2 + (\Delta y)^2} \) by \( |\Delta z| \). Following a computation similar to (1.3), it is easy to get

\[
\Delta f = \frac{\partial f}{\partial z} \Delta z + \frac{\partial f}{\partial \overline{z}} \overline{\Delta z} + o(|\Delta z|). \tag{12.11}
\]

Here, \( o(|\Delta z|) \) is a quantity less than \( \epsilon |\Delta z| \) if \( |\Delta z| \) is small enough, where \( \epsilon \) is an arbitrarily small positive number given beforehand. Dividing both sides by \( \Delta z \), we see that the difference quotient \( \Delta f / \Delta z \) has a limit as \( \Delta z \) tends to zero, provided that \( f \) satisfies the following so-called Cauchy-Riemann equation:

\[
\frac{\partial f}{\partial \overline{z}} = 0,
\]
and, in that case, the complex derivative $f'(z)$ is given by $\partial f/\partial z$. Notice that, when $\Delta z \to 0$, the quotient $\Delta z/\Delta z$ has a limits $+1$ if we let $\Delta z$ move along the real axis and a different limit $-1$ if $\Delta z$ moves along the imaginary axis. Hence, if the Cauchy-Riemann equation fails at a certain point, then the difference quotient $\Delta f/\Delta z$ does not have a limit as $\Delta z$ tends to zero. We have shown,

**(CR-equation).** A (smooth) function $f$ defined in $C$ is complex differentiable if and only if it satisfies the Cauchy-Riemann equation $\partial f/\partial \bar{z} = 0$, and in that case the derivative of $f$ is given by $f' = \partial f/\partial z$.

A (smooth) function $f$ defined on an open set in $C$ is called an **analytic function** or a **holomorphic function** if it satisfies the Cauchy-Riemann equation, or, equivalently, it has a complex derivative at each point in its domain.

The operator $\Delta \equiv 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}$ is the **Laplacian**. Indeed, an easy computation shows

$$\Delta = \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) = \left( \frac{\partial}{\partial x} \right)^2 + \left( \frac{\partial}{\partial y} \right)^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The usual definition of the Laplacian is given by the last expression. Thus

$$\Delta = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \quad (1.6)$$

We call a (smooth) function $f$ defined on an open set in $C$ a **harmonic function** if it satisfies the so-called Laplace equation $\Delta f = 0$. (We apologize for using the symbol $\Delta f$ to denote two completely different notions: increment in $f$ and the Laplacian of $f$. The context will make the meaning of this symbol clear whenever it appears.) Analytic functions are harmonic since a function satisfying the Cauchy-Riemann equation clearly satisfies the Laplace equation. The Laplacian, unlike the $\partial$-operator $\partial/\partial \bar{z}$, is a “real” operator: if $f$ is a real-valued function, then its Laplacian $\Delta f$ is also real-valued. Thus, the real part (or the imaginary part) of a harmonic function is also harmonic. But the real part of an analytic function in general is not analytic. In fact, a real-valued analytic function must be a constant.

**Example 12.3.** To prove the last statement, let $f$ be a real-valued analytic function. Then $2 \frac{\partial f}{\partial \bar{z}} \equiv \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0$. As $f$ is real, $\partial f/\partial x$ and $\partial f/\partial y$ are also real and hence they must vanish. Thus we have $df = (\partial f/\partial x)dx + (\partial f/\partial y)dy = 0$ and hence $f$ is a constant.
The operators $\partial/\partial z$ and $\partial/\partial \bar{z}$ behave like usual partial derivatives, if we consider a function of $x, y$ as a function of $z, \bar{z}$ instead. The following example shows how they work:

**Example 12.4.** Show that the so-called Poisson kernel $P(z, e^{it})$, given below, which is regarded as a function of $z$ for $|z| < 1$, is a harmonic function; (the Poisson kernel is used for describing solutions to Dirichlet’s problem over the unit disk).

$$P(z, e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2}; \quad t \in \mathbb{R}, \quad |z| < 1.$$  \hspace{1cm} (12.9)

(Here $e^{it} = \cos t + i \sin t$.)

**Solution:** First we apply the $\overline{\partial}$-operator to $P(z, e^{it})$. For simplicity, write $w$ for $e^{it}$.

$$\frac{\partial}{\partial \bar{z}} P(z, w) = \frac{\partial}{\partial \bar{z}} \frac{1 - |z|^2}{|w - z|^2} = \frac{\partial}{\partial \bar{z}} \left( \frac{1}{(w - z)(\bar{w} - \bar{z})} \right) = \frac{z\bar{z}}{(w - z)(\bar{w} - \bar{z})}$$

$$= \frac{1}{w - z} \frac{\partial}{\partial \bar{z}} \frac{1}{\bar{w} - \bar{z}} - \frac{z}{w - z} \frac{\partial}{\partial \bar{z}} \frac{\bar{z}}{\bar{w} - \bar{z}}$$

$$= \frac{1}{w - z} \frac{1 - z\bar{w}}{(w - z)(\bar{w} - \bar{z})} = \frac{(w - z)\bar{w}}{(w - z)(\bar{w} - \bar{z})^2} = \frac{w}{(1 - \bar{z}w)^2}.$$  

Since the last expression does not involve $z$, we have $(\partial/\partial z) w/(1 - \bar{z}w)^2 = 0$. Hence

$$\Delta_z P(z, e^{it}) = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} P(z, w) = 4 \frac{\partial}{\partial z} \frac{w}{(1 - \bar{z}w)^2} = 0.$$  

(Here $\Delta_z$ means that the Laplacian is taken with respect to $z$.) Therefore the Poisson kernel $P(z, e^{it})$, as a function of $z$ varying within $|z| < 1$, is a harmonic function.

**Exercise 12.2.** Verify that, when $z = re^{i\theta}$, the Poisson kernel given in (12.9) above is essentially the same as the one given in Chapter 7, by checking the following identity:

$$\frac{1 - |z|^2}{|e^{it} - z|^2} = \frac{1 - r^2}{1 - 2r \cos(t - \theta) + r^2},$$

with $z = re^{i\theta}$.

More precisely, all differential forms we have studied so far should be called 1–forms. Next we study the so called 2–forms. One can get a 2–form by taking the so–called wedge
product or the exterior product $\alpha \wedge \beta$ of 1-forms $\alpha$ and $\beta$. Since every 1-form in two variables $x$ and $y$ can be expressed in the form $f \, dx + g \, dy$, the following rule

$$dx \wedge dx = 0, \quad dy \wedge dy = 0, \quad dy \wedge dx = -dx \wedge dy$$

tells us how to compute $\alpha \wedge \beta$ in practice. For generally, given 1–forms $\alpha$ and $\beta$, we have

$$\alpha \wedge \alpha = 0, \quad \alpha \wedge \beta = -\beta \wedge \alpha. \quad (12.10)$$

**Example 12.5.** Verify the following identities:

$$dz \wedge dz = 0, \quad d\bar{z} \wedge d\bar{z} = 0, \quad \text{and} \quad d\bar{z} \wedge dz = -dz \wedge d\bar{z} = 2i \, dx \wedge dy. \quad (12.11)$$

**Solution.** Here we go:

$$dz \wedge dz = (dx + idy) \wedge (dx + idy)$$

$$= dx \wedge dx + i \, dy \wedge dx + i \, dx \wedge dy - dy \wedge dy$$

$$= 0 + i(-dx \wedge dy) + i \, dx \wedge dy - 0 = 0,$$

$$d\bar{z} \wedge dz = (dx - idy) \wedge (dx + idy)$$

$$= dx \wedge dx - i \, dy \wedge dx + i \, dx \wedge dy + dy \wedge dy$$

$$= 0 - i(-dx \wedge dy) + i \, dx \wedge dy + 0 = 2i \, dx \wedge dy.$$

In the same way we can show that $d\bar{z} \wedge d\bar{z} = 0$ and $-dz \wedge d\bar{z} = 2i \, dx \wedge dy$.

**Example 12.6.** Given 1-forms $\omega_1 = f_1 dz + g_1 d\bar{z}$ and $\omega_2 = f_2 dz + g_2 d\bar{z}$, we wish to find $\omega_1 \wedge \omega_2$. Now

$$\omega_1 \wedge \omega_2 = (f_1 dz + g_1 d\bar{z}) \wedge (f_2 dz + g_2 d\bar{z}) = f_1 g_2 \, dz \wedge d\bar{z} + g_1 f_2 \, d\bar{z} \wedge dz$$

$$= (f_1 g_2 - f_1 h_2) \, dz \wedge d\bar{z} \equiv \begin{vmatrix} f_1 & g_1 \\ f_2 & g_2 \end{vmatrix} \, dz \wedge d\bar{z}.$$

We may replace $dz \wedge d\bar{z}$ by $2i \, dx \wedge dy$ if we wish. This example shows that a 2-form in variables $x \, y$ can be expressed as $f \, dx \wedge dy$ for some function $f$.

Besides taking wedge products, there is another way to obtain 2-forms: taking the differential $d\omega$ of a 1–form, by following the rule:

$$d(f \, dg) = df \wedge dg \quad (12.12)$$
In other words, if \( \omega = f \, dg \), then \( d\omega = df \wedge dg \). A particular case of this rule is

\[
d(df) = 0, \quad \text{or} \quad d^2 = 0. \tag{12.13}
\]

The reason is that we can regard \( df \) as \( 1 \, df \) and hence \( d(df) = d(1 \, df) = d1 \wedge df = 0 \), in view of the fact that the differential of a constant function is always zero.

**Example 12.7.** Verify that 1-form \( \omega = zd\bar{z} + \bar{z}dz \) is closed, that is, \( d\omega = 0 \).

**Solution:** \( d\omega = dz \wedge d\bar{z} + d\bar{z} \wedge dz = dz \wedge d\bar{z} - dz \wedge d\bar{z} = 0 \). Alternatively, we can proceed as follows: \( d(zd\bar{z} + \bar{z}dz) = d(d(z\bar{z})) = 0 \), in view of \( d^2 = 0 \).

We say that a 1–form \( \omega \) is **closed** if \( d\omega = 0 \), and a 1–form \( \omega \) is **exact** if \( \omega \) can be written as \( df \) for some function \( f \). Identity (12.13) tells us: exact forms are closed. We will learn that closed 1–forms are “locally exact” but not necessarily (globally) exact. (This kind of situations is not just a result of mathematically investigation – it actually occurs in nature, e.g. the Aharonov–Bohm effect.)

**Example 12.8.** Recall the angular form

\[
\omega = \frac{xdy - ydx}{x^2 + y^2},
\]

which is defined for all \((x, y) \neq (0, 0)\). We rewrite it as \( \omega = f \wedge dx + gdy \), where \( f = -y/(x^2 + y^2) \) and \( g = x/(x^2 + y^2) \). Then \( d\omega = df \wedge dx + dg \wedge dy \). A brute force computation by using the quotient rule shows

\[
df = \frac{2xydx + (x^2 - y^2)dy}{(x^2 + y^2)^2}, \quad \text{and} \quad dg = -\frac{2xydy + (y^2 - x^2)dx}{(x^2 + y^2)^2}.
\]

Thus we have

\[
d\omega = \frac{2xy}{(x^2 + y^2)^2}dx \wedge dx + \frac{y^2 - x^2}{(x^2 + y^2)^2}dx \wedge dy + \frac{y^2 - x^2}{(x^2 + y^2)^2}dy \wedge dx + \frac{2xy}{(x^2 + y^2)^2}dy \wedge dy = 0,
\]

By using \( dx \wedge dx = 0 \), \( dy \wedge dy = 0 \), and \( dy \wedge dx = -dx \wedge dy \). Alternatively, from example 12.2 above, we know that \( \omega \) is the imaginary part of \( dz/z \). So it is enough to check that \( dz/z \) is closed. Now \( d(dz/z) = d(z^{-1}dz) = d(z^{-1}) \wedge dz = -z^{-2}dz \wedge dz = 0 \). So \( dz/z \) is closed. In the next chapter we will see that the angular form (which is defined for all \((x, y) \neq (0, 0)\)) is not exact.