Del Pezzo Orders with Canonical Singularities

by

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Abstract

In this thesis we work on del Pezzo orders with no worse than canonical singularities. The notion of del Pezzo order is a non-commutative generalization of del Pezzo surfaces.

We recall definitions and some facts about orders. We see that orders are a type of non-commutative surfaces. Then we proceed to classify them following the same procedure of classifying del Pezzo (commutative) surfaces. The types of singularities that we work with are terminal and canonical singularities. Once the order has only terminal singularities the classification is very fluent. But when it comes to canonical singularities it needs more work. However, having the classification of terminal orders helps us classify canonical orders. That is because we first resolve canonical singularities to terminal singularities and then we contract them to minimal terminal models.

Let $\mathcal{X}$ be a terminal del Pezzo order. Running the minimal model program, we get a minimal terminal del Pezzo order over $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$. Then having

the minimal terminal del Pezzo orders classified, we give the classification of terminal del Pezzo orders by blowing up the minimal models which remain del Pezzo.

If we have a canonical del Pezzo order, there is more work to classify it. Let $\mathcal{X}$ be an order with canonical singularities. We see that there is a unique minimal resolution $\mathcal{Y} \to \mathcal{X}$ where $\mathcal{X}$ is a terminal almost del Pezzo order. Now, similarly by running minimal model program, we get a minimal terminal almost del Pezzo order over $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$, $\mathbb{F}_1$, or $\mathbb{F}_2$. Thus we first classify minimal terminal almost del Pezzo orders and then we find all their blowups which remain almost del Pezzo. The resolution $\mathcal{Y} \to \mathcal{X}$ contracts $K_{\mathcal{Y}}$-zero curves. So the blowups of the minimal models must present such curves in order to get contracted to canonical del Pezzo orders that are not terminal.

At the end we give a classification of degree 4 ramification divisors of canonical del Pezzo orders over $\mathbb{P}^2$.

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Chapter 1

Introduction

Working in Algebraic geometry is largely about classifications and in particular classification of varieties (or schemes). However in higher dimensions classifications are less approachable. So it makes sense to work on lower dimensions; in classical algebraic geometry most results are in dimensions one and two, and then generalized to higher dimensions.

Del Pezzo surface is a terminology that was introduced by Pasquale del Pezzo in 1887, [1]. These are the surfaces with ample anti-canonical bundles. More detai was worked out later by others, e.g. M. Demazure and H. C. Pinkham. Later on in 1981, F. Hidaka and K. Watanabe classified such surfaces. They showed that if a del Pezzo surface Z is normal Gorenstein with no more than canonical singularities, then Z is either $\mathbb{P}^1 \times \mathbb{P}^1$, or the contraction of the

Hirzebruch surface $\mathbb{F}_2$ by its minimal section (i.e. Z is the quadric cone), or Z is the contraction of $\tilde{Z}$ by all of its (-2)-curves, where $\tilde{Z} \to \mathbb{P}^2$ is a sequence of blowups at strictly less than 9 points in almost general position. The surfaces $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{F}_2$ are so called the minimal models for such del Pezzo surfaces and it is constructed by minimal model program (MMP) introduced by Castelnuovo.

In modern algebraic geometry non-commutative algebraic varieties are considered. One open problem concerning non-commutative varieties is the classification of non-commutative surfaces. Since it is a difficult problem, people work on it partially; meaning that subclasses of such surfaces are classified instead. A nice subclass of non-commutative surfaces are the ones which are finite over their centres. This ensures that the centre is itself a surface but it is actually commutative. So classification of these non-commutative surfaces relates to and depends on certain commutative surfaces (the centres). Such a non-commutative surface is called an order or more precisely an order over its centre.

In the present work we are interested in classifying del Pezzo orders with no worse than canonical singularities. It is naturally expected that the notions del Pezzo and singularity must be generalizations of del Pezzo (commutative) surfaces and theirs singularities. In 2003 Chan and Kulkarni showed that if an order over a nice enough centre is del Pezzo, then the centre is a del Pezzo surface, [6]. Using this nice fact, they classified del Pezzo orders over normal Gorenstein projective surfaces. Right after that Chan and Ingalls generalized the terminology of minimal model program to orders, [5]. Having an order with terminal singularities, they showed that it can be contracted to a minimal model, where the centre of the minimal model is either a ruled surface, the projective plane, or has a nef canonical sheaf. This helps us give a complimentary result and method to the classification given by Chan and Kulkarni.

In Chapter 2 we will recall the definition of del Pezzo surfaces and then we will see the classification of such surfaces while their singularities are at worst canonical. In Chapter 3 we will see the definition and some facts about orders. This gives a brief background of orders, discriminant of orders and their ramifications. In Chapter 4 we intend to classify del Pezzo orders with only terminal singularities. We will see that the minimal model of a del Pezzo order with terminal singularities is an order over either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$. Thus we first classify the minimal models of such orders and then proceed to classify them by blowing up the minimal models while preserving the ampleness of the anti-canonical sheaf. This will complete the classification of del Pezzo orders with terminal singularities stated in Theorems 4.3.5, 4.3.6, 4.3.7, 4.4.3, and 4.4.5.

Definition 1.0.1. *Let $\Sigma = \{p_1, \cdots, p_n\}$ be a set of distinct points of $\mathbb{P}^2$. We say Σ is in general position if*

1. *No three points lie on a line;*

2. *No six points lie on a conic.*

Definition 1.0.2. *Let Σ be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. The points of Σ are in general position if any curve of the form $aC_0 + bF$ contains less than $2(a+b)$ of the points.*

Theorem 1.0.3. *Let $\mathcal{X}$ be a terminal order on a projective surface S with the discriminant $D = \cup D_i$ and the ramification degrees e_i. Then one of the following occurs*

1. $S = \mathbb{P}^2$, $3 \leq \deg D \leq 5$. *Furtherthe ramification degrees e_i are all equal e. More precisely $e = 2$ when $\deg D = 5$ and $e = 2$ or $e = 3$ when* $\deg D = 4$

2. *There is a blowup $f : S \to \mathbb{P}^2$ at a set of points $\Sigma = \{p_1, \cdots, p_n\} \subset S$, where*

    - $\deg f_* D = 3$, *$n < 9$, and $\Sigma \subset f_* D$ is in general position; or*
    - $\deg f_* D = 3$, *$n = 1$, $p_1 \notin f_* D$, and $e_i = 2$; or*
    - $\deg f_* D = 4$, *$n = 1$, $p_1 \in f_* D$, and $e_i = 2$.*

3. *$S = \mathbb{P}^1 \times \mathbb{P}^1$ and if $D = aC_0 + bF$ for perpendicular fibres C_0 and F is the ramification divisor, then $2 \leq a, b \leq 3$ and ramification degrees are equall and, unless $D \sim 2C_0 + 2F$, they are all 2.*

4. *There is a blowup $f : S \to \mathbb{P}^1 \times \mathbb{P}^1$ at a set of points $\Sigma = \{p_1, \cdots, p_n\} \subset$*

S, where f_*D has bi-degree $(2,2)$, $n < 8$, and $\Sigma \subset f_*D$ is in general position.

The aim of this work is to classify del Pezzo orders with canonical singularities. To do so, in Chapter 5 we need to find their resolution to terminal orders. This is based on the work of Chan, Ingalls, and Hacking, [4]. We will see that the resolution of a canonical del Pezzo order (which is a terminal order) is not necessarily del Pezzo, and further, their minimal models are more various. Theorems 5.2.3 and 5.2.8 give us the following result.

Theorem 1.0.4. *Let $\mathcal{X}$ be a canonical del Pezzo order on Z. Also let $f : \mathcal{Y} \to \mathcal{X}$ be a minimal resolution of $\mathcal{X}$ to an almost del Pezzo terminal order $\mathcal{Y}$ and let $g : \mathcal{Y} \to \mathcal{W}$ be a contraction of $\mathcal{Y}$ to a minimal terminal almost del Pezzo $\mathcal{W}$. Then $\mathcal{W}$ has centre $Z = \mathbb{P}^2$ or $Z = \mathbb{F}_n$ for $n = 0, 1,$ or 2.*

With the same procedure as terminal del Pezzo orders, we classify minimal terminal almost del Pezzo orders, and then in Chapter 6 we find all the blowups when being del Pezzo is preserved. This comes in Theorems 6.1.2, 6.2.2, 6.2.6, and 6.2.9, a summary of which is given below.

Theorem 1.0.5. *Let $\mathcal{W}$ be a minimal terminal almost del Pezzo order over Z with ramification divisor $D = \cup D_i$ and ramification degrees e_i. Also let $g : \mathcal{Y} \to \mathcal{W}$ represent n iterated blowups of $\mathcal{W}$ at the points $\Sigma \subset Z$. Then any of the followings gives us a $K_{\mathcal{Y}}$-zero curve E such that if $f : \mathcal{Y} \to \mathcal{X}$ contracts E, we get a canonical del Pezzo order $\mathcal{X}$ which is not terminal.*

1. $Z = \mathbb{P}^2$, $\deg D = 3$ *and we have one of the followings*

   - Σ *is the set of a double infinitely near point. The exceptional curves are* E_1 *and* E_2*, where* $E_1^2 = -2$ *and* $E_2^2 = -1$*;* $E := E_1$*.*
   - $\Sigma = \{p, q\}$ *where* $p \in D$ *and* $q \notin D$*;* E *is the strict transform of the line going through* p *and* q*.*
   - Σ *contains* 3 *points in* D *(infinitely near points are allowed) where there is a line* L *with multiplicity* 3 *at* Σ*;* $E := l$*.*
   - Σ *contains* 6 *points in* D *(infinitely near points are allowed) where there is a conic* C *with multiplicity* 6 *at* Σ*;* $E := C$*.*
   - Σ *contains* 8 *points in* D *(infinitely near points are allowed) where there is a nodal cubic* C' *with multiplicity* 9 *at* Σ*;* $E := C'$*.*

2. $Z = \mathbb{P}^1 \times \mathbb{P}^1$*, and we have one of the followings*

   - $D \equiv 2C_0 + 2F$*,* $e = 2$*, and* $\Sigma = \{p\}$ *is a single point, where* $p \notin D$*. Then* E *is the proper transform of any fibre (in any direction) passing* p*.*
   - $D \equiv 3C_0 + 2F$*,* $e = 2$*, and* $\Sigma = \{p\}$ *is a single point, where* $p \in D$*. Then* E *is the proper transform of any fibre in* $[F]$ *passing* p*.*
   - $D \equiv 3C_0 + 3F$*,* $e = 2$*, and* $\Sigma = \{p\}$ *is a single point, where* $p \in D$*. Then* E *is the proper transform of any fibre (in any direction) passing* p*.*

- *$D \equiv 2C_0 + 2F$, $e > 1$, and $\Sigma \subset D$ is a set of points in almost general position. Then E is the blowup of any curve in Σ-almost general position.*

3. *$Z = \mathbb{F}_1$, Σ is in almost general position, $D \equiv 2C_0 + 4F$, $e = 2$, and we have one of the followings*

   - *$E := C_0$.*
   - *Any fibre $\tilde{F}$, where multiplicity of F at Σ is 2.*
   - *An exceptional curve E, where $E^2 = -2$.*

4. *$Z = \mathbb{F}_2$, Σ is in almost general position, $D \equiv 2C_0 + 4F$, e is free and we have one of the followings*

   - *$E := C_0$*
   - *$\Sigma = \{p\}$ where $p \notin D$, E is the fibre F passing p.*
   - *$\Sigma = \{p_1, \cdots, p_n\} \subset D$ in almost general position where $n \leq 7$, E is any fibre F with multiplicity e at Σ.*
   - *An exceptional curve E, where $E^2 = -2$.*

At the end, in Appendix A, we give a classification of degree 4 ramification divisors of canonical del Pezzo orders over $\mathbb{P}^2$ which might be of interest to the reader.

We will be always working on algebraically closed fields with characteristic zero. Also the surfaces are irreducible and projective. It is assumed that the reader is familiar with the contents of "Algebraic Geometry" by R. Hartshorne, [9].

Chapter 2

Canonical Del Pezzo Surfaces

In this chapter we look at the classification of del Pezzo surfaces with only canonical singularities. Such a classification involves birational geometry. One way to study this is by running the Minimal Model Program (MMP). We will see that the minimal model of a del Pezzo surface with canonical singularities is $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$. If there is no ambiguity we write canonical surface and refer to a surface with canonical singularities

2.1 Minimal Model Program (MMP)

Minimal Model Program is a part of the birational classification of algebraic varieties. In this work it is a tool for the classification of algebraic surfaces.

It is used to construct a birational model for any projective surface which is as canonical as possible. Meaning that the canonical bundle is nef, or if $K.E$ is negative for some irreducible curve E and canonical divisor K, then $E^2 \geq 0$. Note that if the canonical bundle is not nef, then $K.E < 0$ for an irreducible curve E. If $E^2 < 0$, then $K.E = -1$ and $E^2 = -1$. Such a curve is called a (-1)-curve and $E \simeq \mathbb{P}^1$. The following Proposition, called Castelnuovo's Contractibility Criterion, is a key point of MMP for surfaces.

Proposition 2.1.1. *[12, Theorem 1.1.6] Let S be a nonsingular projective surface and let $E \subset S$ be a (-1)-curve. Then there exists a nonsingular surface S' and a single blowup at a point p, $g : S \to S'$ such that E is the corresponding exceptional curve.*

Having Castelnuovo's contraction for (-1)-curves, we can construct minimal models of nonsingular surfaces. The following proposition shows that the minimal model always exists for smooth surfaces, however it may not be unique.

Proposition 2.1.2. *[9, Theorem 5.8] Every nonsingular surface admits a birational morphism to a minimal model.*

The goal of minimal model program is to contract nonsingular surfaces to simpler surfaces which are also nonsingular. But what if we have a singular surface? In this work we consider only canonical singularities for surfaces. Canonical singularities for surfaces are only rational double points and they are sometimes called du Val or Kleinian singularities as they were studied by Patrick du Val and Felix Klein. For more details and types of singularities see [13].

Remark 2.1.3. *If a normal Gorenstein surface T is not smooth, there is a resolution $f : S \to T$ where f is a sequence of blowups and S is a nonsingular surface. Then we have the following equation for canonical divisors.*

$$K_S \equiv f^*(K_T) + \sum_i a_i E_i$$

where the sum is over all f-exceptional curves and the a_i are rational numbers. In the case that singularities of T are only rational double points then $a_i \geq 0$ for every i. Further f is called minimal if $a_i \leq 0$ for every i and we can see that the minimal resolution always exists [5, Corrolary 3.6] and is unique [11, Theorem 3.52]. So if a surface T has only canonical singularities and $f : S \to T$ is a minimal resolution, then

$$K_S \equiv f^*(K_T).$$

Thus for a singular surface we first resolve it to a nonsingular surface and then run MMP. This gives the following diagram.

$$\begin{tikzcd}
S \arrow[dr, "f"] \arrow[dd, "g"'] & \\
& T \arrow[dl, dashed, "gof^{-1}"] \\
S' &
\end{tikzcd}$$

Figure 2.1: Contraction of the minimal resolution of a canonical surface to a minimal model

2.2 Del Pezzo Surfaces

In this section we define del Pezzo surfaces and recall their classification when they have no worse than canonical singularities. We will see that any such del Pezzo surface can be resolved to a smooth surface and the contracted to one of the surfaces $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{F}_2$. These three surfaces are the minimal models for del Pezzo surfaces.

2.2.1 Smooth Del Pezzo Surfaces

Definition 2.2.1. *Let S be a surface and let K_S denote the canonical divisor on S. S is called del Pezzo if $-K_S$ is ample or equivalently if*

1. $K_S^2 > 0$

2. $-K_S.C > 0$ *for every effective curve* C*.*

If 2 is replaced by $-K_S.C \geq 0$ *for every effective curve, then* S *is called almost del Pezzo.*

Now let S be a smooth del Pezzo surface and let $g : S \to S'$ be a contraction to a minimal model S'. Then is S' also del Pezzo?

Proposition 2.2.2. *Let* S *be a smooth del Pezzo surface and let* $g : S \to S'$ *be a contraction to a minimal model* S'*. Then* S' *is also del Pezzo.*

Proof. By [9, Proposition 3.3] we know $K_{S'}^2 = K_S^2 + 1 > 0$ and so $K_{S'}^2 > 0$. Now let C' be an effective curve in S'. We let C denote the proper transform of C'. Then $C' = g_*C$ and

$$\begin{aligned} K_{S'}.C' &= K_{S'}.g_*C \\ &= g^*K_{S'}.C \\ &= \left(K_S - \sum_i r_iE_i\right) C \\ &< \left(-\sum_i r_iE_i\right) C. \end{aligned} \tag{$*$}$$

The following claim finishes the proof. □

Claim 2.2.3. *We claim that* $(-\sum_i r_iE_i)\, C \leq 0$*. So then* S' *is del Pezzo.*

Remark 2.2.4. *Note that the same calculation works for almost del Pezzo surfaces. The only difference is in the equation* $(*)$ *where* $<$ *shall be replaced by* $\leq$ *which proves* S' *is almost del Pezzo.*

Proof of Claim 2.2.3. Let $g : S \to S'$ be the contraction of S to the minimal model S'. Then we have the following equation.

$$K_S = g^* K_{S'} + \sum_i r_i E_i,$$

where the r_i are non negative as S' is nonsingular. Since $E_i C \geq 0$ for every i, then $(\sum_i r_i E_i)C$ is non negative.

Let S be a smooth del Pezzo surface and let S' be a minimal model of S. As S' is del Pezzo $K_{S'}$ can not be nef, then minimality here means that there is no (-1)-curve in S'. Using [10, Corollary 3.6] and the fact that $\mathbb{F}_2$ is not del Pezzo we see that S' is the projective surface $\mathbb{P}^2$ or the ruled surface $\mathbb{P}^1 \times \mathbb{P}^1$. This gives us the classification for minimal del Pezzo surfaces. To classify all smooth del Pezzo surfaces we need to blow up the minimal models and keep it del Pezzo. Theorem 2.2.10 indicates how many and what type of points in S' can be blown up while the surface remains del Pezzo. But before that we need to state the definition of almost general position for points on the projective surface.

Definition 2.2.5. *Let* $p \in S$ *be a single point in the surface* S *and let* $f : T \to S$ *be a blowup of* S *at* p. *Now let* q *be a point in the exceptional* *curve* $E = f^{-1}(p)$*. Then* q *is determined by a point in* $\mathbb{P}(T_pS)$*. The points* p *and* q *are called infinitely near points as they are both mapped to the same point in* S*. This can be inducted to a set of* n *infinitely near points* $p_1, \cdots, p_n$ *blown up from a point* $p = p_1 \in S$*. If* $E_1, \cdots, E_m$ *are the corresponding exceptional curves and the blowups are done in order, meaning that the* i*-th blowup replaces the point* p_i *by the exceptional curve* E_i*, then* $\{E_1, \cdots, E_n\}$ *are in a tree of exceptional curve, see the following Figure as an example. By the figure we mean that* E_1 *is obtained by the blowup* f *at* p*.*

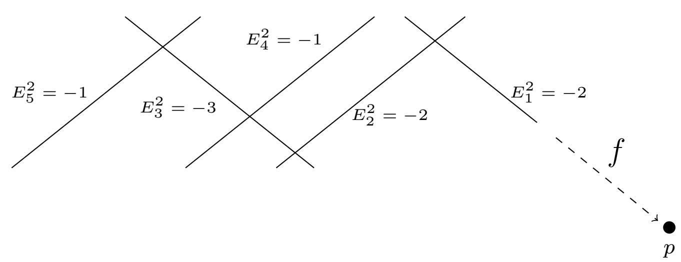


Figure 2.2: Example of an exceptional tree corresponding to $p \in S$

Definition 2.2.6. *Let* $p_1, \cdots, p_m$ *be infinitely near points blown up from a point* $p \in S$ *where* S *is a smooth surface. Further let* $t_p = \{E_1, \cdots, E_m\}$ *be the exceptional tree corresponding to* p*. If* C *is an irreducible curve* C *passing through* p*, then we define the multiplicity of* C *at the infinitely near points* $p_1, \cdots, p_m$ *as the following.*

$$m_p(C) = \sum_{j=1}^{m} \ell_j E_j.\tilde{C}. \tag{2.1}$$

$\tilde{C}$ *is the* f*-proper transform of* C *where* f *is the birational morphism repre-*

senting the sequence of blowups and $\ell_j - 1$ is the least number of exceptional curves in t_p which connect E_j to p. For instance in Figure 2.2, $\ell_5 = 4$ and $\ell_2 = 2$. Note that if C is a smooth curve at p, then $\tilde{C}$ intersect exactly one of the exceptional curves in t_p.

Proposition 2.2.7. *Let C be a curve in the surface S. Multiplicity of C at the infinitely near points is a generalization of the well-known multiplicity defined for C at a single point p.*

Proof. Let C be a curve in the smooth surface S and let $f : S' \to S$ denote a single blowup at a point $p \in S$. To refuse any confiusion, for now we denote the well-know multiplicity of C at a point p by $m'_p(C)$. In Equation 2.1 we have $j = 1$ and $\ell_1 = 1$. So

$$m_p(C) = \ell_1 E_1.\tilde{C} = E_1.\tilde{C}. \tag{$*$}$$

On the other hand

$$E_1.\tilde{C} = E_1(f^*C - m'_p(C)E_1) = 0 + m'_p(C). \tag{$**$}$$

comparing $(*)$ and $(**)$ we get $m'_p(C) = m_p(C)$ $\square$

Definition 2.2.8. *Let Σ be a set of points in $\mathbb{P}^2$ where infinitely near points are allowed. Σ is in almost general position if*

1. *No four points (counting the multiplicities) are on a line.*

2. *No seven points (counting the multiplicities) are on a conic.*

3. *No point is on a* (-2)*-curve.*

Note that the above definition is a recall of the definition of "in almost general position points" given in [6]. We only use a different language for the last condition. One can compare our definition with the one in [6] stated in the following.

Definition 2.2.9. *[6, Definition 15] Let* $\Sigma = \{p_1, \cdots, p_n\}$ *be a set of closed points in* $\mathbb{P}^2$ *where we allow infinitely near points. Denote by* Σ_j *the subset* $\Sigma = \{p_1, \cdots, p_j\}(1 \leq j \leq n)$ *and let* $V(\Sigma_j) \to \mathbb{P}^2$ *be the blowup of* $\mathbb{P}^2$ *with the centre* Σ_j*. Then there exists the sequence*

$$V(\Sigma) = V(\Sigma_n) \to \cdots V(\Sigma_1) \to \mathbb{P}^2.$$

Let E_j *be the exceptional curve for* $V(\Sigma_j) \to V(\Sigma_{j-1})$*. Then* Σ *is in almost general position if*

1. *No four points (counting the multiplicities) are on a line.*

2. *No seven points (counting the multiplicities) are on a conic.*

3. *No point* p_{j+1} *of* $V(\Sigma_j)$ *lies on the strict transform of an exceptional curve* $E_i(1 \leq i \leq j)$ *with* $\hat{E}_i^2 = -2$*.*

Theorem 2.2.10. *[10, Theorem 3.4.v] Let $g : S \to \mathbb{P}^2$ be blowups at points $\Sigma = \{p_1, \cdots, p_n\}$. Then S is almost del Pezzo if and only if $n \leq 8$ and Σ is in almost general position.*

2.2.2 Del Pezzo Surfaces with Canonical Singularities

If a del Pezzo surface has singularities, then we need to resolve it first to a smooth surface and then find the minimal model for the smooth surface. But note that if a surface T is del Pezzo with canonical singularities and $f : S \to T$ is a resolution, then S is not necessarily del Pezzo. More precisely we will see that if f is the minimal resolution, then S is almost del Pezzo.

Proposition 2.2.11. *Let T be a del Pezzo surface with canonical singularities and let $f : S \to T$ be the minimal resolution. Then S is almost del Pezzo.*

Proof. Let $f : S \to T$ be the minimal resolution of the del Pezzo surface T with canonical singularities. So

$$0 < K_T^2 = f^*(K_T)^2 = K_S^2$$

and for any irreducible curve $C \subset S$

$$K_S.C = f^*(K_T).C = K_T.f_*C \qquad (*)$$
$$\begin{cases} = 0 & \text{if } f_*C \text{ is a point} \\ < 0 & \textit{otherwise.} \end{cases}$$

□

The equations in $(*)$ indicates that $K_S.C = 0$ if and only if C is an f-exceptional curve. Further in this case $C^2 = -2$. Having Remark 2.2.4, we see that the minimal model S' is almost del Pezzo and minimality here means that if there is any curve $C' \subset S'$ with $(C')^2 < 0$, then $(C')^2 = -2$. Moreover the (-2)-curves are the ones which we can contract to get a canonical del Pezzo surface.

Proposition 2.2.12. *[10, Corollary 3.6 and Theorem 3.4] Let T be a del Pezzo surface with canonical singularities and let $f : S \to T$ and $g : S \to S'$ be the minimal resolution of T and the contraction of S to the minimal model S' respectively. Then S' is $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{F}_2$. Further, in the first case g is a sequence of blowups at points $\Sigma = \{p_1, \cdots, p_n\}$ where $n < 9$ and Σ contains at least one of*

- *three points on a line l and l is an f-exceptional curve, or*
- *six points on a conic C and C is an f-exceptional curve, or*

- *eight points on a nodal cubic C where the singularity of C is at one of the points. Then C is an f-exceptional curve, or*
- *two infinitely near points p_i and p_j and the line going through p_i with direction p_j is an f-exceptional curve.*

Chapter 3

Orders

In this chapter we define and work with (maximal) orders over surfaces. To have a good understanding of orders, we can say that in this work orders are non-commutative surfaces which are finite over their centres. If a non-commutative surface is finite over its centre, then the centre should be a surface as well, and of course it is a commutative surface. In algebraic geometry, when the notion of non-commutativity comes up, there should be some non-commutative algebras involved. However, working with commutative algebras is always easier, and that is why people try to find a nice relation between commutativity and non-commutativity in a geometric sense. In this chapter we define the canonical bundle of an order and we give a description of it which involves the canonical bundle of the centre. During this work we let $\mathbf{k}$ be an algebraically closed field with characteristic zero and also we always assume the centres are normal and Gorenstein surfaces.

3.1 Maximal Orders

Definition 3.1.1. *Let R be a noetherian integrally closed domain with fraction field K. An R-order is an R-module $\mathcal{A}$ such that:*

1. *$\mathcal{A}$ is finitely generated as an R module,*
2. *$\mathcal{A}$ is torsion free as an R-module and $K \otimes_R \mathcal{A}$ is a central simple K-algebra.*

Since $\mathcal{A}$ is torsion free, it is a subring of $D := K \otimes_R \mathcal{A}$, therefore it is sometimes said that $\mathcal{A}$ is an order in D. We say $\mathcal{A}$ is maximal if it is maximal with respect to inclusion among orders in D. Note that it is always assumed that $\mathcal{A}$ is a central R-module, i.e. $R \subseteq Z(\mathcal{A})$.

Definition 3.1.2. *Let Z be a scheme. Also let $\mathcal{A}$ be a sheaf of rings which is at the same time a quasi-coherent sheaf of $\mathcal{O}_Z$-modules. Namely, for $U \subset Z$, $\mathcal{A}(U)$ is an $\mathcal{O}_Z(U)$-central algebra. Then we say $\mathcal{A}$ is an $\mathcal{O}_Z$-central algebra.*

Definition 3.1.3. *Let Z be an integral noetherian scheme with function field K. An $\mathcal{O}_Z$-order $\mathcal{A}$ over Z is a coherent torsion free $\mathcal{O}_Z$-central algebra such that $\mathcal{A} \otimes_Z K$ is a central simple K-algebra. An $\mathcal{O}_Z$-order is called maximal if it is maximal with respect to inclusion amongst orders in $\mathcal{A} \otimes_Z K$.*

Example 3.1.4. *Let* $\mathbf{k}$ *be an algebraically closed field with characteristic zero. Consider the following algebra*

$$\mathcal{A} = \frac{\mathbf{k}\langle x, y\rangle}{xy + yx}.$$

The centre of $\mathcal{A}$ *is* $R := Z(\mathcal{A}) = \mathbf{k}[x^2, y^2]$ *and*

$$\mathcal{A} = R \oplus xR \oplus yR \oplus xyR.$$

Therefore $\mathcal{A}$ *is finitely generated over its centre. Further, it is easy to check that* $\mathcal{A}$ *is torsion free. Since*

$$K \otimes_R \mathcal{A} = K \oplus xK \oplus yK \oplus xyK,$$

$K \otimes_R \mathcal{A}$ *is a central simple* K*-algebra.*

Let $u := x^2$ *and* $v := y^2$*. Then* $Z = \mathrm{Spec}(\mathbf{k}[u, v]) = \mathbb{A}^2$ *and for an open subset* $U \subset Z$ *we have* $\mathcal{A}(U) = \mathcal{O}_Z(U) \otimes_R \mathcal{A}$*. Therefore,* $\mathcal{A}$ *is an* $\mathcal{O}_Z$*-algebra.*

We are interested in maximal orders. Maximality of orders generalizes the notion of normality for schemes. Therefore the concept of normalization extends to embedding in a maximal order. But unfortunately the maximal order is not unique unless the order is over a discrete valuation ring which is in this case unique up to conjugation.

Throughout when we say order we refer to Definition 3.1.3. For us an order $\mathcal{A}$ is defined over a normal Gorenstein scheme Z as the centre. Also during this work, orders are assumed to be maximal unless specified differently. Moreover, the term commutative is usually ignored and surfaces refer to commutative surfaces.

Let $\mathcal{A}$ be an order (not necessarily maximal). We are looking for a maximal order containing $\mathcal{A}$. One way is to take its double dual (or reflexive hull), however it may not be maximal yet.

Definition 3.1.5. *Let R be a commutative noetherian normal domain and let $\mathcal{A}$ be a finitely generated R-module. Then the dual of $\mathcal{A}$ is $\mathcal{A}^* := \mathrm{Hom}_R(\mathcal{A}, R)$. We say $\mathcal{A}$ is reflexive if it is isomorphic to its double dual $\mathcal{A}^{**}$.*

Let R be a commutative noetherian normal domain and let K be its fraction field. If $\mathcal{A}$ is an R-order in $D = K \otimes_R \mathcal{A}$, then $\mathcal{A}^{**}$ is also an order in D, [3, Lemma 6.3]. Since $\mathcal{A}$ and $\mathcal{A}^*$ are torsion free R-modules, therefore $\mathcal{A} \subseteq \mathcal{A}^{**}$.

There is a criteria given by Auslander and Goldman for an order to be maximal. The idea is to look at the localization of $\mathcal{A}$ at the prime ideals of Z rather that $\mathcal{A}$ itself. But before that we need to study the localization of a module. Note that it is easier to check maximality of $\mathcal{A}_p$ as an R_p-order because in this case the order is over a discrete valuation ring.

Definition 3.1.6. *Let R be a commutative ring and let A be an R-module. If p is a prime ideal in R, then localization of A at p as a module is $(R-p)^{-1}A$*

which is an R_p-module.

Proposition 3.1.7. *[3, Proposition 6.4] An R-order $\mathcal{A}$ is maximal if and only if $\mathcal{A}$ is reflexive and $\mathcal{A}_p$ is maximal for every irreducible divisor $p \in Z$, where $Z = \operatorname{Spec}(R)$.*

Let $\mathcal{A}$ be an order in $K \otimes_R \mathcal{A}$. The double dual $\mathcal{A}^{**}$ is always a good candidate for the maximal order containing $\mathcal{A}$. Since $\mathcal{A}^{**}$ is reflexive by Proposition 3.1.7 we only need to check if for every prime divisor p the R_p-module $\mathcal{A}_p$ is maximal.

3.2 Del Pezzo Orders

In the present work we are interested in a class of orders called del Pezzo. Since del Pezzo orders are the generalization of del Pezzo surfaces it is very natural to define del Pezzo orders to be the ones with ample anti-canonical sheaf. So we need to define the canonical sheaf for orders. Later on we also talk about types of singularities for orders as we want to classify del Pezzo orders with only canonical singularities.

Definition 3.2.1. *Let $\mathcal{A}$ be an order over a surface Z. Since Z is normal it is Cohen-Macaulay, therefore the dualizing sheaf exists. Since Z is Gorenstein the dualizing sheaf is actually the canonical sheaf $\omega_Z \equiv \mathcal{O}_Z(K_Z)$ where K_Z is* *a canonical divisor. Then the canonical sheaf of $\mathcal{A}$ is defined as the following*

$$\omega_{\mathcal{A}} := \mathscr{H}om_{\mathcal{O}_Z}(\mathcal{A}, \omega_Z).$$

Definition 3.2.2. *An order $\mathcal{A}$ is called del Pezzo if the anti-canonical sheaf $\omega^*_{\mathcal{A}} = \mathscr{H}om_{\mathcal{A}}(\omega_{\mathcal{A}}, \mathcal{A})$ is ample, see [6, p: 152] for details about the ampleness on orders.*

In Definition 2.2.1 we saw that a surface is del Pezzo if and only if some numerical inequalities hold for the canonical divisors which is significantly helpful for working on del Pezzo surfaces. We are looking for a similar equivalence for del Pezzo orders. To this end we need to define the ramification divisor. following discussion is recalled from [6, p:151].

Let Z^1 be the set of irreducible divisors in $Z = \mathrm{Spec}(R)$ which are irreducible curves in our case. Let D_0 be in Z^1 and let $p_0 \sim D_0$ be the height 1 prime ideal so that $D_0 = \mathrm{Spec}\left(\frac{R}{p_0}\right)$. Moreover let $K(D_0)$ denote the function field of D_0 and let F be the centre of the residue ring $\frac{\mathcal{A}_{p_0}}{\mathrm{rad}\,\mathcal{A}_{p_0}}$. Then we get the field extension $[F : K(D_0)]$ which is almost always trivial. When the field extension is not trivial for a curve D_0, we say that $\mathcal{A}$ ramifies at D_0 with (ramification) degree $e_0 = [F : K(D_0)]$. Also D_0 is called a ramified divisor with degree e_0 and it has a cyclic cover $\widetilde{D_0}$ of degree e_0. Let $D = \cup_i D_i$ be the union of all ramification divisors and let $\{e_i\}$ be the ramification degrees. We call D the discriminant of the order $\mathcal{A}$.

Proposition 3.2.3. *[6, Lemma 8] Let $\mathcal{A}$ be an order over Z and let $(D = \cup_i D_i, \{e_i\}_i)$ be its discriminant and the ramification degrees. Then $\mathcal{A}$ is del Pezzo if and only if the divisor*

$$-K_{\mathcal{A}} := -\left(K_Z + \sum_i \left(1 - \frac{1}{e_i}\right) D_i\right),$$

is ample. We call $K_{\mathcal{A}}$ the canonical divisor on Z corresponding to $\mathcal{A}$.

Proposition 3.2.3 suggests that numerical calculations for the canonical sheaf $\omega_{\mathcal{A}}$ of $\mathcal{A}$ depend only on the centre and the ramification divisors. Canonical sheaves of the centres of del Pezzo orders are easy and straightforward to find. So it is crucial to find the ramification degrees of the ramified divisors in order to do numerical calculations for orders. We know all e_i are strictly greater than 0 but they are more restricted when $\mathcal{A}$ is del Pezzo. This will come in the next chapters where we talk about terminal and canonical singularities for orders. We will see that classification of singularities for orders depends on the centre of the orders as well as the configuration of the ramified divisors.

3.3 Ramifications and Blowups

In this section, $\mathcal{X}$ and $\mathcal{Y}$ denote maximal orders over normal Gorenstein surfaces. Let $(D = \cup_i D_i, \{e_i\}_i)$ be the discriminant and the ramification degrees of an order $\mathcal{X}$ and let μ denote the lowest common multiple of the e_i. The points that two or more of the ramification divisors intersect are called branch points. Beside the degrees for the ramification curves, at each branch point p we define a ramification index for each divisor D_i passing p, denoted by $\bar{\mu_{i,p}}$. We denote the ramification index by a line above because for every i and p, the ramification index $\bar{\mu_{i,p}}$ belongs to the cyclic group $\mathbb{Z}_\mu$. If a divisor does not pass a branch point, we can define its ramification index at the point to be 0. We also denote the order of $\bar{\mu_{i,p}}$ in the group $\mathbb{Z}_\mu$ by $r_{i,p}$. There are also some compatibilities between $\bar{\mu_{i,p}}$, $r_{i,p}$, e_i, and μ which come in Definition 3.3.1.

Definition 3.3.1. *Let $D = \cup_i D_i$ be the discriminant of an order $\mathcal{X}$ and let e_i be its ramification degrees. Also let $\mu_{i,p}$ denote the ramification index of the ramification divisor D_i at the branch point p. A ramification diagram is the data $(D = \cup_i D_i, \{e_i\}_i, \{\mu_{i,p}\})$, such that the following properties hold.*

1. *For every branch point p of any ramification divisor D_i, $r_{i,p} | e_i$.*

2. *On any ramification curve D_i, the sum $\sum_{\nu \in D_i} \bar{\mu_{i,\nu}}$ is zero.*

3. *At any branch point ν, the sum $\sum_{\{i | \nu \in D_i} \bar{\mu_{i,\nu}}\}$ is zero.*

4. *Let D_i be a rational ramification curve and let $\{\bar{\mu_{i,\nu}}\}_\nu$ be the indices for D_i at its branch points. Then the lowest common multiple of the $r_{i,\nu}$ is e_i.*

We say D_i ramifies at p if $r_{i,p}$ is not 1 and we say it totally ramifies at p if $r_{i,p} = e_i$.

Example 3.3.2. *Consider the diagrams in Figure 3.1. The left figure is the ramification diagram of a configuration of four divisors (in this case lines). The numbers near the curves are the ramification degrees for the divisors and the numbers near branch points are the ramification indices for the curves passing the branch point. The right diagram is a more detailed figure for only one branch point as an example to describe what we mean by the numbers in the left diagram.*

One can check that the numbers in the left diagram satisfy all the hypothesis in Definition 3.3.1. Note that the choice of the numbers for a configuration of divisors is not unique and there may be other possibilities. In the right diagram there are two irreducible divisors (lines) with ramification degrees 6 and 2. Then they have sextuple and double covers respectively. Further, the order of $\bar{3}$ in both $\mathbb{Z}_6$ and $\mathbb{Z}_2$ is 2. Therefore the covers at the branch point are irreducible curves of multiplicities 2.

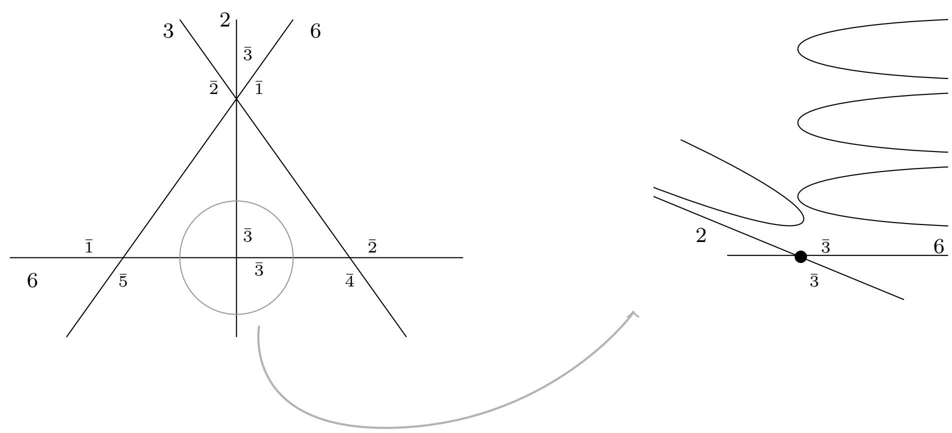


Figure 3.1: A ramification diagram of four lines

Proposition 3.3.3. *[6, Corollary 20] Given any ramification diagram, there exists an order with that ramification.*

We seek to classify del Pezzo orders with canonical singularities. To this end we will show that any such order can be contracted by a finite number of blowdowns to a minimal (almost) del Pezzo order with terminal singularities. The definition of almost del Pezzo orders is just a generalization of the definition of almost del Pezzo surfaces, see Definition 2.2.1, we will define it more precisely in Definition 5.2.2. To give the desired classification we will first classify minimal del Pezzo orders with terminal singularities and then we blow them up. The idea is to find all the blowups that remain del Pezzo. So one needs to know the canonical divisor of the orders after the blowups explicitly.

Let $\mathcal{X}$ be a minimal terminal del Pezzo order, we will see that in the case of minimal terminal del Pezzo orders the centre and the discriminant are both very restrictive and easy to classify. So we basically need to classify the blowups and this is what we do in the next chapters. Here we provide an introduction about blowups of orders.

Talking about singularities of orders is not as easy as commutative surfaces. Singularities for an order not only depend on the centre, but also depend on the ramification diagram of the order. However when an order has terminal or canonical singularities its centre should have no worse than canonical singularities, see table 5.1. During this work when we say canonical singularities we include terminal singularities too, unless specified differently. In the commutative case, surfaces with terminal singularities are smooth and surfaces with canonical singularities have rational singularities. Canonical singularities of surfaces are also called simple surface singularities or du Val singularities. When we have an order with terminal singularities the centre has to be smooth, see Definition 4.1.1. We talk about the canonical singularities for surfaces in Chapter 5. In this setting, when we say resolution of an order $\mathcal{X}$, we say a sequence of blowups of its centre to a smooth surface such that the ramification diagram is also resolved to a terminal ramification diagram. In Chapters 4 and 5 will see the definition of terminal and canonical singularities for orders.

To classify del Pezzo orders with canonical singularities we need to know how the blowups and blowdowns change the discriminant or more precisely the ramification diagram. Note that blowups and blowdowns of orders are indeed birational morphisms of the centres.

Proposition 3.3.4. *[3, Proposition 6.4] Let $\mathcal{X}$ be an R-order, then $\mathcal{X}$ is maximal if and only if $\mathcal{X}$ is reflexive and $\mathcal{X}_p$ is maximal for all height 1 primes $p \in \operatorname{Spec} R$.*

Let $\mathcal{X}$ be an order over Z with the discriminant $D = \cup_i D_i$ and the ramification degrees $\{e_i\}$. And let $f : Z \to Z'$ represent blowing down a contractible curve E. Then $f_*(\mathcal{O}_\mathcal{X})$ is an $f_*(\mathcal{O}_Z)$-module. Having the morphism $f^\# : \mathcal{O}_{Z'} \to f_*\mathcal{O}_Z$, we find a natural structure of $\mathcal{O}_{Z'}$-module for $f_*\mathcal{O}_\mathcal{X}$. So $f_*\mathcal{O}_\mathcal{X}$ is an order (not necessarily maximal) over Z'. Using Proposition 3.3.4, we see that $\mathcal{Y} := (f_*\mathcal{X})^{**}$ is a maximal order. Now let $g : Z'' \to Z$ represent blowing up a point $p \in Z$ to the exceptional curve E. Then $g^{-1}(\mathcal{O}_\mathcal{X})$ is a $g^{-1}(\mathcal{O}_Z)$-module. We define $f^*(\mathcal{O}_\mathcal{X})$ to be $f^{-1}(\mathcal{O}_\mathcal{X}) \otimes_{f^{-1}(\mathcal{O}_Z)} \mathcal{O}_{Z''}$. In the case of blowups, there is not always a unique maximal order containing $f^*(\mathcal{O}_\mathcal{X})$. Then when blowup a maximal order there may be several maximals containing $f^*(\mathcal{O}_\mathcal{X})$ to choose. For more details about the maps and notations see [9, p:109, Definitions].

Further, the discriminant in Z' is $D' = f_*D$ and the ramification degrees are preserved when the centre is blown down, i.e. $e'_i = e_i$ where e'_i is the ramification degree for f_*D_i. For the blowup $g : Z'' \to Z$ at the point p, the discriminant on Z'' depends on the choice of p. If $p \notin D$ or if p is a smooth point of D, then the discriminant D'' in Z'' is the proper transform of D and the ramification degrees do not change. But if $p \in D$ is a singular point of D, then the g-exceptional curve E may belong to the discriminant in Z'', however the ramification degree for E needs some discussion which will come in Chapter 4. Therefore we have the following equation

$$\Delta_{Z''} = \sum_i \left(1 - \frac{1}{e_i}\right) D_i'' + \left(1 - \frac{1}{e}\right) E,$$

where as we said before e can vary and D_i'' is the proper transform of D_i. Note that if E does not ramify or equivalently $e = 1$, then E disappears from the equation.

Definition 3.3.5. *Let $\mathcal{X}$ be an order over Z. If $f : Z \to Z'$ represents blowing down a contractible curve E, then we call $\mathcal{Y} = (f_*\mathcal{X})^{**}$ a blowdown of $\mathcal{X}$ by contracting E. Further if $g : Z'' \to Z$ represents blowing up a point $p \in Z$ to the exceptional curve E, then we choose a maximal order $\mathcal{W}$ containing $f^*(\mathcal{O}_{\mathcal{X}})$ and we call it a blowup of $\mathcal{X}$ at a point p.*

Let $f : Z \to Z'$ be a birational morphism between normal Gorenstein surfaces. Often by abuse of notation, we say f is a morphism between orders $\mathcal{X}$ and $\mathcal{Y}$ and we have the following equation

$$K_{\mathcal{Y}} = K_{\mathcal{X}} + \sum_i a_i E_i,$$

where $K_{\mathcal{X}} = K_Z + \Delta_Z$, $K_{\mathcal{Y}} = K_{Z'} + \Delta$, E_i are f-exceptional curves, and the numbers a_i depend on f.

Chapter 4

Del Pezzo Terminal Orders

In this chapter we will introduce orders with terminal singularities. In the commutative case, surfaces with terminal singularities and smooth surfaces are the same, but it is not in general true for orders. We will see that when we have an order with terminal singularities the centre should be a smooth surface, however the discriminant may also have singularities. We seek to classify del Pezzo orders with terminal singularities. Throughout when we say terminal orders we mean orders with terminal singularities.

Chan and Ingalls in [5], generalized the notion of minimal model program (MMP) to terminal orders over surfaces. We introduced the term "minimal model program" (or Mori's program) in classical algebraic geometry and specifically for del Pezzo surfaces in Chapter 2. We have a similar procedure for the non-commutative cases or more precisely for terminal orders. We let $\mathcal{X}$ and $\mathcal{Y}$ denote orders over centres $Z_{\mathcal{X}}$ and $Z_{\mathcal{Y}}$.

Let $\mathcal{X}$ be a terminal order. Then it can be contracted to a minimal terminal order $\mathcal{Y}$, meaning that there is no $K_{\mathcal{X}}$-negative curve. We show that if the terminal order $\mathcal{X}$ is del Pezzo, then so is its minimal terminal order. This helps us classify terminal del Pezzo orders via classification of minimal terminal del Pezzo orders.

For a minimal terminal del Pezzo order the centre is very restrictive and we show that it is either $Z = \mathbb{P}^1 \times \mathbb{P}^1$ or $Z = \mathbb{P}^2$. Thus to classify terminal del Pezzo orders, we give a classification for minimal terminal del Pezzo orders on these well-known surfaces. Then all we need to do is to blowup these orders and at the same time keep them del Pezzo.

4.1 Orders with Terminal Singularities

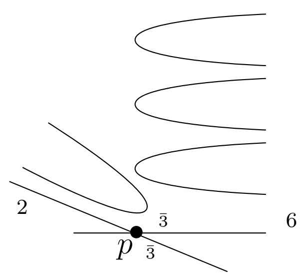
Definition 4.1.1. *Let $\mathcal{X}$ be an order on Z and let $D = \cup_i D_i$ be its ramification divisor with ramification degrees e_i and ramification covers $\widetilde{D_i}$. Then $\mathcal{X}$ has terminal singularities if*

- *Z is a smooth surface,*
- *the ramification divisor has normal crossings*

- *the cyclic cover $\widetilde{D_i}$ ramifies at all the nodes of D_i*

- *locally at any branch point p, only two divisors intersect, one cover $\widetilde{D_i}$ ramifies totally with ramification degree e_i (see Definition 3.3.1), and the other one, $\widetilde{D_j}$, ramifies with ramification degree $e_j = ne_i$ for some positive integer n. Moreover $r_{i,p} = r_{j,p} = e_i$.*

The following diagram clarifies the last condition for terminal singularities. For the oblique line the ramification degree $e_o = 2$ and the order of the branch index at the node is $r_{o,p} = 2$ and for the horizontal line the order of the branch index $r_{h,p} = 2$ and $e_h = 3.2 = 6$.



Definition 4.1.2. *Let $\mathcal{X}$ be a terminal order over Z and let $K_{\mathcal{X}} = K_Z + \Delta$ be its canonical divisor. Then $\mathcal{X}$ is a minimal terminal order if for every irreducible curve $C \in Z$, either $K_{\mathcal{X}}C \geq 0$ or $C^2 \geq 0$.*

Proposition 4.1.3. *[5, Theorem 3.10] Let $\mathcal{X}$ be a terminal order over Z. Suppose there is an irreducible curve $E \in Z$ such that $E^2 < 0$ and $K_{\mathcal{X}}.E < 0$. Then there exists a map $\pi : Z \to Z'$ that contracts exactly E and the order $\mathcal{X}'$ over Z' is terminal.*

The following proposition proves that every terminal order over a surface is minimal or it can be blown down to a minimal terminal order. This is as a result of generalizing the minimal model program to terminal orders over surfaces by Chan and Ingalls, [5]. Before stating the proposition we refer reader to [11, Definition 1.15] for details and the definition of extremal curves.

Proposition 4.1.4. *Let $\mathcal{X}$ be a terminal order over Z. Then $K_{\mathcal{X}}$ is nef or there exists an extremal curve E such that $K_{\mathcal{X}}.E < 0$ and one of the following occurs.*

- $E^2 < 0$*: E is a (-1)-curve and there exists $\pi : Z \to Z'$ contracting exactly E.*
- $E^2 = 0$*: $\pi : Z \to C$ is a ruled surface over a smooth curve C with E a fibre. Moreover, $K_{\mathcal{X}}.E < 0$.*
- $E^2 > 0$*: $Z \simeq \mathbb{P}^2$ and $-K_{\mathcal{X}}$ is ample.*

Let $K_{\mathcal{X}}$ be not nef. If for an extremal curve E the self intersection is negative, then $\pi : Z \to Z'$ contracts E. Proposition 4.1.3 proves that the order $\mathcal{X}'$ over Z' is a terminal order. Then $\mathcal{X}$ can be replaced by $\mathcal{X}'$ and we can repeat the proposition for $\mathcal{X}'$. This ensures that we end with a minimal terminal order. Further, note that if for an extremal curve E the self intersection is zero, then the order is not necessarily minimal. More precisely there are some smooth surfaces with extremal curves E and E' where $E^2 = 0$ and $E'^2 < 0$ and E' is contractible. For instance in Hirzebruch surface $\mathbb{F}_1$ there are two extremal curves C_0 and F where C_0 is the negative section and F is any fibre.

Corollary 4.1.5. *Let $\mathcal{X}$ be a terminal order over Z. Then there exists a sequence of blowdowns of (-1)-curves*

$$f : Z \to Z_1 \to \cdots \to Z_n = Z'$$

and a maximal order $\mathcal{X}'$ over Z'. Let $K_{\mathcal{X}'} = K_{Z'} + \Delta'$. Then one of the following holds,

- *$K_{\mathcal{X}'}$ is nef.*
- *$\pi : Z' \to C$ is a ruled surface and $-K_{\mathcal{X}'}.F > 0$ for F a fibre. Further, Z' contains no irreducible curve C such that $C^2 < 0$ and $K_{\mathcal{X}'}.C < 0$.*
- *$Z' \simeq \mathbb{P}^2$ and $-K_{\mathcal{X}'}$ is ample.*

where $\mathcal{X}'$ is the terminal order over Z' and $K_{\mathcal{X}'} = K_{Z'} + \Delta'$.

4.2 Minimal Terminal Del Pezzo Orders

In this section we seek to give a classification of minimal terminal del Pezzo orders. But the classification is actually for the centres of minimal terminal orders. This is still very helpful as del Pezzo orders are significantly dependent on their centres. We will show that the centre of a del Pezzo order is either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$. The key point of the proof is based on a theorem by Chan and Kulkarni. They showed that if an order over a surface is del Pezzo, then the centre is necessarily del Pezzo, Proposition 4.2.2. Then we aim to classify all terminal orders by blowups of minimal terminal orders. But before that we need to show that any terminal del Pezzo order can be contracted to a minimal terminal order.

Lemma 4.2.1. *Let $\mathcal{X}$ be a del Pezzo order over Z and let $f : Z \to Z'$ be a birational morphism which contracts exactly an irreducible curve E suth that $E^2 = -1$. Then the order $\mathcal{X}'$ over Z' is del Pezzo.*

Proof. Consider the equations

$$K_{\mathcal{X}} = K_Z + \Delta$$
$$K_{\mathcal{Y}} = K_{Z'} + \Delta'$$
$$K_Z + \Delta \equiv f^*(K_{Z'} + \Delta') + aE.$$

As $\mathcal{X}$ is a del Pezzo order, $(K_Z + \Delta)^2 > 0$ and $(K_Z + \Delta).C < 0$ for any effective curve $C \in Z$. So

$$\begin{aligned}(K_Z + \Delta)^2 &= (f^*(K_{Z'} + \Delta') + aE)^2 \\ &= (K_{Z'} + \Delta')^2 - (a)^2 > 0,\end{aligned}$$

so $(K_{Z'} + \Delta')^2 > 0$.

If C is an effective curve in Z, then f^*C is an effective curve in Z'.

$$\begin{aligned} 0 > (K_{Z'} + \Delta')f^*C &= (f^*(K_Z + \Delta) + aE)f^*C \\ &= f^*(K_Z + \Delta)f^*C' + aEf^*C \\ &= (K_Z + \Delta)f_*f^*C + 0 \\ &= (K_Z + \Delta)C. \end{aligned}$$

□

Proposition 4.2.2. *[6, Theorem 12] Let $\mathcal{X}$ be a del Pezzo order on the centre Z. Then Z is a del Pezzo surface.*

Now if we add the assumption of ampleness of the anti-canonical bundle $-K_{\mathcal{X}}$ to Corollary 5.3.4 we get the following result.

Theorem 4.2.3. *Let $\mathcal{X}$ be a minimal terminal del Pezzo order on Z. Then $Z = \mathbb{P}^1 \times \mathbb{P}^1$ or $Z = \mathbb{P}^2$.*

Proof. Using Corollary 5.3.4 for terminal del Pezzo orders we get the following sequence of contraction

$$f : Z \to Z_1 \to \cdots \to Z_n = Z'$$

such that $\mathcal{X}'$ is a minimal terminal del Pezzo order. So Z' is a del Pezzo surface and since $\mathcal{X}'$ is del Pezzo $K_{\mathcal{X}'}$ is not nef, thus one of the following holds,

- $\pi : Z' \to C$ is a ruled surface.
- $Z' \simeq \mathbb{P}^2$.

All we need is to show that if the first case occurs, then $Z = \mathbb{P}^1 \times \mathbb{P}^1$. So let S be a ruled surface and let C_0 be the minimal section, as in [9] , and F a fibre. Then by the genus formula for ruled surfaces we have

$$K_Z C_0 = n + 2g - 2 \text{ and}$$
$$K_Z^2 = 8(1 - g),$$

where g denotes the genus of C_0 and $n = -C_0^2$. Since Z' is del Pezzo, we have $1 - g > 0$ and $n + 2g - 2 < 0$. Then $g < 1$, i.e. C_0 is a rational ruled surface, and $n < 2$. Now we only need to show that $n \neq 1$.

Let $n = 1$ and $Z' = \mathbb{F}_1$ and let C_0 be the (-1)-section in Z. So $C_0^2 = -1$ and as $\mathcal{X}'$ is del Pezzo, $C.K_{\mathcal{X}'} < 0$ which violates the hypothesis of $\mathcal{X}$ to be a minimal order.

On the other hand, if $Z = \mathbb{P}^1 \times \mathbb{P}^1$ or $Z = \mathbb{P}^2$ then for every effective curve $C \in Z$, $C^2 \geq 0$ and the hypothesis of Definition 4.1.2 hold. $\square$

So all terminal orders are blowups of minimal terminal orders over $\mathbb{P}^1 \times \mathbb{P}^1$ or

$\mathbb{P}^2$. Therefore, to classify terminal del Pezzo orders we first classify the ones over $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ and then we blow them up. We give the restrictions for the number and the types of the blowups which keep the order del Pezzo.

4.3 Terminal Del Pezzo Orders on Blowups of $\mathbb{P}^2$

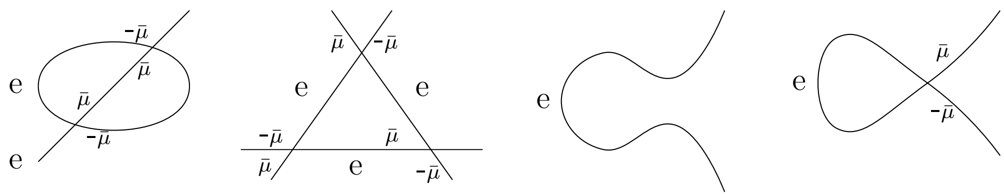
In this section we present the classification of terminal del Pezzo orders over blowups of $\mathbb{P}^2$. To this end we first classify minimal teminal del Pezzo orders over $\mathbb{P}^2$. Then we can classify terminal del Pezzo orders as they can be constructed by blowups of the minimal orders over $\mathbb{P}^2$. We let $\mathcal{X}$ denote a del Pezzo order on $\mathbb{P}^2$ and $D = \cup D_i,\ \{e_i\}$ denote the ramification configuration of $\mathcal{X}$. Using the results in [5] and [6], we have that $3 \leq \deg(D) \leq 5$. Additionally for a ramification configuration $D = \cup D_i$, all the indices e_i are equal.

$deg(D)$	e
3	≥ 2
4	$2, 3$
5	2

Table 4.1: Degrees and indices for terminal orders on $\mathbb{P}^2$

Degree 3 ramification divisors

Let $Z = \mathbb{P}^2$ and let $\mathcal{X}$ be a maximal order on Z. Moreover let D be a degree 3 ramification divisor on Z and define $\Delta = \left(1 - \frac{1}{e}\right) D$. Then D is of one of the types in Figure 4.1. The number e represents the ramification degree of the curves and μ is any generator of the cyclic group $\frac{\mathbb{Z}}{e\mathbb{Z}}$ and represents the ramification index of the curves at the branch points.



Transverse line & conic   Transverse lines   A smooth cubic   A nodal cubic

Figure 4.1: Cubic ramification configurations

When we blowup an order at a point p the canonical bundle of the new order depends on p and the discriminant. There are always three cases, the point can be out of D, or it can be a smooth or a singular point of the discriminant.

Lemma 4.3.1. *Let $\mathcal{X}$ be a terminal order over Z and let $f : Z' \to Z$ be a blowup at a point p. Then we have the equation*

$$K_{Z'} + \Delta' \equiv f^*(K_Z + \Delta) + aE,$$

where

1. $a = 1$*, if* p *is not in* $D = \lceil \Delta \rceil$*.*

2. $a = \dfrac{1}{e}$*, if* p *is a smooth point of* D*, where* e *is the ramification degree of* D*.*

3. $a = \dfrac{1}{ne}$*, if* p *is a singular point of* D*, where the ramification degrees of the ramification curves crossing at* p *are* e *and* ne*.*

Proof. Let $f : Z' \to Z$ be a blowup at a point p. We have the following equations

$$
\begin{aligned}
aE &\equiv K_{Z'} + \Delta' - f^*(K_Z + \Delta) \\
K_{Z'} &\equiv f^*K_Z + E \\
\Delta' &\equiv \begin{cases} \widetilde{\Delta} + \left(1 - \dfrac{1}{e}\right) E; & \text{if } p \text{ is a singular point of } D \\ \widetilde{\Delta}; & \text{otherwise} \end{cases} \qquad (*) \\
f^*(K_Z + \Delta) &\equiv f^*K_Z + f^*\Delta \\
f^*\Delta &\equiv \begin{cases} \widetilde{\Delta}; & \text{if } p \text{ is not a point of } D \\ \widetilde{\Delta} + \left(1 - \dfrac{1}{e}\right) E; & \text{if } p \text{ is a smooth point of } D \\ \widetilde{\Delta} + \left(1 - \dfrac{1}{e}\right) E + \left(1 - \dfrac{1}{ne}\right) E; & \text{if } p \text{ is a singular point of } D \end{cases}
\end{aligned}
$$

where $\widetilde{\Delta}$ represents the proper transform of Δ and equation in $(*)$ comes from [4, Remark 2.8]. These give us the desired result. □

We start with blowing up a point $p \notin D$. We show that if a del Pezzo order with a discriminant of degree 3 in $\mathbb{P}^2$ is blown up at a point $p \notin D$, then the order remains del Pezzo only if $e = 2$. Our work here is independent from but similar to what is done in [2].

Lemma 4.3.2. *Let $\mathcal{X}$ be a terminal order on $Z = \mathbb{P}^2$ and let D be a discriminant of degree 3 with the ramification degree e. Also let $f : Z' \to Z$ be a blowup at a point not in D. Then the associated order $\mathcal{Y}$ over Z' is del Pezzo if and only if $e = 2$.*

Proof. Let $e = 2$ and let $f : Z' \to Z$ be a blowup at the point $p \notin D$ and let C' be an effective curve in Z'. Then

$$K_{Z'} + \Delta' \equiv f^*(K_Z + \Delta) + E,$$

and

$$\begin{aligned}(K_{Z'} + \Delta')C' &= (f^*(K_Z + \Delta) + E)C' \\ &= f^*(K_Z + \Delta)C' + EC' \\ &= \left(-3H + \left(1 - \frac{1}{2}\right) 3H\right) f_*C' + EC' \\ &= -\frac{3}{2}d + r,\end{aligned}$$

where $d = \deg(f_*C')$ and r is the multiplicity of C' at p which is not greater than d. So $(K_{Z'} + \Delta')C' < 0$. Now we need to find $(K_{Z'} + \Delta')^2$.

$$\begin{aligned}(K_{Z'} + \Delta')^2 &= (f^*(K_Z + \Delta) + E)^2 \\ &= (K_Z + \Delta)^2 - 1 \\ &= (-3H + (1 - \frac{1}{e})3H)^2 - 1 \\ &= (\frac{-3}{e}H)^2 - 1 \\ &= \frac{9}{e^2} - 1.\end{aligned}$$

Therefore $\frac{9}{e^2} - 1 > 0$ if and only if $e < 3$. $\square$

Proposition 4.3.3. *Let $\mathcal{X}$ be a terminal order on $Z = \mathbb{P}^2$ and let D be a discriminant of degree 3. Also Let $Z' \xrightarrow{f} Z$ be two blowups at the points p and q and let $\mathcal{Y}$ be the associated maximal order on Z' obtained by the blowups. If any of p or q is not in D then $\mathcal{Y}$ is not del Pezzo.*

Proof. By Lemma 4.3.2 we know if $e \neq 2$, then the order is not del Pezzo. So let $e = 2$ and without loss of generality let us assume $p \notin D$. Then

$$K_{Z''} + \Delta'' \equiv h^*(K_{Z'} + \Delta') + E_p + aE_q,$$

where a depends on q whether it is in the ramification divisor or not.

$$\begin{aligned}(K_{Z'} + \Delta')^2 &= (f^*(K_Z + \Delta) + E_p + aE_q)^2 \\ &= (K_Z + \Delta)^2 - 1 - a^2 \\ &= \frac{9}{2^2} - 1 - a^2 > 0.\end{aligned}$$

For any effective curve $C' \in Z'$,

$$\begin{aligned}(K_{Z'} + \Delta')C' &= (f^*(K_Z + \Delta) + E_p + aE_q)C' \\ &= f^*(K_Z + \Delta)C' + E_pC' + aE_qC' \\ &= (-3H + (1 - \frac{1}{2})3H)h_*C' + E_pC' + aE_qC' \\ &= -\frac{3}{2}d + r_p + ar_q,\end{aligned}$$

where r_p and r_q are multiplicities of $C := f_*C'$ at p and q, respectively, and $d = \deg(C)$. So $r_p + ar_q \leq (1 + a)d$ and equality holds if C is a d-tuple line going through p and q. So for a d-tuple line C going through p and q, $(K_{Z'} + \Delta')\widetilde{C} \geq 0$ since $a \geq \frac{1}{2}$, i.e. $\mathcal{Y}$ is not del Pezzo. $\square$

Proposition 4.3.3 indicates that if a point $p \notin D$ is blown up, then the order remains del Pezzo if $e = 2$ and further, no more blowups are allowed either in or not in D. So the only remaining case here for degree 3 discriminants is to know how many points in D and what configurations of blowups preserve the order being del Pezzo. We will see that the classification of the points is very similar to the commutative case, which says blowups of up to 8 points in general position is allowed.

Definition 4.3.4. *Let $\Sigma = \{p_1, \cdots, p_n\}$ be a set of distinct points of $\mathbb{P}^2$. We say Σ is in general position if*

1. *No three points lie on a line;*

2. *No six points lie on a conic.*

Theorem 4.3.5. *Let $\mathcal{X}$ be a terminal order on $Z = \mathbb{P}^2$ with the ramification divisor D of degree 3 and the ramification degree e. Let $f : Z' \to Z$ be a sequence of blowups at the points $\Sigma = \{p_1, \cdots, p_n\}$. Then the associated maximal order $\mathcal{Y}$ over Z' is del Pezzo if and only if one of the following occurs.*

1. *$\Sigma \subset D$, then Σ is in general position and $n < 9$;*

2. *$\Sigma \not\subset D$, then $n = 1$, $p_1 \notin D$ and $e = 2$.*

Proof. The second case is actually is what we showed in Proposition 4.3.3. So we only need to prove the case that $\Sigma \subset D$.

Let $f : Z' \to Z$ be a sequence of blowups at the points $\Sigma = \{p_1, \cdots, p_n\} \subset D$ and let E_i be the expetional curve obtained by the blowup at p_i. By abuse of notation, we assume $E_i \subseteq Z'$, but we actually mean the proper transform of E_i. Then

$$\begin{aligned}
K_{Z'} + \Delta' &\equiv f^*(K_{Z'} + \Delta') + \frac{1}{e}\sum_{i=1}^{n} E_i \\
(K_{Z'} + \Delta')^2 &= (f^*(K_Z + \Delta) + \frac{1}{e}\sum_{i=1}^{n} E_i)^2 \\
&= (K_Z + \Delta)^2 - \frac{n}{e^2} \\
&= \frac{9}{e^2} - \frac{n}{e^2} > 0.
\end{aligned}$$

Therefor $n < 9$. Now let $C' = C_0 + \sum_{i=1}^{n} a_i E_i$ be an effective curve in Z', where for every i, $C_0 - E_i$ is not effective. Let $C := f_* C' = f_* C_0$, then

$$\begin{aligned}
(K_{Z'} + \Delta')C' &= (f^*(K_Z + \Delta) + \frac{1}{e}\sum_{i=1}^{n} E_i)C' \\
&= (K_Z + \Delta) f_* C' + \frac{1}{e}\sum_{i=1}^{n}(E_i C') \\
&= \frac{-3}{e} d + \frac{1}{e}\sum_{i=1}^{n}(E_i C_0) - \frac{1}{e}\sum_{i=1}^{n} a_i \\
&= \frac{-3}{e} d + \frac{1}{e}\sum_{i=1}^{n} m_{p_i}(C) - \frac{1}{e}\sum_{i=1}^{n} a_i,
\end{aligned}$$

where $d = \deg(C)$, and $m_{p_i}(C)$ is the multiplicity of C at p_i. We need to find $\sum_{i=1}^{n} m_{p_i}(C)$ which is the total multiplicities of C at Σ. Let $C = n_1 C_1 + n_2 C_2 + \cdots + n_k C_k$ be an effective curve in $Z = \mathbb{P}^2$ with n_1 lines, n_2 conics, etc. Then as $n < 9$ and the points are in general position, each line intersects at most 2 points, each conic intersects at most 5 points and the other can go through all the n points. Then

$$m_{p_1,\ldots,p_8}(C) \leq 2n_1 + 5n_2 + 8n_3 + \cdots 8n_k.$$

Moreover $\deg(D) = n_1 + 2n_2 + \cdots + kn_k$, thus

$$\begin{aligned}(K_{Z'} + \Delta')C' &= \frac{-3}{e}d + \frac{1}{e}\sum_{i=1}^{n} m_{p_i} - \frac{1}{e}\sum_{i=1}^{n} a_i \\ &\leq \underbrace{-3\frac{n_1 + \cdots + kn_k}{e} + \frac{2n_1 + 5n_2 + 8n_3 + \cdots 8n_k}{e}}_{\text{strictly less than } 0} - \frac{1}{e}\sum_{i=1}^{8} a_i.\end{aligned}$$

□

Degree 4 and 5 Ramification Divisors

In this section we blowup maximal orders on $\mathbb{P}^2$ with degree 4 or 5 ramification divisors. We denote the ramification divisors by $D = \cup D_i$, and d indicates the degree of D. According to Table 4.1, ramification degrees are all equal, say e, and $e = 2$ or $e = 3$ if $d = 4$ and $e = 2$ if $d = 5$.

For terminal del Pezzo order $\mathcal{X}$ on $Z = \mathbb{P}^2$ let $D = \cup D_i$ denote the ramification divisor and let $d = \deg D$. If $f : Z' \to Z$ is a blowup at a point p. Then we get the following equation

$$K_{Z'} + \Delta' \equiv f^*(K_Z + \Delta) + aE',$$

where by Lemma 4.3.1 $a = 1$ if $p \notin D$ and $a = \dfrac{1}{e}$ if $p \in D$.

Theorem 4.3.6. *Let $\mathcal{X}$ be a terminal del Pezzo order on $Z = \mathbb{P}^2$ with the ramification divisor $D = \cup D_i$ of degree 4. Also let $f : Z' \to Z$ be a blowup at a point p and let $\mathcal{Y}$ be the associated maximal order over Z'. Then $\mathcal{Y}$ is del Pezzo if and only if $p \in D$ and $e = 2$. Moreover $\mathcal{Y}$ can not be blown up to a del Pezzo order.*

Proof. Let $\mathcal{X}$ be a terminal del Pezzo order on $Z = \mathbb{P}^2$ with the ramification divisor $D = \cup D_i$ of degree 4. Let $f : Z' \to Z$ be a blowup at a point p and let $\mathcal{Y}$ be the associated maximal order over Z'.

$$\begin{aligned}
(K_{Z'} + \Delta')^2 &= (f^*(K_Z + \Delta) + aE')^2 \\
&= (K_Z + \Delta)^2 - a^2 \\
&= \left(-3H + \left(1 - \frac{1}{e}\right) 4H\right)^2 - a^2 \\
&= \left(\left(1 - \frac{4}{e}\right) H\right)^2 - a^2 \\
&= 1 - \frac{8}{e} + \frac{16}{e^2} - a^2.
\end{aligned}$$

$e = 2$ or $e = 3$. If $a = 1$, then $(K_{Z'} + \Delta')^2 \leq 0$. Therefore $a = \dfrac{1}{e}$ and $(K_{Z'} + \Delta')^2 = 1 - \dfrac{8}{e} + \dfrac{16}{e^2} - \dfrac{1}{e^2}$ which is zero for $e = 3$ and greater than zero if $e = 2$. Now we show $\mathcal{Y}$ is del Pezzo if $e = 2$. Let C' be an effective curve in Z' and let $C = f_*C'$.

$$\begin{aligned}
(K_{Z'} + \Delta')C' &= (f^*(K_Z + \Delta) + \frac{1}{2}E)C' \\
&= f^*(K_Z + \Delta)C' + \frac{1}{2}EC' \\
&= \left(-3H + \left(1 - \frac{1}{2}\right) 4H\right) f_*C' + \frac{1}{2}EC' \\
&= (-H)f_*C' + \frac{1}{2}EC' \\
&\leq -d + \frac{1}{2}m_p(C) \\
&\leq -\frac{1}{2}d,
\end{aligned}$$

where $d = \deg(C)$ and $m_p(C)$ is the multiplicity of C at p.

Now let $f : Z' \to Z$ be blowups at the points p and q in D. And let C' be an effective curve in Z' and let $C = f_*(C')$

$$\begin{aligned}
K_{Z'} + \Delta' &\equiv f^*(K_Z + \Delta) + \frac{1}{2}E_p + \frac{1}{2}E_q; \\
(K_{Z'} + \Delta')^2 &= \left(f^*(K_Z + \Delta) + \frac{1}{2}E_p + \frac{1}{2}E_q\right)^2 \\
&= (K_Z + \Delta)^2 - \frac{1}{4} - \frac{1}{4} \\
&= 1 - \frac{8}{2} + \frac{16}{4} - \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

and

$$
\begin{aligned}
(K_{Z'} + \Delta')C' &= \left(f^*(K_Z + \Delta) + \frac{1}{2}E_p + \frac{1}{2}E_q \right) C' \\
&= f^*(K_Z + \Delta)C' + \frac{1}{2}(E_p + E_q)C' \\
&\leq -d + d,
\end{aligned}
$$

where equality occurs when C is a d-tuple line passing p and q, $d = \deg(C)$ and $m_p(C)$ and $m_q(C)$ are the multiplicities of C at p and q, respectively. $\square$

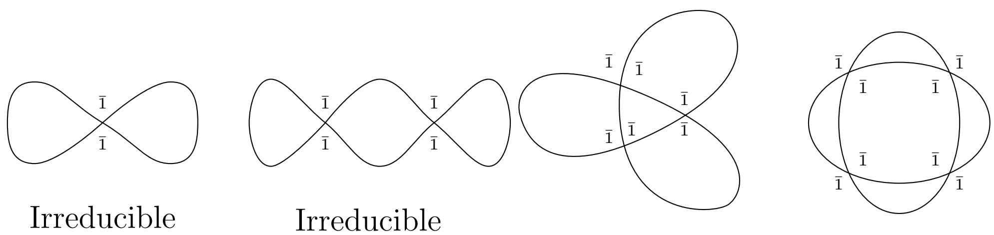
Theorem 4.3.7. *Let $\mathcal{X}$ be a terminal del Pezzo order on $Z = \mathbb{P}^2$ with the ramification divisor $D = \cup D_i$ of degree 5. Also let $f : Z' \to Z$ be a blowup at a point p and let $\mathcal{Y}$ be the associated maximal order over Z'. Then $\mathcal{Y}$ is not del Pezzo.*

Proof. Let $\mathcal{X}$ be a terminal del Pezzo order on $Z = \mathbb{P}^2$ with the ramification divisor $D = \cup D_i$ of degree 5. And let $f : Z' \to Z$ be a blowup at a point p and let $\mathcal{Y}$ be the maximal order over Z'.

$$
\begin{aligned}
(K_{Z'} + \Delta')^2 &= (f^*(K_Z + \Delta) + aE)^2 \\
&= (K_Z + \Delta)^2 - a^2 \\
&= (-3H + (1 - \frac{1}{e})5H)^2 - a^2 \\
&= (-\frac{1}{2}H)^2 - a^2 \\
&= \frac{1}{4} - a^2,
\end{aligned}
$$

where $a = 1$ or $a = \frac{1}{2}$, both of which make the self intersection $(K_{Z'} + \Delta')^2$ less than or equal to zero. $\square$

Theorem 4.3.8. *The followings figures give us a complete list of degree* 4 *terminal ramification divisors on* $\mathbb{P}^2$.



Irreducible quartic with one node	Irreducible quartic with two nodes	Irreducible quartic with three nodes	Two conics crossing at 4 points.

Figure 4.2: Quartic ramification configurations with $e = 2$

Before proving, the reader should note that since there are no branch points for smooth cases they are not interesting. So the smooth ramification divisors are ignored. See Appendix A for more details about the types of singularities.

Proof. Having Definition 3.3.1, by Proposition 3.3.3 the given ramification divisors are ramification of orders and one can check that they satisfy the hypothesis of ramification divisors of terminal orders. We only need to show that other cases are not possible. Let $e = 2$ and let D contains a line ℓ. Then ℓ intersects $D - \ell$ three times. By the hypothesis of terminal orders we know

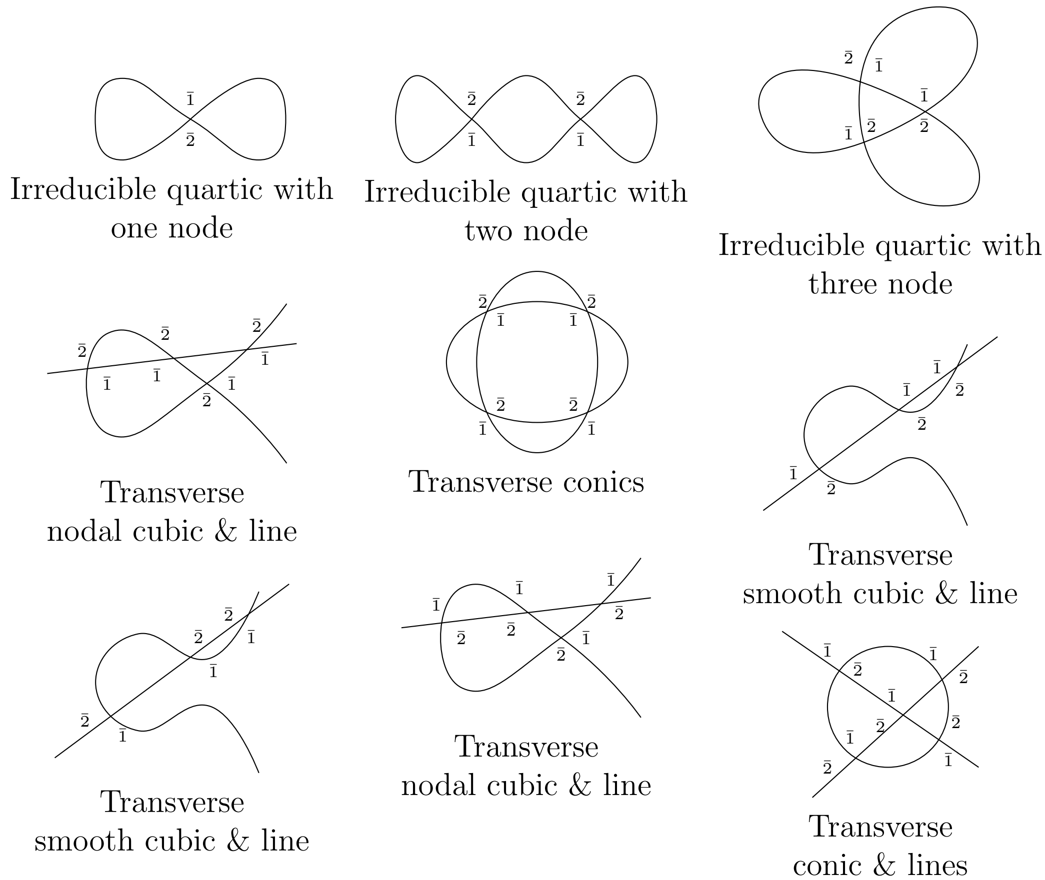


Figure 4.3: Quartic ramification configurations with $e = 3$

ℓ should ramify at all the three branch points. So the ramification indices for ℓ at its branch points are 1. But it can not occur since they must add up to 2. This finishes the proof for the case $e = 2$.

For $e = 3$ the proof is similar and straight forward. We work on the case where D is a configuration of four lines as an example.

Let D be a configuration of four lines. So each line intersect each of the other lines once. Meaning that each line has exactly three branch points. Since $e = 3$, then at each branch point the ramification indices of the two intersecting curves must add up to 3. So one of them has ramification degree 2 and the other one has ramification degree 1. Further, since the ramification indices of any curve must add up to 3 then each curve has ramification indices as $(1, 1, 1)$ or $(2, 2, 2)$. Now let ℓ_1, ℓ_2 and ℓ_3 intersect at points p_{12}, p_{13}, and p_{23}. Without loss of generality let the ramification degree of ℓ_1 at p_{12} and consequently at p_{13} be 1. Then the ramification indices of ℓ_2 and ℓ_3 are 2. But it is not possible as we get a contradiction at the point p_{23}. $\square$

4.4 Terminal Del Pezzo Orders on $\mathbb{P}^1 \times \mathbb{P}^1$

In this section we classify minimal terminal del Pezzo orders on $\mathbb{P}^1 \times \mathbb{P}^1$ and then we find blowups of the minimal models that remain del Pezzo. We let $\mathcal{X}$ denote a minimal terminal del Pezzo order on $\mathbb{P}^1 \times \mathbb{P}^1$ and let $D = \cup_i D_i$ be the discriminant with ramification degrees e_i. The following is a nice restriction for the discriminants of del Pezzo orders on $\mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 4.4.1. *[6, Proposition 30] If D is the discriminant of a del Pezzo order on the ruled surface $\mathbb{P}^1 \times \mathbb{P}^1$, then $2C_0 + 2F \leq [D] \leq 3C_0 + 3F$ where C_0 and F denote fibres of $\mathbb{P}^1 \times \mathbb{P}^1$ in the rwo different directions of ruling, say C_0 is a section of the projection to the first component and F is a section of the projection to the second component, and $[D]$ denotes the* *divisor class of D. Moreover the ramification degrees for D are equal, say e, and $e = 2$ if $2C_0 + 2F < [D] \leq 3C_0 + 3F$. Conversely any maximal order with such ramification data is del Pezzo.*

Proposition 4.4.2. *Let $\mathcal{X}$ be a terminal del Pezzo order over $\mathbb{P}^1 \times \mathbb{P}^1$ and let D be its discriminant. Then if $f : Z' \to \mathbb{P}^1 \times \mathbb{P}^1$ is a blowup at a point $p \notin D$, then the order $\mathcal{X}'$ associated to Z' is not del Pezzo.*

Proof. Let $\mathcal{X}$ be a terminal del Pezzo order on $\mathbb{P}^1 \times \mathbb{P}^1$ and let D be its discriminant with the ramification degree e, then the canonical divisor is as the following

$$\begin{aligned} K_{\mathcal{X}} = K_{\mathbb{P}^1 \times \mathbb{P}^1} + \Delta &\equiv (-2C_0 - 2F) + \left(1 - \frac{1}{e}\right)(aC_0 + bF), \text{ for suitable } a \text{ and } b, \\ &= \left(a - 2 - \frac{a}{e}\right) C_0 + \left(b - 2 - \frac{b}{e}\right) F. \end{aligned}$$

Further, let $f : Z' \to \mathbb{P}^1 \times \mathbb{P}^1$ represents a blowup at a point $p \notin D$. Then if E is the corresponding exceptional curve, we have the following equations

$$\begin{aligned} K_{Z'} + \Delta' &\equiv f^*(K_{\mathbb{P}^1 \times \mathbb{P}^1} + \Delta) + E, \\ &= f^*\left(\left(a - 2 - \frac{a}{e}\right) C_0 + \left(b - 2 - \frac{b}{e}\right) F\right) + E. \end{aligned}$$

Let $a = b = 2$. Then

$$\begin{aligned}(K_{Z'} + \Delta')^2 &= (f^*(K_{\mathbb{P}^1\times\mathbb{P}^1} + \Delta))^2 - 1,\\ &= \left(f^*\left(-\frac{2}{e}C_0 - \frac{2}{e}F\right)\right)^2 - 1,\\ &= \frac{8}{e^2} - 1.\end{aligned}$$

Therefore in order for a blowup at a point out of D to keep the order del Pezzo the ramification degree of the discriminant should be $e = 2$. Also we can see that if $f : Z' \to \mathbb{P}^1 \times \mathbb{P}^1$ represents two blowup at the points $p, q \notin D$ then $(K_{Z'} + \Delta')^2 = 0$ and the order will not be del Pezzo. But for the first blowup we still need to check if the intersection of the effective curves with the canonical divisor is negative. So let $C = C' + rE$ be an effective curve in Z' where $C' - E$ is not effective. Also let $f_*C' = aC_0 + bF$ for non-negative integers a and b. Then

$$\begin{aligned}(K_{Z'} + \Delta')C &= (f^*(K_{\mathbb{P}^1\times\mathbb{P}^1} + \Delta))C + E.C,\\ &= (-C_0 - F)f_*C + (f_*C)_p - r,\\ &= (-C_0 - F)(aC_0 + bF) + (aC_0 + bF)_p - r,\\ &\leq -a - b + (aC_0 + bF)_p,\end{aligned}$$

where $(aC_0 + bF)_p$ is the multiplicity of $(aC_0 + bF)$ at p. In the last equation equality holds if C is the proper transform of an effective curve in $\mathbb{P}^1 \times \mathbb{P}^1$. Now let C be the proper transform of a line $l \equiv F$ going through p, then we see that $(K_{Z'} + \Delta')C = 0$. Therefore the order will not remain del Pezzo after a blowup at a point out of the discriminant $D \equiv 2C_0 + 2F$. Now let $a = b = 3$. Then

$$\begin{aligned}(K_{Z'} + \Delta')^2 &= (f^*(K_{\mathbb{P}^1 \times \mathbb{P}^1} + \Delta))^2 - 1, \\ &= \left(f^*\left(-\frac{1}{2}C_0 + -\frac{1}{2}F\right)\right)^2 - 1, \\ &= 2\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) - 1, \\ &= \frac{1}{2} - 1 < 0\end{aligned}$$

Now without loss of generality let us assume that $a = 2$ and $b = 3$. Then $e = 2$ and we have the following equations

$$\begin{aligned}(K_{Z'} + \Delta')^2 &= (f^*(K_{\mathbb{P}^1 \times \mathbb{P}^1} + \Delta))^2 - 1, \\ &= \left(f^*\left(-C_0 - \frac{1}{2}F\right)\right)^2 - 1, \\ &= 2\,(-1)\left(-\frac{1}{2}\right) - 1 = 0\end{aligned}$$

$\square$

Now let us blow up the order $\mathcal{X}$ at a point p in the discriminant. If $f : Z' \to$

$\mathbb{P}^1 \times \mathbb{P}^1$ represents such a blowup, then

$$\begin{aligned} K_{Z'} + \Delta' &\equiv f^*(K_{\mathbb{P}^1\times\mathbb{P}^1} + \Delta) + \frac{1}{e}E, \\ &= f^*\left(\left(a - 2 - \frac{a}{e}\right)C_0 + \left(b - 2 - \frac{b}{e}\right)F\right) + \frac{1}{e}E. \end{aligned}$$

We again work on three different cases, which are $a = b = 2$, $a = 2$ and $b = 3$, and $a = b = 3$. Note that in the last two cases $e = 2$.

Theorem 4.4.3. *Let $\mathcal{X}$ be an order over $\mathbb{P}^1 \times \mathbb{P}^1$ and let D be its discriminant such that $2C_0 + 2F < [D]$ where C_0 and F are fibres of the centre of the two different projections of $\mathbb{P}^1 \times \mathbb{P}^1$. Then any blowup of the order at a point in D fails to be del Pezzo.*

Proof. Let $f : Z' \to \mathbb{P}^1 \times \mathbb{P}^1$ represent a blowup at a point $p \in D$ with the exceptional curve E where $2C_0 + 2F < [D \equiv aC_0 + bF] \leq 3C_0 + 3F$. Then we know the ramification degrees for D are $e = 2$. Without loss of generality we assume $a = 3$ and consider the effective curve $C = \tilde{F}$ where F is a fibre passing p.

$$
\begin{aligned}
(K_{Z'} + \Delta')C &= (f^*(K_{\mathbb{P}^1\times\mathbb{P}^1} + \Delta))C + \frac{1}{2}E.C \\
&= \left(-\frac{1}{2}C_0 + \left(b - 2 - \frac{b}{2}\right)F\right) f_*C + \frac{1}{2}m_p(f_*C) \\
&= \left(-\frac{1}{2}C_0 + \left(b - 2 - \frac{b}{2}\right)F\right) F + \frac{1}{2}m_p(F) \\
&\leq -\frac{1}{2} + \frac{1}{2}m_p(F) \\
&= 0
\end{aligned}
$$

$\square$

Now the only remaining case for orders obtained by the blowups of the order over $\mathbb{P}^1 \times \mathbb{P}^1$ is where

Definition 4.4.4. *Let Σ be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. The points of Σ are in general position if any curve of the form $aC_0 + bF$ contains less than $2(a+b)$ of the points.*

Theorem 4.4.5. *Let $\mathcal{X}$ be an order over $\mathbb{P}^1 \times \mathbb{P}^1$ and let D be its discriminant such that $[D] = 2C_0 + 2F$ where C_0 and F are two intersecting fibres of the centre. Further let $f : Z' \to \mathbb{P}^1 \times \mathbb{P}^1$ be a sequence of blowups at the points $\Sigma = \{p_1, \cdots, p_n\} \subset D$. Then the associated maximal order $\mathcal{Y}$ over Z' is del Pezzo if and only if Σ is in general position and $n \leq 7$.*

Before to start the proof we need to define a notation for our convenience. If

C is a curve and Σ a set of points we define $m_\Sigma(C)$ to be the totall multiplicity of C at the points of Σ.

Proof. Let $f : Z' \to \mathbb{P}^1 \times \mathbb{P}^1$ be blowups at points $\Sigma = \{p_1, \cdots, p_n\}$ and let $E_1, \cdots, E_n$ be the corresponding exeptional curves. Then

$$\begin{aligned} K_{Z'} + \Delta' &\equiv f^*(K_{\mathbb{P}^1\times\mathbb{P}^1} + \Delta) + \sum_{i=1}^n \frac{1}{e} E_i, \\ &= f^*\left(-\frac{2}{e}C_0 - \frac{2}{e}F\right) + \sum_{i=1}^n \frac{1}{e}E_i. \end{aligned}$$

So the self intersection is as the following

$$\begin{aligned} (K_{Z'} + \Delta')^2 &= (f^*(K_{\mathbb{P}^1\times\mathbb{P}^1} + \Delta))^2 + (\sum_{i=1}^n \frac{1}{e}E_i)^2, \\ &= \left(f^*\left(-\frac{2}{e}C_0 - \frac{2}{e}F\right)\right)^2 - \sum_{i=1}^n \frac{1}{e^2}, \\ &= \frac{8}{e^2} - \frac{n}{e^2}. \end{aligned}$$

Therefore $n \leq 7$. Further, let $C = C' + \sum_{i=1}^n r_i E_i$ be an effective curve where

C' is the proper transform of an effective curve in Z, say $C' = \widetilde{aC_0 + bF}$.

$$
\begin{aligned}
(K_{Z'} + \Delta')C &= (f^*(K_{\mathbb{P}^1\times\mathbb{P}^1} + \Delta))C + \left(\sum_{i=1}^{n} \frac{1}{e} E_i\right) C \\
&= (-\frac{2}{e}C_0 - \frac{2}{e}F)f_*C + \frac{1}{e}m_\Sigma(f_*C) - \sum_{i=1}^{n} \frac{r_i}{e} \\
&= (-\frac{2}{e}C_0 - \frac{2}{e}F)(aC_0 + bF) + \frac{1}{e}m_\Sigma(aC_0 + bF) - \sum_{i=1}^{n} \frac{r_i}{e} \\
&\leq \frac{2}{e}(-a-b) + \frac{1}{e}m_\Sigma(aC_0 + bF) \\
&< 0 \qquad\qquad (*)
\end{aligned}
$$

where inequality in $(*)$ comes from the fact that the point of Σ are in general position. $\square$

Followings are the lists of possible minimal terminal del Pezzo orders on $\mathbb{P}^1 \times \mathbb{P}^1$. In the descriptions, by (a, b) we mean a curve in the divisor class $aC_0 + bF$.

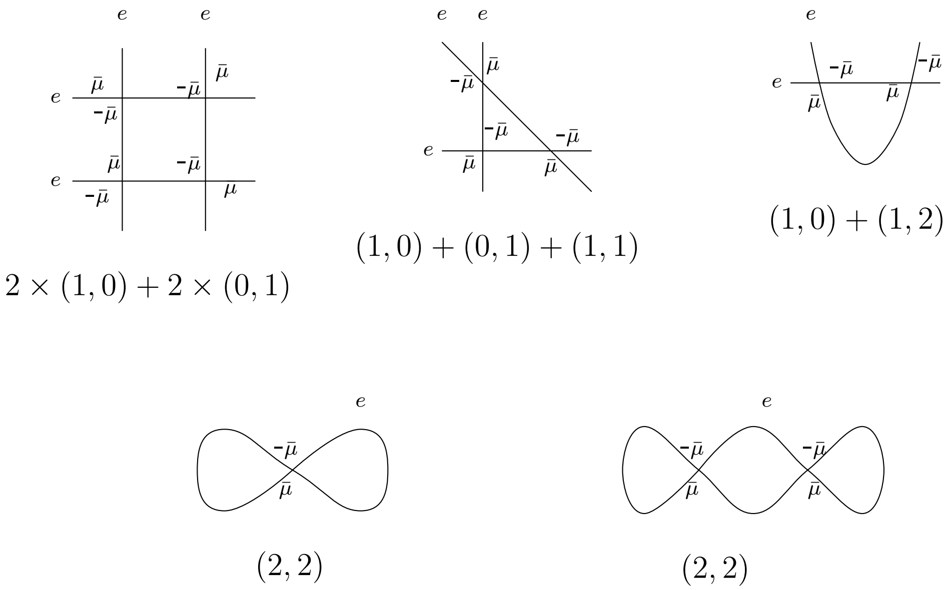


Figure 4.4: $[D] = 2C_0 + 2F$

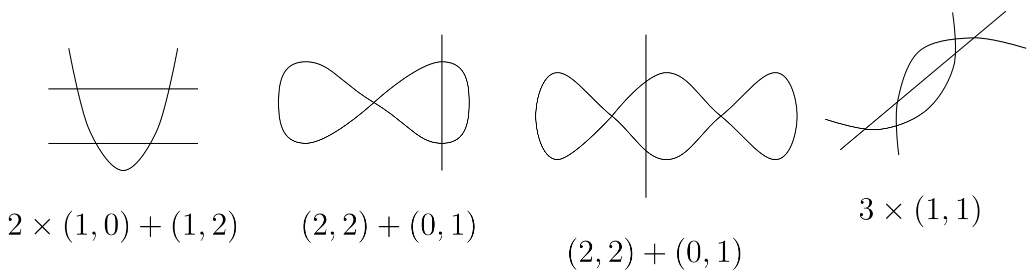


Figure 4.5: $2C_0 + 2F < [D] \leq 3C_0 + 3F, e = 2$

In Figure 4.5 all the ramification degrees are $e = 2$ and all the indices for the branch points are $\bar{1}$.

Chapter 5

Minimal Models of Canonical Del Pezzo Orders

In this chapter we will classify del Pezzo orders with canonical (not terminal) singularities. We will follow the approach in the commutative case for our classifications where it is appropriate, however it may not work the same in general. We saw in Chapter 3 that for an order to be del Pezzo it only depends on the centre and the ramification divisors, further, we mentioned that when we have an order with canonical singularities its centre should have no worse than canonical singularities. So the classification for orders with canonical singularities is doubtlessly related to classification of canonical surfaces.

Now Let X denote a surface with canonical singularities. Then the following diagram exists

$$\begin{tikzcd}
Y \arrow[dr, "f"] \arrow[dd, "g"'] & \\
& X \arrow[dl, dashed, "gof^{-1}"] \\
W &
\end{tikzcd}$$

Figure 5.1: Resolution of canonical surfaces to the minimal model

where $f : Y \to X$ is the unique minimal resolution of X to a smooth surface Y, and $g : Y \to W$ contracts Y to a minimal model. Minimal model means that W is one of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{F}_n$ for $n \neq 1$. For more details about minimal model program for surfaces see [11].

5.1 Resolution of Canonical Orders

Let $\mathcal{X}$ be a maximal order on Z and let D_Z be its discriminant. Also let $\Delta_Z = \sum_i \left(1 - \frac{1}{e_i}\right) D_i$ be the ramification configuration of $\mathcal{X}$. Whenever it is clear we simply write D and Δ to refer to D_Z and Δ_Z. Also we are always working on Gorenstein surfaces as the centres of the orders.

Definition 5.1.1. *Let $\mathcal{X}$ be a maximal order on Z. Then by [5, Corollary 3.6] there exists a sequence of blowups $f : \mathcal{Y} \to \mathcal{X}$, called resolution of $\mathcal{X}$, where $\mathcal{Y}$ is a terminal order. The resolution $f : \mathcal{Y} \to \mathcal{X}$ is called minimal if $(K_{\mathcal{Y}})E \geq 0$ for every f-exceptional curve E. See the proof of existence of the minimal resolution in [4, Theorem 2.15].*

Definition 5.1.2. *Let $\mathcal{X}$ be a maximal order and let $f : \mathcal{Y} \to \mathcal{X}$ be a resolution of $\mathcal{X}$. Then we have the following equation*

$$K_{\mathcal{Y}} = K_{\mathcal{X}} + \sum_i a_i E_i,$$

where E_i are f-exceptional curves. Also $K_{\mathcal{X}} = K_Z + \Delta_Z$ and $K_{\mathcal{Y}} = K_{Z'} + \Delta_{Z'}$. The order $\mathcal{X}$ has canonical singularities if $a = \min\{a_i\} \geq 0$. We shortly call $\mathcal{X}$ a canonical order.

Proposition 5.1.3. *[4, Theorem 6.5] Let $\mathcal{X}$ be an order over Z with ramification divisor D and let $p \in Z$ be a closed point. Then the singularity at p is canonical if D in an étale neighbourhood of p is of one of the types listed in the Table 5.1. Here e_D denotes the ramification degree and e_P denotes branch ramification index. Note that the list excludes terminal singularities that we discussed before but they are also considered as canonical singularities. Further, an order $\mathcal{X}$ is canonical if at every point it has no worse that canonical singularities.*

Type	Centre	Discriminant	e_D	e_P
$A_{1,2,\xi}$	smooth	xy	$2e, 2e$	e, e
BL_n	smooth	$y^2 - x^{2n+1}$	2	1
B_n	smooth	$(y + x^n)(y - x^n)$	$2, 2$	$1, 1$
L_n	smooth	$(y + x^{n+1})(y - x^{n+1})$	$2, 2$	$2, 2$
DL_n	smooth	$x(y^2 - x^{2n-1})$	$2, 2$	$2, 2$
BD_n	smooth	$x(y + x^{n-1})(y - x^{n-1})$	$2, 2, 2$	$2, 2, 1$
A_n	$x^2 + y^2 + z^{n+1}$	1	$-$	$-$
D_n	$x^2 + zy^2 + z^{n-1}$	1	$-$	$-$
E_6	$x^2 + y^3 + z^4$	1	$-$	$-$
E_7	$x^2 + y^3 + yz^3$	1	$-$	$-$
E_8	$x^2 + y^3 + z^5$	1	$-$	$-$
$A_{n,\xi}$	$xy = z^{n+1}$	z	e, e	e, e

Table 5.1: Configurations of Exceptional and Ramification Curves in Canonical Orders

In Table 5.1, note that for A_n, D_n, E_6, E_7, and E_8 the centres are singular, indeed the singularities are in the centres and there is no ramification curve at the single point.

5.2 The Minimal Model of Terminal Orders for Del Pezzo Canonical Orders

Let us have a canonical order $\mathcal{X}$. Reviewing the diagram in Figure 5.1, as in the commutative case, we now need to find the minimal model for the terminal order $\mathcal{Y}$ defined in 5.1.1. In this section we recall the work of

Chan and Ingalls in [5] to define a minimal model $\mathcal{W}$ of a terminal order $\mathcal{Y}$. This gives a complete diagram for minimal terminal resolutions of canonical orders. Later we seek to repeat the same procedure for del Pezzo canonical orders. We will show that if a canonical order $\mathcal{X}$ is del Pezzo, then its minimal model denoted by $\mathcal{W}$ is almost del Pezzo, see Definition 5.2.2. We will classify canonical del Pezzo orders via classification of minimal terminal almost del Pezzo orders.

Theorem 5.2.1. *[4, Proposition 6.1] Let $\mathcal{X}$ be a canonical order and let $f : \mathcal{Y} \to \mathcal{X}$ be its minimal resolution. Then $K_{\mathcal{Y}} = f^*K_{\mathcal{X}}$.*

Definition 5.2.2. *Let $\mathcal{X}$ be a maximal order over a Gorenstein surface Z. We call $\mathcal{X}$ almost del Pezzo if $K_{\mathcal{X}}^2 > 0$, and for every effective curve $C \in Z$, $-K_{\mathcal{X}}C \geq 0$.*

Theorem 5.2.3. *Let $\mathcal{X}$ be a del Pezzo canonical order and let $f : \mathcal{Y} \to \mathcal{X}$ be its minimal resolution. Then $\mathcal{Y}$ is almost del Pezzo.*

Proof. Let $\mathcal{X}$ be a del Pezzo canonical order and let $f : \mathcal{Y} \to \mathcal{X}$ be its minimal resolution. Then by theorem 5.2.1 we have

$$K_{\mathcal{Y}}^2 = (f^*K_{\mathcal{X}})^2 = K_{\mathcal{X}}^2 > 0.$$

Now let C be an effective curve in $Z_{\mathcal{Y}}$. f_*C is either an effective divisor or some finite points in $Z_{\mathcal{X}}$ depending on whether C is a non exceptional curve.

Then we have

$$-K_{\mathcal{Y}}C = -f^*K_{\mathcal{X}}C = -K_{\mathcal{X}}f_*C \geq 0,$$

where equality occurs only if f_*C is a set of points or equivalently only if C contains only f-exceptional curves. $\square$

Theorem 5.2.4. *Let $\mathcal{Y}$ be a terminal almost del Pezzo order over $Z_{\mathcal{Y}}$ and let $\mathcal{W}$ over $Z_{\mathcal{W}}$ be a minimal terminal order obtained by running the minimal model program on $\mathcal{Y}$. Then $\mathcal{W}$ is almost del Pezzo.*

Proof. If $\mathcal{Y}$ is a minimal terminal order it is almost del Pezzo by the assumption and we are done. So assume that $\mathcal{Y}$ is not minimal, then by blowing down some (-1)-curves, E_i's, we get a minimal terminal order. So by induction we only need to show that if $\mathcal{Y}$ is a terminal almost del Pezzo order and $g : Z_{\mathcal{Y}} \to Z_{\mathcal{W}}$ contracts a (-1)-curve E, then $\mathcal{W} := (g_*\mathcal{Y})^{**}$ is almost del Pezzo. To do so, we have

$$(g^*K_{\mathcal{W}})^2 > K_{\mathcal{Y}}^2 > 0$$

and for any effective curve C in $Z_{\mathcal{W}}$, we know g^*C is effective in $Z_{\mathcal{Y}}$. So

$$-K_{\mathcal{W}}C = -g^*K_{\mathcal{W}}g^*C = -(K_{\mathcal{Y}} - aE)g^*C = -K_{\mathcal{Y}}g^*C \geq 0.$$

$\square$

Lemma 5.2.5. *[6, Theorem 1] Let Z be a surface with canonical singularities and let C be an irreducible curve. If $(K_Z + C)C < 0$ and $K_Z C \geq 0$, then C is a smooth rational curve.*

Chan and Kulkarni showed that if an order $\mathcal{X}$ on a normal Gorenstein surface Z is del Pezzo, then the centre is del Pezzo. We want to generalize their result to almost del Pezzo orders. The proof is mostly the same, however we need to prove the following Lemma.

Lemma 5.2.6. *Let $\mathcal{X}$ be a maximal order on a normal Gorenstein surface Z. If $\mathcal{X}$ is almost del Pezzo, then for every irreducible curve C, $K_Z C \leq 0$.*

Proof. Chan and Kulkarni showed that if an order $\mathcal{X}$ on the centre Z is del Pezzo, then $K_Z C < 0$ for every irreducible curve $C \in Z$, [6, Theorem 12]. To do so, by contradiction it is assumed that there is an irreducible curve C, such that $K_{\mathcal{X}} C < 0$ but $K_Z C \geq 0$. Then the curve C is a smooth rational curve and it leads to a contradiction. Here we only need to show that if for any curve C, $K_{\mathcal{X}} C \leq 0$ and $K_Z C > 0$, then C is smooth rational. For then, the same contradiction would be reached.

Thus let $\Delta = \sum_i (1 - \frac{1}{e_i}) D_i$ be the ramification configuration for the order $\mathcal{X}$ and let C be an irreducible curve in Z. If C is not one of the ramification divisors D_i, then $\Delta C \geq 0$ and the arguments is proved by the following equation

$$K_X C = (K_Z + \Delta)C \leq 0.$$

If C is a ramification divisor, then without loss of generality we can assume that $C = D_1$ and the ramification degree for C, $e = e_1$. $\mathcal{X}$ is almost del Pezzo, so $K_{\mathcal{X}}C \leq 0$.

$$\begin{aligned} 0 \geq K_{\mathcal{X}}C &= \left(K_Z + \sum_i \left(1 - \frac{1}{e_i}\right) D_i\right) C \\ &= \left(\frac{1}{e}(K_Z C) + \left(1 - \frac{1}{e}\right)(K_Z + C)C + \sum_{i \neq 1} \left(1 - \frac{1}{e_i}\right) D_i\right) C. \end{aligned}$$

By contradiction, let $K_Z C > 0$. As $\left(\sum_{i \neq 1} \left(1 - \frac{1}{e_i}\right) D_i\right) C \geq 0$, then $(K_Z + C)C < 0$. So by Lemma 5.2.5 we conclude that C is a smooth rational curve. □

Theorem 5.2.7. *Let $\mathcal{X}$ be a maximal order on a normal Gorenstein surface Z. Then if $\mathcal{X}$ is almost del Pezzo, so is Z.*

Proof. $\mathcal{X}$ is almost del Pezzo, so $K_{\mathcal{X}}\Delta \leq 0$ and $0 < K_{\mathcal{X}}^2$, also by Lemma 5.2.6, $K_Z\Delta \leq 0$. Then

$$\begin{aligned} 0 < K_{\mathcal{X}}^2 &= K_{\mathcal{X}}(K_Z + \Delta) \\ &= K_{\mathcal{X}}K_Z + K_{\mathcal{X}}\Delta \\ &\leq K_{\mathcal{X}}K_Z \\ &= (K_Z + \Delta)K_Z \\ &= K_Z^2 + K_Z\Delta \\ &\leq K_Z^2. \end{aligned}$$

This together with Lemma 5.2.6 finishes the proof. □

Theorem 5.2.8. *Let $\mathcal{W}$ be a minimal terminal almost del Pezzo order on Z. Then we have one of the followings*

1. *$Z = \mathbb{P}^2$, and $\mathcal{W}$ is del Pezzo;*
2. *Z is a rational ruled surface. More precisely, $Z = \mathbb{F}_n$ for $n = 0, 1,$ or 2.*

Proof. By [5, Corollary 3.20] we know for minimal terminal orders, we have one of the following

1. $Z = \mathbb{P}^2$, and $\mathcal{W}$ is del Pezzo;
2. $Z \to C$ is a ruled surface for a smooth rationa curve C.

So we only need to show if for a minimal terminal almost del Pezzo order the later occurs, then the surface is rationally ruled and moreover it is $\mathbb{F}_0, \mathbb{F}_1$, or $\mathbb{F}_2$.

Let $\mathcal{W}$ be a minimal almost del Pezzo order on Z and let $\pi : Z \to C$ be the morphism surjecting Z to the curve C. We note the arithmetic genus of C

by g. By the genus formula for ruled surfaces we have

$$K_Z C_0 = n + 2g - 2 \text{ and}$$
$$K_Z^2 = 8(1-g),$$

where $n = -C_0^2$. By Theorem 5.2.7 we have that Z is almost del Pezzo, so $K_Z^2 > 0$ and $K_Z C_0 \leq 0$. Therefore $g = 0$ and $n \leq 2$, i.e. $Z_{\mathcal{W}} = \mathbb{F}_n$ for $0 \leq n \leq 2$. □

Classification of del Pezzo orders over $\mathbb{P}^2$ with singularities is given in [6] and we also described in Chapter 4 the classification for del Pezzo orders with terminal singularities. In this section we firstly resolved a canonical order and then contracted it to a minimal model which is almost del Pezzo and has terminal singularities. We can classify minimal terminal almost del Pezzo orders first and then classify canonical orders via this classification. That is what we do in the next section.

5.3 Minimal Almost Del Pezzo Terminal Orders on Rational Ruled Surfaces

In this section we classify minimal almost del Pezzo terminal orders on rational ruled surfaces $\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{F}_1$ and $\mathbb{F}_2$. We let Z denote any of these ruled surfaces if it is not specified which. Let us recall first some results from [5, 6].

Proposition 5.3.1. *[5, p: 450] Let $\mathcal{X}$ denote a minimal terminal order over a rational ruled surface Z. Also let $\cup D_i$ and $\{e_i\}$ denote its discriminant and ramification degrees. Also let C_0 be a minimal section and F a fixed fibre of the ruled surface Z. If $D = \cup D_i \equiv aC_0 + bF$ for non-negative integers a and b, then $a = 2$ or $a = 3$. Moreover the ramification degrees for divisors intersecting F are equal and when $a = 3$ they are 2.*

Proposition 5.3.2. *[6, Lemma 23] Let p be a prime integer dividing some ramification degree e_i and let p^{max} be the largest power of p dividing any of the ramification degrees. Let D_p be the union of all ramification divisors D_i whose ramification degrees are divisible by p^{max}, then $p_a(D_p) \geq 1$, where $p_a(D_p)$ denotes the arithmetic genus of D_p.*

Now let D be a ramification divisor of some order over the rational ruled surface $\mathbb{F}_n$. If p is a prime number dividing any ramification degree and $D_p = a_pC_0 + b_pF$ for some a_p and b_p, then by genus formula and Proposition 5.3.2 we have

$$2p_a(D_p) = (a_p - 1)(2b_p - na_p - 2) \geq 2. \qquad (5.1)$$

Remark 5.3.3. *Let $\mathcal{X}$ denote a minimal terminal order over a rational ruled surface Z and let $D = \cup D_i \equiv 3C_0 + bF$ be its ramification divisor. Then by Proposition 5.3.1 we know the ramification degree of the divisors intersecting F is 2. Further, by Proposition 5.3.2 and Equation 5.1 we know all other* *ramification degrees divide 2, therefore they are all equal.*

$\mathbf{Z}=\mathbb{P}^1 \times \mathbb{P}^1$

Let $\mathcal{X}$ be a minimal terminal order on $Z = \mathbb{P}^1 \times \mathbb{P}^1$ and let $D = \cup D_i \equiv aC_0 + bF$ be the discriminant and the ramification degrees, where C_0 and F indicate the fibres for the two different ways of ruling $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 5.3.4. *Let $\mathcal{X}$ be a minimal terminal order on $Z = \mathbb{P}^1 \times \mathbb{P}^1$ and let $D = \cup D_i \equiv aC_0 + bF$ be its discriminant. Then $2 \leq a, b \leq 3$ and all the ramification degrees $e_i = e$. Further, $e = 2$ if $2C_0 + 2F < [D]$.*

Proof. By Proposition 5.3.1 we know $2 \leq a, b \leq 3$. If $2C_0 + 2F < D$, then Remark 5.3.3 gives us the desired result. If $D \equiv 2C_0 + 2F$, then Proposition 5.3.1 finishes the proof.

□

Remark 5.3.5. *Let $\mathcal{W}$ be a minimal terminal almost del Pezzo order on $\mathbb{P}^1 \times \mathbb{P}^1$ with the discriminant D. Then by Theorem 5.3.4 we know $2C_0 + 2F \leq [D] \leq 3C_0 + 3F$ and all the ramification degrees are the same. Considering Proposition 4.4.1, we see that $\mathcal{W}$ is actually a del Pezzo order.*

$\mathbf{Z}=\mathbb{F}_1$

Now assume that $\mathcal{X}$ denotes a minimal terminal almost del Pezzo order on $Z = \mathbb{F}_1$. And let $\cup D_i$ and $\{e_i\}$ denote its discriminant and the ramification degrees. Also let C_0 be a fixed section and F a fixed fibre of the ruled surface, where $C_0^2 = -1$, $F^2 = 0$, $C_0.F = 1$, and the canonical bundle is $K_Z \equiv -2C_0 - 3F$. Then for suitable a_i and b_i we have $D_i \equiv a_iC_0 + b_iF$. We also set $D := \cup D_i \equiv aC_0 + bF$ for non negative integers a and b.

As $\mathcal{X}$ is a minimal terminal order and $C_0^2 < 0$, then $K_{\mathcal{X}}C_0 \geq 0$. On the other hand $\mathcal{X}$ is almost del Pezzo, so $K_{\mathcal{X}}C_0 \leq 0$, thus $K_{\mathcal{X}}C_0 = 0$.

$$\begin{aligned}
0 = K_{\mathcal{X}}C_0 &= \left(-2C_0 - 3F + \sum(1-\frac{1}{e_i})(a_iC_0 + b_iF)\right) C_0 \\
&= 2 - 3 + \sum(1-\frac{1}{e_i})(-a_i + b_i)
\end{aligned}$$

$$\Rightarrow \quad \sum(1-\frac{1}{e_i})b_i = \sum(1-\frac{1}{e_i})a_i + 1 \tag{5.2}$$

Theorem 5.3.6. *Let $\mathcal{X}$ be a minimal almost del Pezzo terminal order on $Z = \mathbb{F}_1$ and let $D = \cup D_i \equiv aC_0 + bF$ be its discriminant. Then D satisfies*

1. *$D \equiv 2C_0 + 4F$ or*
2. *$D \equiv 3C_0 + 5F$.*

Furthermore, the ramification degrees are all equal to 2*.*

Proof. Let $\mathcal{X}$ be a minimal almost del Pezzo terminal order over $Z = \mathbb{F}_1$ and let $D = \cup D_i \equiv \cup(a_i C_0 + b_i F) = aC_0 + bF$ and $\{e_i\}$ be its ramification configuration. We firstly show all the ramification degrees are equal, for then by Equation 5.2 we have

$$\begin{aligned} & a(1-\frac{1}{e})+1 = b(1-\frac{1}{e}) \\ \Rightarrow \quad & (b-a)(\frac{e-1}{e}) = 1 \\ \Rightarrow \quad & b-a = \frac{e}{e-1} \in \mathbb{Z} \\ \Rightarrow \quad & e = 2 \ \& \ b = a+2. \end{aligned}$$

By Proposition 5.3.1 we know $2 \leq a \leq 3$. Moreover if $a = 3$, then by Remark 5.3.3 we know all the ramification degrees are 2 and so $b = 5$.

Now let $a = 2$. By Proposition 5.3.1, we know all the ramification degrees for ramification divisors intersecting F are equal, say e, and Remark 5.3.2 ensures that all other ramification degrees divide e. Then by Equation 5.2

we have the following inequality

$$
\begin{aligned}
3(1-\frac{1}{e}) &< 3-\frac{2}{e} \\
&= 2(1-\frac{1}{e})+1 \\
&= \sum(1-\frac{1}{e_i})b_i \\
&\leq \sum(1-\frac{1}{e})b_i \qquad\qquad (*) \\
&= b(1-\frac{1}{e}) \\
\Rightarrow \quad 4 &\leq b,
\end{aligned}
$$

where the inequality in $(*)$ is because all ramification degrees divide e.

On the other hand, we know $e_i \geq 2$ for all i. Using Equation 5.2 once more we get

$$
\begin{aligned}
\frac{1}{2}b &\leq \sum(1-\frac{1}{e_i})b_i \\
&= 2(1-\frac{1}{e})+1 \\
&< 3 \\
\Rightarrow \quad b &\leq 5.
\end{aligned}
$$

Let $e = e_i t_i$ for some t_i's. Then t_i is one if $e = e_i$, and t_i is greater than one if $e \neq e_i$.

$$\begin{aligned} 2(1-\frac{1}{e})+1 &= \sum b_i - \sum \frac{b_i}{e_i} \\ &= b - \frac{b_i t_i}{e} \\ \Rightarrow \quad \frac{(b-3)e+2}{e} &= \frac{\sum b_i t_i}{e} \\ \Rightarrow \quad (b-3)e+2 &= \sum b_i t_i. \end{aligned}$$

We allow repetition in the numbers t_i, so we can assume that all the b_i are one. Then

$$(b-3)e+2 = \sum t_i, \tag{5.3}$$

where $t_i = \dfrac{e}{e_i}$.

Let $b = 5$. If e is a power of a prime number, say $e = p^n$, then $D_p = a_p C_0 + b_p F$ is the union of ramification divisors whose ramification degree is equal to p^n. By Equation 5.1 and Remark 5.3.2 we see that $b_p \geq 3$. Namely, $t_i = 1$ for at least three different i, say $t_1 = t_2 = t_3 = 1$. Then Equation 5.3 is simplified to

$$2e = 1 + \frac{e}{e_4} + \frac{e}{e_5} \leq 1 + e,$$

which is not true.

So it only remains to discuss the case that there are two or more prime numbers dividing e. Let 2 and 3 divide e. Then by Equation 5.1 and Remark 5.3.2 we see that at least 3 of the divisors are divisible by 2 and at least 3 of the divisors are divisible by 3. Therefore we get the following contradiction

$$\begin{aligned}
2e+2 &= \frac{e}{e_1}+\frac{e}{e_2}+\frac{e}{e_3}+\frac{e}{e_4}+\frac{e}{e_5} \\
&\le \begin{cases} \frac{e}{6}+\frac{e}{6}+\frac{e}{6}+\frac{e}{2}+\frac{e}{2}; \text{ or} \\ \frac{e}{6}+\frac{e}{6}+\frac{e}{2}+\frac{e}{2}+\frac{e}{3}; \text{ or} \\ \frac{e}{6}+\frac{e}{2}+\frac{e}{2}+\frac{e}{3}+\frac{e}{3} \end{cases} \\
&< 2e
\end{aligned}$$

If the prime numbers dividing e are different from 2 and 3, then the contradiction is even more clear. Therefore $b \neq 5$. Now we need to find what the ramification degrees are.

$b = 4$, so we have

$$e+2 = \frac{e}{e_1}+\frac{e}{e_2}+\frac{e}{e_3}+\frac{e}{e_4}.$$

Let e_1 be the smallest degree. If $e_1 \geq 4$,

$$e+2 = \frac{e}{e_1}+\frac{e}{e_2}+\frac{e}{e_3}+\frac{e}{e_4} \leq 4\frac{e}{e_1} \leq \frac{4e}{4}.$$

So $e_1 = p$ where p is one of the prime numbers 2 or 3. We claim that p divides all the degrees. If so, by Proposition 5.3.2 we have $e_2 = e_3 = e_4 = e$ and we can see that it results in equality of all the degrees and $e = 2$. If by contradiction, for instance $p \nmid e_2$, then e_2 divide both e_3 and e_4, we get

$e_3 = e_4 = e$. Since $(p, e_2) = 1$ we have

$$e = \frac{e}{p} + \frac{e}{e_2} < e.$$

which is not true. $\square$

Z=$\mathbb{F}_2$

Now assume that we have a minimal almost del Pezzo order $\mathcal{X}$ over $Z = \mathbb{F}_2$. As before we show the ramification configuration of the order by $(\cup D_i, \{e_i\})$. Let C_0 be a fixed section and F a fixed fibre of the ruled surface, where $C_0^2 = -2$, $F^2 = 0$, $C_0.F = 1$. Recall that the canonical divisor is $K_Z \equiv -2C_0 - 4F$. Then for suitable a_i and b_i we have $D_i \equiv (a_iC_0 + b_iF)$. We also set $D := \cup D_i$ and $aC_0 + bF = \cup(a_iC_0 + b_iF)$ for non negative integers a and b.

Theorem 5.3.7. *Let $\mathcal{X}$ be a minimal almost del Pezzo terminal order over $Z = \mathbb{F}_2$ and let D, $\{e_i\}$ be its ramification configuration. Then D satisfies*

1. *$D \equiv 2C + 4F$ or*

2. *$D \equiv 3C + 6F$.*

Furthermore, the ramification degrees are all equal and in the second case they are 2.

Proof. Let $D \equiv aC + bF = \cup(a_iC + b_iF)$, $\{e_i\}$ be a ramification configuration for a minimal almost del Pezzo terminal order over $\mathbb{F}_2$. Let $D = aC_0 + bF$ be a curve in $\mathbb{F}_2$, then we have

$$\begin{aligned}
0 = K_{\mathcal{X}}C_0 &= \left(-2C_0 - 4F + \sum(1 - \frac{1}{e_i})(a_iC_0 + b_iF)\right) C_0 \\
&= 4 - 4 + \sum(1 - \frac{1}{e_i})(-2a_i + b_i) \\
\Rightarrow \quad 2\sum(1 - \frac{1}{e_i})a_i &= \sum(1 - \frac{1}{e_i})b_i, \qquad (*)
\end{aligned}$$

By Proposition 5.3.1, we know $a = 2$ or 3. Further if $a = 3$, by Remark 5.3.3 we know all the ramification degrees are 2. So, by Equation $(*)$, $b = 6$. Now let $a = 2$. We know by Proposition 5.3.1 that the ramification degrees of all the divisors in D intersecting F are equal, say e.

Since $e_i \geq 2$ for every i, we have $b \leq 7$ by the following equation

$$\frac{1}{2}b \leq \sum(1 - \frac{1}{e_i})b_i = 4(1 - \frac{1}{e}) < 4.$$

Furthermore we know ramification degrees for divisors intersecting F are all equal, say e, and other ramification degrees e_i divide e. Therefore for every i, $(1 - \frac{1}{e_i}) \leq (1 - \frac{1}{e})$ and we have

$$\begin{aligned}
4(1 - \frac{1}{e}) &= \sum(1 - \frac{1}{e_i})b_i \leq b(1 - \frac{1}{e}) \\
\Rightarrow \qquad & 4 \leq b,
\end{aligned}$$

Where equality holds if and only if all the ramification degrees are equal. We claim that $b = 4$. To prove the claim we firstly let e be a power of a prime number, p^n. And then we prove the general case by contraditions.

Let $e = p^n$ for a prime number p and a positive integer n, Note that $n = 1$ is allowed. Then $D_p = a_p C_0 + b_p F$ is the union of ramification divisors whose ramification degree is equal to $p^n = e$. By Equation 5.1 and Remark 5.3.2 we see that $b_p \geq 4$. Recall Equation $(*)$

$$4\left(1-\frac{1}{e}\right) = \sum \left(1-\frac{1}{e_i}\right) b_i = 4\left(1-\frac{1}{e}\right) + \left(1-\frac{1}{e_i}\right) b_i,$$

for some i. Therefore $b_i = 0$, i.e. $b = 4$

Now we prove the general case. We let $e = p^n q^m r$ where p and q are distinct prime numbers and r is coprime to p and q and it can be 1. Simplifying equation $(*)$ we have

$$4(1-\frac{1}{e}) = \sum b_i - \sum \frac{b_i}{e_i}. \qquad (**)$$

We know each e_i divides e, so let $e = e_i t_i$ where t_i is a positive integer. Therefore we have

$$\sum t_i b_i = \left(\sum b_i - 4\right) e + 4.$$

We allow repetition in the numbers t_i and we assume all b_i are one. So we have $\sum t_i = (\sum b_i - 4)\, e + 4$. Let in the factorization $e = p^n q^m r$, p and

q be the biggest prime numbers dividing e and let $D_p = a_pC_0 + b_pF$ and $D_q = a_qC_0 + b_qF$ be their corresponding sub divisors of D. By Equation 5.1 and remark 5.3.2 we have $b_p \geq 4$ and $b_q \geq 4$. Namely each of b_p and b_q divide at least four of the ramification degrees e_i in the sum $\sum t_i = \frac{e}{e_1} + \frac{e}{e_2} + \frac{e}{e_3} + \frac{e}{e_4} + \frac{e}{e_5}$.

Now by contradiction let $\sum b_i = b = 5$. Then

$$\sum t_i b_i = e + 4.$$

Therefore we have one of the followings

$$\begin{aligned}
\sum t_i &= \frac{e}{e_1} + \frac{e}{e_2} + \frac{e}{e_3} + \frac{e}{e_4} + \frac{e}{e_5} \\
&= \begin{cases} \dfrac{p^nq^mr}{p^nr_1} + \dfrac{p^nq^mr}{p^nq^mr_2} + \dfrac{p^nq^mr}{p^nq^mr_3} + \dfrac{p^nq^mr}{p^nq^mr_4} + \dfrac{p^nq^mr}{q^mr_5}; \text{ or} \\ \dfrac{p^nq^mr}{p^nq^mr_1} + \dfrac{p^nq^mr}{p^nq^mr_2} + \dfrac{p^nq^mr}{p^nq^mr_3} + \dfrac{p^nq^mr}{p^nq^mr_4} + \dfrac{p^nq^mr}{r_5} \end{cases} \\
&\leq \begin{cases} q^mr + r + r + r + p^nr; \\ r + r + r + r + \dfrac{p^nq^mr}{2} \end{cases} \\
&= \begin{cases} r(q^m + 3 + p^n); \\ r(4 + \dfrac{p^nq^m}{2}). \end{cases}
\end{aligned}$$

If $r = 1$, then we have $q^m+3+p^n < p^nq^m+4$ and $4+\dfrac{p^nq^m}{2} < p^nq^m+4$. If $r > 1$, then $\{p, q\} \neq \{2, 3\}$. Thus $r(p^n+q^m+3) < rp^nq^m$ and $r(4+\dfrac{p^nq^m}{2}) < rp^nq^m$. So we get $\sum t_i < e + 4$ which can not be true.

Now let $b = 6$. Then we have

$$\begin{aligned} 4(1 - \frac{1}{e}) &= \sum b_i - \sum \frac{b_i}{e_i} \\ &= 6 - \frac{\sum t_i b_i}{e} \\ \Rightarrow \quad \sum t_i b_i &= 2e + 4. \end{aligned}$$

Since e is divisible by two distinct prime numbers, say p and q, then by remark 5.3.2 we have the following possibilities.

$$\begin{aligned} 2e + 4 &= \sum t_i \\ &= \frac{e}{e_1} + \frac{e}{e_2} + \frac{e}{e_3} + \frac{e}{e_4} + \frac{e}{e_5} + \frac{e}{e_6} \\ &\leq \begin{cases} \frac{e}{pq} + \frac{e}{pq} + \frac{e}{pq} + \frac{e}{pq} + \frac{e}{2} + \frac{e}{2}; \text{ or} \\ \frac{e}{pq} + \frac{e}{pq} + \frac{e}{pq} + \frac{e}{p} + \frac{e}{q} + \frac{e}{2}; \text{ or} \\ \frac{e}{pq} + \frac{e}{pq} + \frac{e}{p} + \frac{e}{p} + \frac{e}{q} + \frac{e}{q} \end{cases} \\ &\leq 2e. \end{aligned}$$

So b can not be 6.

For $b = 7$ we have

$$\begin{aligned} 4(1-\frac{1}{e}) &= \sum b_i - \sum \frac{b_i}{e_i} \\ &= 7 - \frac{\sum t_i b_i}{e} \\ \Rightarrow \quad \sum t_i b_i &= 3e + 4. \end{aligned}$$

Since e is divisible by two distinct prime numbers, say p and q, then by remark 5.3.2 we have the following possibilities.

$$\begin{aligned} 3e + 4 &= \sum t_i \\ &= \frac{e}{e_1} + \frac{e}{e_2} + \frac{e}{e_3} + \frac{e}{e_4} + \frac{e}{e_5} + \frac{e}{e_6} + \frac{e}{e_7} \\ &\leq \begin{cases} \frac{e}{pq} + \frac{e}{pq} + \frac{e}{pq} + \frac{e}{pq} + \frac{e}{2} + \frac{e}{2} + \frac{e}{2}; \text{ or} \\ \frac{e}{pq} + \frac{e}{pq} + \frac{e}{pq} + \frac{e}{p} + \frac{e}{q} + \frac{e}{2} + \frac{e}{2}; \text{ or} \\ \frac{e}{pq} + \frac{e}{pq} + \frac{e}{pq} + \frac{e}{p} + \frac{e}{q} + \frac{e}{2} + \frac{e}{2}; \text{ or} \\ \frac{e}{pq} + \frac{e}{p} + \frac{e}{p} + \frac{e}{p} + \frac{e}{q} + \frac{e}{q} + \frac{e}{q} \end{cases} \\ &\leq 3e. \end{aligned}$$

So b can not be 7. This finishes the proof of $b = 4$. $\square$

Chapter 6

Classification of Canonical Del Pezzo Orders

In this section we will give the classification of all canonical del Pezzo orders. Let $\mathcal{X}$ be a canonical order, in the previous chapter we showed there is the following diagram

$$
\begin{tikzcd}
\mathcal{Y} \arrow[dd, "g"'] \arrow[dr, "f"] & \\
& \mathcal{X} \arrow[dl, dashed, "gof^{-1}"] \\
\mathcal{W} &
\end{tikzcd},
$$

Figure 6.1: Resolution of canonical orders to the minimal model

where $f : \mathcal{Y} \to \mathcal{X}$ is the unique minimal resolution for $\mathcal{X}$; meaning that $\mathcal{X}$ is resolved to a terminal order $\mathcal{Y}$ over a smooth surface $Z_{\mathcal{Y}}$, 5.1.1, and $\mathcal{W}$ is a minimal terminal order on $Z_{\mathcal{W}} = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{F}_1$, or $\mathbb{F}_2$, 5.2.4. Further, if $\mathcal{X}$ is a del Pezzo order, then $\mathcal{W}$ is also del Pezzo when $Z_{\mathcal{W}} = \mathbb{P}^2$ and it is almost del Pezzo otherwise. So we can blowup a minimal terminal (almost) del Pezzo order $\mathcal{W}$ on Z to classify del Pezzo orders with canonical singularities. But it is clear that not any blowups are allowed. Once the order $\mathcal{W}$ is blowup by $f : \mathcal{Y} \to \mathcal{W}$, then $\mathcal{Y}$ should be an almost del Pezzo order. Moreover we need to have a $K_{\mathcal{Y}}$-zero curve in order to do the contraction $f : \mathcal{Y} \to \mathcal{X}$ where $\mathcal{X}$ is a canonical del Pezzo order.

6.1 Canonical Orders Obtained by Terminal Orders on $\mathbb{P}^2$

Let $Z = \mathbb{P}^2$, and we let D be the discriminant on Z corresponding to an order $\mathcal{W}$. Then $3 \leq \deg(D) \leq 5$. We firstly let D be a of degree 3. So all the ramification degrees are the same, say e. We skip some steps and we refer the reader to Chapter 4 for more detailed calculations. Here we denote birational morphisms for blowups by $f : \mathcal{Y} \to \mathcal{W}$ where f is a sequence of blowups.

Let $f : \mathcal{Y} \to \mathcal{W}$ be a blowup at a point p not in D, then we know the (unique) obtained maximal order $\mathcal{Y}$ is almost del Pezzo if the ramification degree for D is 2; however it actually remains del Pezzo, so there is no $K_{\mathcal{Y}}$-zero curve. So Let $f : \mathcal{Y} \to \mathcal{W}$ denote a blowup at a point p out of D twice with the exeptionals E_p and E'_p such that $E_p^2 = -2$, $(E'_p)^2 = -1$. Then $K_{\mathcal{Y}}^2 = K_{\mathcal{W}}^2 - 2 = \frac{1}{4}$ and further, we have the following equation.

$$K_{\mathcal{Y}} \equiv f^*(K_{\mathcal{W}}) + 2E_q + E'_p.$$

Let $C = C_0 + aE_p + bE_q$ be an effective curve in $Z_{\mathcal{Y}}$ where neither $C_0 - E_p$

nor $C_0 - E_{p'}$ is effective. Then

$$\begin{aligned}
K_{\mathcal{Y}} D &= (f^*(K_{\mathcal{W}}) + 2E_{p'} + E_p)C \\
&= f^*(K_{\mathcal{W}})C + (2E_{p'} + E_p)C \\
&= K_{\mathcal{W}} f_* C + (2E_{p'} + E_p)C \\
&= \frac{-3}{2}d + (2E_{p'} + E_p)(C_0 + aE_p + bE_{p'}) \\
&= \frac{-3}{2}d + (2E_{p'} + E_p)C_0 + (2E_{p'} + E_p)(aE_p + bE_{p'}) \\
&\leq \frac{-3}{2}d + d - b \\
&\leq 0.
\end{aligned}$$

Note that the only case that $(K_{\mathcal{Y}} + \Delta_{\mathcal{Y}})C = 0$ is when $C = aE_p$ for any positive integer a.

If we blowup one more point, then $K^2 \leq 0$ where by K we mean the canonical order after the third blowup. So the order would not be (almost) del Pezzo.

Now let $f : \mathcal{Y} \to \mathcal{W}$ refer to a birational morphisms which blowup the points

$p \notin D$ and $q \in D$. Then

$$\begin{aligned} K_{\mathcal{Y}} &\equiv f^*(K_{\mathcal{W}}) + E_p + \frac{1}{2}E_q; \\ K_{\mathcal{Y}}^2 &= \frac{9}{4} - 1 - \frac{1}{4} > 0; \\ K_{\mathcal{Y}}D &= (f^*(K_{\mathcal{W}}) + E_p + \frac{1}{2}E_q)D \\ &= -\frac{3d}{2} + E_pC + \frac{1}{2}E_qC - a - \frac{b}{2} \\ &\leq -a - \frac{b}{2}, \end{aligned}$$

where $D \equiv C + aE_p + bE_q$ and $d = \deg(f_*C)$. In order for the equality to occur, a and b should be *zero* and multiplicity of C at both p and q should be d. Therefore, $D := \tilde{L}$ where L is the line going through p and q.

Now we focus only on blowups at the points in the ramification divisors. It is the main and actually the only remaining part for degree 3 divisors to be covered. Before to start we need to state the definition of "in almost general position points".

Definition 6.1.1. *Let $f : Z \to \mathbb{P}^2$ be a sequence of blowups at points $\Sigma = \{p_1, \cdots, p_n\}$ in order and let $E_1, \cdots, E_n$ be the corresponding exceptional curves, $1 \leq i \leq n$. Note that the points are not in the same surfaces; however they are all in blowups of $\mathbb{P}^2$ and we allow infinitely near points. The set of points $p_1, \cdots, p_n$ is in almost general position if:*

1. *No four points (counting the multiplicities) are on a line.*

2. No seven points (counting the multiplicities) are on a conic.

3. No point of a (-2)-exceptional curve is blown up.

Let $f : Z \to \mathbb{P}^2$ be blowups of points $\Sigma = \{p_1, \cdots, p_8\}$ as follows. We denote a degree 3 ramification divisor in $\mathbb{P}^2$ by D_0 and we blowup a point $p_0 \in D_0$ to get the exceptional curve E_1. By induction for $1 \leq i \leq 7$ we define $D_i := D_{i-1} + r_i E_i$, where r_i equals 1 or 0 depending on the ramification degree of E_i. Then we blowup $p_i \in D_i$ and get the exceptional curve E_{i+1}. Thus we have the following equation.

$$
\begin{aligned}
K_Z + \Delta_Z &= f^*(K_{\mathbb{P}^2} + \Delta_{\mathbb{P}^2}) + \sum_1^8 \frac{b_i}{e} E_i \\
(K_Z + \Delta_Z)^2 &= f^*(K_{\mathbb{P}^2} + \Delta_{\mathbb{P}^2}) - \frac{8}{e^2} \\
&= \frac{1}{e^2} > 0.
\end{aligned}
$$

Where b_i's are as follows. Let $E_{i_1}, \cdots, E_{i_m}$ are exeptional curves which make the following tree.

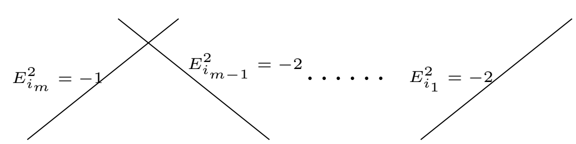


Figure 6.2: Configuration of exceptional trees for canonical singularities

Then $b_{i_j} = j$.

Now let the points $\Sigma = \{p_1, \cdots, p_8\}$ be in almost general position and let $f : Z \to \mathbb{P}^2$ denote the blowups of these points. Also let $D = C + \sum_1^8 a_i E_i$ be any effective divisor where for every i, $C - E_i$ is not effective. Then

$$\begin{aligned}
(K_Z + \Delta_Z)D &= (f^*(K_{\mathbb{P}^2} + \Delta_{\mathbb{P}^2}) + \sum_1^8 \frac{b_i}{e} E_i)D \\
&= f^*(K_{\mathbb{P}^2} + \Delta_{\mathbb{P}^2})C + (\sum_1^8 \frac{b_i}{e} E_i)D \\
&= (-\frac{3}{e}H) f_* C + (\sum_1^8 \frac{b_i}{e} E_i)(C + \sum_1^8 a_i E_i) \\
&= -\frac{3}{e}d + (\sum_1^8 \frac{b_i}{e} E_i)C + (\sum_1^8 \frac{b_i}{e} E_i)(\sum_1^8 a_i E_i) \\
&= -\frac{3}{e}d + (\frac{1}{e}\sum_1^8 b_i E_i)C + \frac{1}{e}\sum_{i,j=1}^8 b_i a_j E_i E_j \\
&= -\frac{3}{e}d + (\frac{1}{e}\sum_1^8 b_i E_i)C + \frac{1}{e}\sum_{k|E_k^2=-1} -a_k \\
&\leq -\frac{3}{e}d + (\frac{1}{e}\sum_1^8 b_i E_i)C \\
&\leq 0, \qquad (*)
\end{aligned}$$

where $d = deg(f_* C)$. Now we need to prove the last inequality, for then we see that if D contains any exceptional curve, then $(K_Z + \Delta_Z)D$ is strictly less than zero. So in order for some effective curve D to be a $(K_Z + \Delta_Z)$-zero curve, it is necessary to be the proper transform of some effective curve in $\mathbb{P}^2$, i.e. $D = \widetilde{f_* D} = C$.

To prove $(*)$, let D be as above and let $C = f_* D$. We only need to show the inequality for irreducible curves. For now let C be a line in $\mathbb{P}^2$. Then $d = 1$ and we have

$$\begin{aligned}(\frac{1}{e}\sum_1^8 b_i E_i)C &= (\frac{1}{e}\sum_1^8 b_i E_i)C \\ &\frac{1}{e}m(C)_p \\ &\leq \frac{1}{e}3 \\ &= \frac{3}{e}d.\end{aligned}$$

If C is a conic, then $d = 2$ and

$$\begin{aligned}(\frac{1}{e}\sum_1^8 b_i E_i)C &= (\frac{1}{e}\sum_1^8 b_i E_i)C \\ &\frac{1}{e}m(C)_p \\ &\leq \frac{1}{e}6 \\ &= \frac{3}{e}d.\end{aligned}$$

We see that if C is a conic or a line, the equality occurs if the multiplicity of C at p is exactly 6 for a conic and 3 for a line. We can also see that when C

is a curve of higher degrees, then we get the strict inequality.

Now we can state the following theorem.

Theorem 6.1.2. *Let $\mathcal{W}$ be a minimal terminal almost del Pezzo order on $Z = \mathbb{P}^2$ with ramification divisor D and let $\deg(D) = 3$. Also let $g : \mathcal{Y} \to \mathcal{W}$ represent the blowups of $\mathcal{W}$. Then each of the followings gives us the contractible mentioned $K_{\mathcal{Y}}$-zero curve E such that if $f : \mathcal{Y} \to \mathcal{X}$ contracts E, we get a canonical del Pezzo order $\mathcal{X}$.*

1. *Blowing up a point $p \in D$ twice, to get the exceptional curves E_1 and E_2, where $E_1^2 = -2$ and $E_2^2 = -1$; $E := E_1$.*

2. *Blowing up points $p \in D$ and $q \notin D$; E is the line going through p and q.*

3. *Blowing up 3 points in D, counting multiplicities, where all the points belong to a line l; and $E := l$.*

4. *Blowing up 6 points in D, counting multiplicities, where all the points belong to a conic C; and $E := C$.*

Now let the degree of the ramification divisor D be equal to 4.

Theorem 6.1.3. *Let $\mathcal{W}$ be a minimal terminal almost del Pezzo order on $Z = \mathbb{P}^2$. Let D be the ramification divisor and let $\deg(D) = 4$. If $g : \mathcal{Y} \to \mathcal{W}$ represents the blowups of $\mathcal{W}$ at points $\Sigma = \{p_1, \cdots, p_n\}$, then $\Sigma \subset D$, $n \leq 3$ and the point are not linear.*

Proof. By Theorem 4.3.6 we know all the blowups should be at points in D and also $e = 2$. Let $\Delta = \frac{1}{2}D$, where D is a ramification divisor in $Z = \mathbb{P}^2$ and $\deg(D) = 4$. Then we have the following equation

$$\begin{aligned}(K_Z + \Delta)^2 &= (-3H + (1 - \frac{1}{2})4H)^2 \\ &= (1 - \frac{4}{2}H)^2 \\ &= 1 - \frac{8}{2} + \frac{16}{2^2} \\ &= 1\end{aligned}$$

It is easy to see each blowup, reduces $(K_Z + \Delta)^2$ by $\frac{1}{4}$. So there are only 3 blowups allowed. Now let$f : Z_{\mathcal{Y}} \to Z$ be blowups at n points, $n \leq 3$, with E_i, $1 \leq i \leq n$. Also let $C = C' \sum_1^n a_i E_i$ be an effective divisor such that for every i, $C' - E_i$ is not effective.

$$\begin{aligned}
(K_{Z_{\mathcal{Y}}} + \Delta_{Z_{\mathcal{Y}}})C &= (f^*(K_Z + \Delta_Z) + \sum_1^n \frac{b_i}{e} E_i)C \\
&= f^*(K_Z + \Delta_Z)C + (\sum_1^n \frac{b_i}{e} E_i)C \\
&= -Hf_*C + (\sum_1^n \frac{b_i}{e} E_i)(C' + \sum_1^n a_i E_i) \\
&= -d + (\sum_1^n \frac{b_i}{e} E_i)C' + (\sum_1^n \frac{b_i}{e} E_i)(\sum_1^n a_i E_i) \\
&= -d + (\frac{1}{2}\sum_1^n b_i E_i)C' + \frac{1}{2}\sum_{i,j=1}^n b_i a_j E_i E_j \\
&= -d + (\frac{1}{2}\sum_1^n b_i E_i)C' + \frac{1}{2}\sum_{k|E_k^2=-1} -a_k \\
&\leq -d + (\frac{1}{2}\sum_1^n b_i E_i)C' \\
&\leq 0.
\end{aligned}$$

So in order to get $K_{\mathcal{Y}}$-zero curves it is necessary that $a_i = 0$ for every i. Let C' be an irreducible line. $d = 1$ so then $\sum_1^n b_i E_i C' = 2$. So the multiplicity of $f_*(C')$ at Σ should be 2. So in the diagram in Figure 6.1, $f : \mathcal{Y} \to \mathcal{X}$ should contract lines with multiplicity two at Σ. □

The last case to check for the orders on the projective plane is the orders with degree 5 discriminants. But looking at calculations in the proof of Theorem 4.3.7 we see that the first blowup $g : \mathcal{Y} \to \mathcal{W}$ results in $(K_{Z_{\mathcal{Y}}} + \Delta_{Z_{\mathcal{Y}}})^2 \leq 0$.

So the order $\mathcal{Y}$ can not be almost del Pezzo.

6.2 Canonical Orders Obtained by Terminal Orders over Rational Ruled Surfaces

The case $Z = \mathbb{P}^1 \times \mathbb{P}^1$

Let $\mathcal{W}$ be a minimal terminal almost del Pezzo order on $\mathbb{P}^1 \times \mathbb{P}^1$. And let D denote the discriminant corresponding to $\mathcal{W}$ and let C_0 and F denote two fixed fibres in different ruling directions. Then $2C_0 + 2F \leq [D] \leq 3C_0 + 3F$ and $\Delta = \left(1 - \frac{1}{e}\right) D$. We know that $e = 2$ if $2C_0 + 2F < [D]$. Further by Remark 5.3.5 $\mathcal{W}$ is actually del Pezzo. Recall that the canonical divisor of $\mathcal{W}$ is as the following.

$$K_{\mathcal{W}} = K_{\mathbb{P}^1 \times \mathbb{P}^1} + \Delta \equiv (-2C_0 - 2F) + \left(1 - \frac{1}{e}\right)(aC_0 + bF), \text{ for suitable } a \text{ and } b,$$
$$= \left(a - 2 - \frac{a}{e}\right) C_0 + \left(b - 2 - \frac{b}{e}\right) F.$$

Recalling the diagram in Figure 6.1, we seek to find a $K_{\mathcal{Y}}$-zero curve $E \in Z_{\mathcal{Y}}$. This lets us blow down $\mathcal{Y}$ to a canonical del Pezzo order $\mathcal{X}$ by contracting E. Before that we need the following definition.

Definition 6.2.1. *Let Σ be a set of point in $\mathbb{P}^1 \times \mathbb{P}^1$. The points of Σ are in* *almost general position if any curve of the form $aC_0 + bF$ contains no more than than $2(a+b)$ of the points. If C is a (a,b)-curve, we call C in Σ-almost general position if it contains exactly $2(a+b)$ points of Σ*

Theorem 6.2.2. *Let $\mathcal{W}$ be a minimal terminal (almost) del Pezzo order on $\mathbb{P}^1 \times \mathbb{P}^1$ with the discriminant D and the ramification degree e. Let $g : \mathcal{Y} \to \mathcal{W}$ be a sequence of blowups at points $\Sigma = \{p_1, \cdots, p_n\}$. Then each of the followings gives us a contractible $K_{\mathcal{Y}}$-zero curve E such that if $f : \mathcal{Y} \to \mathcal{X}$ contracts E, then $\mathcal{X}$ is a del Pezzo canonical order. And this actually classifies all the canonical del Pezzo orders.*

1. *$D \equiv 2C_0 + 2F$, $e = 2$, and $\Sigma = \{p\}$ is a single point, where $p \notin D$. Then E is the proper transform of any fibre (in any direction) passing p.*
2. *$D \equiv 3C_0 + 2F$, $e = 2$, and $\Sigma = \{p\}$ is a single point, where $p \in D$. Then E is the proper transform of any fibre in $[F]$ passing p.*
3. *$D \equiv 3C_0 + 3F$, $e = 2$, and $\Sigma = \{p\}$ is a single point, where $p \in D$. Then E is the proper transform of any fibre (in any direction) passing p.*
4. *$D \equiv 2C_0 + 2F$, e is free, and $\Sigma \subset D$ is a set of points in almost general position. Then E is the blowup of any curve in Σ-almost general position.*

Note that if D is a $(2,3)$-curve, then the same statement as the second case is true just by exchanging C_0 and F.

Proof. For the proof we just refer the reader to the calculations in Chapter 4. More specifically look at Remark 4.4.2 and Theorems 4.4.3 and 4.4.5 $\square$

The cases $Z = \mathbb{F}_1$

Let $\mathcal{W}$ be an almost del Pezzo order on Hirzebruch surface $\mathbb{F}_1$ with the discriminant D. By Theorem 5.3.6, $D \equiv 2C_0 + 4F$ or $D \equiv 3C_0 + 5F$ and the ramification degrees are all equall to 2. Further we have the following equations for the canonical divisors.

$$D \equiv 2C_0 + 4F : \; K_{\mathcal{W}} = -2C_0 - 3F + \frac{1}{2}(2C_0 + 4F) = -(C_0 + F) \qquad (6.1)$$

$$D \equiv 3C_0 + 5F : \; K_{\mathcal{W}} = -2C_0 - 3F + \frac{1}{2}(3C_0 + 5F) = -\frac{1}{2}(C_0 + F) \qquad (6.2)$$

Lemma 6.2.3. *Let $\mathcal{W}$ be a minimal terminal (almost) del Pezzo order on $\mathbb{F}_1$ with the discriminant $D \equiv 3C_0 + 5F$. Let $g : \mathcal{Y} \to \mathcal{W}$ be any blowup of $\mathcal{W}$. Then Y is not almost del Pezzo.*

Proof. Let $g : \mathcal{Y} \to \mathcal{W}$ dontes a blowup at a point p. Then we have the following equations

$$
\begin{aligned}
p \notin D: \quad K_{\mathcal{Y}}^2 &= (g^*(K_{\mathcal{W}}) + E)^2 \\
&= \frac{1}{4}(C_0 + F)^2 - 1 < 0 \\
p \in D: \quad K_{\mathcal{Y}}^2 &= (g^*(K_{\mathcal{W}}) + \frac{1}{2}E)^2 \\
&= \frac{1}{4}(C_0 + F)^2 - \frac{1}{4} = 0
\end{aligned}
$$

which can not occur for almost del Pezzo surfaces. □

Reviewing calculations in the proof of Theorem 5.3.6 we see that if $\mathcal{W}$ is an almost del Pezzo order on Hirzebruch surface $\mathbb{F}_1$ with the discriminant D. Then C_0 is a $K_{\mathcal{W}}$-zero curve. Therefore contracting C_0 gives a blowdown to a canonical del Pezzo order.

If $D \equiv 3C_0 + 5F$, then by Lemma 6.2.3 there is no blowup to almost del Pezzo orders. Thus C_0 is the only contractible curve. So $\mathcal{X}$ is a canonical del Pezzo order over $\mathbb{P}^2$ with a discriminant of degree 5 and ramification degree $e = 2$.

If $D \equiv 2C_0 + 4F$, it needs more detailed discussion. We claim that if $\mathcal{W}$ is blown up to an almost del Pezzo order, then the blowups are at points in D.

Otherwise if we blowup a point $p \notin D$ then

$$\begin{aligned} K_{\mathcal{Y}}^2 &= (g^*(K_{\mathcal{W}}) + E)^2 \\ &= (-C_0 - F)^2 - 1 \\ &= 0, \end{aligned}$$

which is not true for almost del Pezzo orders.

Definition 6.2.4. *Let $\Sigma = \{p_1, \cdots, p_n\}$ be a set of point in $\mathbb{F}_1$. Σ is in almost general position if*

1. *Non of the points p_i lie on (proper transform of) the section C_0*
2. *No more that 2 points (counting multiplicities) lie on (proper transform of) a fibre.*
3. *No point is on a (-2)-curve.*

Now we would like to classify del Pezzo orders with canonical singularities for which the minimal terminal del Pezzo order is over $\mathbb{F}_1$ and $D \equiv 2C_0 + 4F$. In the next two theorems we assume that the blowups are done in orders. Namely the i-th blowup is at the point p_i for $1 \leq i \leq n$ and $\Sigma \subset D$. Note that if p_1 and p_2 are infinitely near, then depending on p_1 if it is a singular point or not E_1 may be in the discriminant. Since p_2 must be in D if $E_1 \in D$, then any point on E_1 can be blown up, but if E_1 is not in the discriminant, then p_2 is the only point of intersection of E_1 and D.

Theorem 6.2.5. *Let $\mathcal{W}$ be a minimal terminal almost del Pezzo order on $\mathbb{F}_1$ with the discriminant $D \equiv 2C_0 + 4F$. Also Let $g : \mathcal{Y} \to \mathcal{W}$ be a sequence of blowups at points $\Sigma = \{p_1, \cdots, p_n\}$. Then $\mathcal{Y}$ is almost del Pezzo if and only if $n \leq 3$ and Σ is in almost general position.*

Proof. Let $g : \mathcal{Y} \to \mathcal{W}$ be a sequence of blowups at points $\Sigma = \{p_1, \cdots, p_n\}$. Since all the blowups are at points in D and $e = 2$, each blowup reduces the self intersection $K_{\mathcal{W}}$ by $\frac{1}{4}$. Moreover, $K_{\mathcal{W}}^2 = (C_0 + F)^2 = 1$. Then $K_{\mathcal{Y}}^2 > 0$ implies $n < 4$. For the rest of the proof we first show that if Σ is in almost general position, then $\mathcal{Y}$ is almost del Pezzo. And then we show that if Σ is not in almost general position, then $\mathcal{Y}$ is not almost del Pezzo. Let $C = a\widetilde{C_0 + b}F + r_1E_1 + r_2E_2 + r_3E_3$ be an effective curve in $Z_{\mathcal{Y}}$, where r_2 and r_3 can be zero depending on the number of blowups n. Then

$$K_{\mathcal{Y}} = g^*(-C_0 - F) + a_1E_1 + a_2E_2 + a_3E_3,$$

where $a_i \in \{\frac{1}{2}, 1, \frac{3}{2}\}$ depend on the blowups and the tree of exceptional curves.

$$\begin{aligned}
K_{\mathcal{Y}}.C &= (g^*(-C_0 - F) + a_1E_1 + a_2E_2 + a_3E_3).C \\
&= (-C_0 - F)(aC_0 + bF) + (a_1E_1 + a_2E_2 + a_3E_3)(aC_0 + bF) + \underbrace{\sum_1^3 a_iE_i.\sum_1^3 r_iE_i}_{\leq 0} \\
&\leq -b + b\underbrace{(a_1E_1 + a_2E_2 + a_3E_3).F}_{\leq 1} \\
&\leq 0.
\end{aligned}$$

Now if Σ is not in almost general position, then at least one of the three conditions fails. If $g : \mathcal{Y} \to \mathcal{W}$ denotes a blowup at a point $p \in C_0$. Then

$$\begin{aligned}
K_{\mathcal{Y}}\tilde{C_0} &= \left(g^*(K_{\mathcal{W}}) + \frac{1}{2}E\right)\tilde{C_0} \\
&= g^*(K_{\mathcal{W}}).\tilde{C_0} + \frac{1}{2}E.\tilde{C_0} \\
&= (-C_0 - F).C_0 + \frac{1}{2} \\
&= \frac{1}{2}.
\end{aligned}$$

Which is agains the definition for almost del Pezzo surfaces.

If there is a fibre F with multiplicity more than 2 at Σ, then

$$\begin{aligned} K_{\mathcal{Y}}\tilde{F} &= \left(g^*(K_{\mathcal{W}}) + \sum_i a_i E_i\right)\tilde{F} \\ &= g^*(K_{\mathcal{W}}).\tilde{F} + \left(\sum_i a_i E_i\right)\tilde{F} \\ &\geq (-C_0 - F).F + \frac{3}{2} \\ &= \frac{1}{2}. \end{aligned}$$

And finally there is no blowup at a point in a (-2)-exceptional curve for the obvious reasean that there is no (-3)-curve in the resolution of canonical orders.

□

Theorem 6.2.6. *Let $\mathcal{W}$ be a minimal terminal almost del Pezzo order on $\mathbb{F}_1$ with the discriminant $D \equiv 2C_0 + 4F$. Also Let $g : \mathcal{Y} \to \mathcal{W}$ be a sequence of blowups at points $\Sigma = \{p_1, \cdots, p_n\}$ in almost general position. Each of the followings is a $K_{\mathcal{Y}}$-zero curve E such that if $f : \mathcal{Y} \to \mathcal{X}$ contracts E, then $\mathcal{X}$ is a canonical del Pezzo order.*

1. *The section C_0.*
2. *Any fibre $\tilde{F}$, where multiplicity of F at Σ is 2.*
3. *An exceptional curve E, where $E^2 = -2$.*

Proof. Let $g : \mathcal{Y} \to \mathcal{W}$ be a sequence of blowups at points $\Sigma = \{p_1, \cdots, p_n\}$ in almost general position. We only need to find the intersection of the canonical divisor $K_{\mathcal{Y}}$ with each of the listed curve. The first case where $E = C_0$ is proved in Theorem 5.3.6. Now let there is a fibre F with multiplicity 2 at Σ. Then

$$\begin{aligned}
\text{two single points on } F: \quad K_{\mathcal{Y}}.\tilde{F} &= \left(g^*(-C_0 - F) + \frac{1}{2}(E_1 + E_2)\right).\tilde{F} \\
&= -1 + 1 = 0 \\
\text{a double point on } F: \quad K_{\mathcal{Y}}.\tilde{F} &= \left(g^*(-C_0 - F) + \frac{1}{2}E_1 + E_2\right).\tilde{F} \\
&= -1 + 1 = 0.
\end{aligned}$$

Now let E_1 and E_2 be a tree of exceptional curves where $E_1^2 = -2$. Then

$$\begin{aligned}
K_{\mathcal{Y}}.E_1 &= \left(g^*(-C_0 - F) + \frac{1}{2}E_1 + E_2\right).E_1 \\
&= 0 + \frac{1}{2}(-2) + 1 = 0
\end{aligned}$$

□

The cases $Z = \mathbb{F}_2$

Let $\mathcal{W}$ be an almost del Pezzo order on the surface $\mathbb{F}_2$ with the discriminant D. By Theorem 5.3.7, $D \equiv 2C_0 + 4F$ or $D \equiv 3C_0 + 6F$ and the ramification degrees are all equal to e and further in the second case $e = 2$. Then we have the following equations for the canonical divisors.

$$\begin{aligned} D \equiv 2C_0 + 4F: \quad K_{\mathcal{W}} &= -\frac{1}{e}\left(2C_0 + 4F\right) \\ K_{\mathcal{W}}^2 &= \frac{8}{e^2} \qquad (*) \\ D \equiv 3C_0 + 6F: \quad K_{\mathcal{W}} &= -\frac{1}{2}C_0 - F \\ K_{\mathcal{W}}^2 &= \frac{1}{2}. \qquad (**) \end{aligned}$$

We want to know how many what types of blowups give a $K_{\mathcal{Y}}$-zero curve where we denote the sequence of blowups by $f : \mathcal{Y} \to \mathcal{W}$. We start to classify the case $D \equiv 3C_0 + 6F$ as it is very restrictive. By $(**)$ we know only one single blowup keeps the order almost del Pezzo and it has to be at a point $p \in D$. By the calculations in the proof of Theorem 5.3.7 we know that $K_{\mathcal{Y}}.C_0 = 0$ and so $p \notin C_0$. Let $C = \widetilde{bF} + rE$ be an effective curve in $Z_{\mathcal{Y}}$. Then

$$\begin{aligned} K_{\mathcal{Y}}C &= \left(f^*\left(-\frac{1}{2}C_0 - F\right) + \frac{1}{2}E\right)C \\ &= f^*\left(-\frac{1}{2}C_0 - F\right).C + \frac{1}{2}E.C \\ &= \left(-\frac{1}{2}C_0 - F\right).bF + \frac{1}{2}E.(\widetilde{bF} + rE) \\ &= -\frac{b}{2} + \frac{b}{2} - \frac{r}{2}. \end{aligned}$$

So if $g : \mathcal{Y} \to \mathcal{X}$ contracts C_0 or the fibre F where the blowup is at a point $p \in F$, then $\mathcal{X}$ is a del Pezzo order with canonical singularities.

The classification for minimal terminal almost del Pezzo orders over $\mathbb{F}_2$ with the discriminant $D \equiv 2C_0 + 4F$ is more enormous. This is actually in two extents, a blowup at a point out of D and on the other hand, more blowups at points in D keep the order almost del Pezzo.

Definition 6.2.7. *Let $\Sigma = \{p_i\}_i$ be a set of points in $\mathbb{F}_2$. Σ is in almost general position respect to m if*

1. *Non of the points p_i lie on (proper transform of) the section C_0*
2. *No more than m points (counting multiplicity) lie on (proper transform of) a fibre.*
3. *No point is on a (-2)-curve.*

Proposition 6.2.8. *Let $\mathcal{W}$ be a minimal terminal almost del Pezzo order on $Z = \mathbb{F}_2$ with ramification divisor $D \equiv 2C_0 + 4F$. Also let $g : \mathcal{Y} \to \mathcal{W}$ represent a sequence of blowups at the point $\Sigma = \{p_i\}_i$. Then $\mathcal{Y}$ is almost del Pezzo if and only if all the followings hold.*

1. *None of the points p_i is in C_0.*
2. *If there is a blowup at a point $p_i \notin D$, then there are at most three more blowups, they should be in D, and also they should be in almost general*

position respect to 2. Further non of them should lie on the same fibre as the one p_i does.

3. *There are no more than 7 blowups and the points should be in almost general position respect to the ramification degree e.*

Proof. Since C_0 is a (-2)-curve and in the resolution of canonical orders there is no (-3)-curve, then no point of C_0 can be blown up. By Equation $(*)$ we know that if there is any blowup at a point out of then then $e = 2$ and therefore $K_{\mathcal{W}}^2 = 2$. Further each blowup at a point out D reduces $K_{\mathcal{Y}}$ by one, thus only one point p_i can lie out of D. Now let $\mathcal{W}' \to \mathcal{W}$ and $\mathcal{Y} \to \mathcal{W}'$ represent respectively a blowup at $p_1 \notin D$ and a sequence of blowups at the points $\Sigma - \{p_1\} \subset D$. Then $e = 2$ and we know each blowup at a point out of D reduces $K_{\mathcal{W}'}^2 = 1$ by $\dfrac{1}{4}$. Thus we can only have three more blowups and they are all at points of D.

Let $g : \mathcal{Y} \to \mathcal{W}$ represent a sequence of blowups at the points $p_1 \notin D$ and $\{p_2, p_3, p_4\} \subset D$ with the exceptional curves $\{E_i\}$ respecting indices. If by contradiction p_1 and $p_2 \in F$ for some fibre F, then

$$\begin{aligned}
K_{\mathcal{Y}}.\tilde{F} &= \left(f^*\left(-C_0 - 2F\right) + E_1 + \frac{1}{2}\sum_{i=1}^{3} a_i E_i\right)\tilde{F} \\
&= -1 + E_1\tilde{F} + \underbrace{\frac{1}{2}\sum_{i=1}^{3} a_i E_i \tilde{F}}_{>0} \\
&> 0,
\end{aligned}$$

which is a contradiction. With the same calculation we can see that if F is any fibre where $p_1 \notin F$, then $\sum_{i=1}^{3} a_i E_i \tilde{F} \leq 2$ which means that no more than two points of $\{p_2, p_3, p_4\}$ lie on F. Further we can see that for any j

$$\begin{aligned}
K_{\mathcal{Y}}.E_j &= \left(f^*\left(-C_0 - 2F\right) + E_1 + \frac{1}{2}\sum_{i=1}^{3} a_i E_i\right) E_j \\
&= \left(E_1 + \frac{1}{2}\sum_{i=1}^{3} a_i E_i\right) E_j < 0
\end{aligned}$$

Finally we need to show that if all the blowups are at points out of D, then item 3 has to be the case and conversely if item 3 occurs as well as the condition in item 1, then the order is almost del Pezzo. So let $g : \mathcal{Y} \to \mathcal{W}$ represent a sequence of blowups at the points $\Sigma = \{p_1, \cdots, p_n\} \subset D$ with the exceptional curves $\{E_i\}$ respecting indices. Then by Equation $(*)$ and the fact that each blowup reduces $K_{\mathcal{W}}$ by $\dfrac{1}{e^2}$ we see $n \leq 7$.

Moreover by the following equation we see that the multiplicity of any fibre

F at Σ can not be more than e.

$$
\begin{aligned}
K_{\mathcal{Y}}.\tilde{F} &= \left(f^* \left(-C_0 - 2F\right) + \frac{1}{e}\sum_{i=1}^{7} a_i E_i \right) \tilde{F} \\
&= -1 + \frac{1}{e}\sum_{i=1}^{7} a_i E_i \tilde{F}
\end{aligned}
$$

□

The classification for $K_{\mathcal{Y}}$-zero curves is as the following.

Theorem 6.2.9. *Let $\mathcal{W}$ be a minimal terminal almost del Pezzo order on $Z = \mathbb{F}_2$ with ramification divisor $D \equiv 2C_0 + 4F$ and ramification degree e. Also let $g : \mathcal{Y} \to \mathcal{W}$ represent the blowups of $\mathcal{W}$. Then each of the followings gives a $K_{\mathcal{Y}}$-zero curve E such that if $f : \mathcal{Y} \to \mathcal{X}$ contracts E, then the order $\mathcal{X}$ is a del Pezzo order with canonical singularities.*

1. *The sectionC_0*
2. *Blowing up points $p \notin D$, $E := F$ where F is the fibre passing p.*
3. *Blowing up a set of points $\Sigma = \{p_1, \cdots, p_n\} \subset D$ in almost general position where $n \leq 7$, $E := F$ is any fibre with multiplicity e at Σ.*
4. *$E := E_j$ where E_j is (-2)-exceptional curve.*

Note that all the blowups are assumed to satisfy conditions of Proposition 6.2.8.

Proof. See the calculations of the proofs of Theorem 5.3.7 and Proposition 6.2.8. □

Appendix A

Quadric Ramification Curves

Let D be the discriminant for an order $\mathcal{A}$ over $\mathbb{P}^2$. And let

$$\Delta = \sum_i \left(1 - \frac{1}{e}\right) D_i,$$

where D is the union of the irreducible ramification curves D_i. If the order is del Pezzo, then $3 \leq \deg(D) \leq 5$. When $\deg(D) = 3$ it is easy to find all possible configurations for the discriminant, see the list in Figure 4.1. But for higher degrees the possible configurations are more numerous especially when the order has canonical singularities. For the configuration of the discriminants of canonical orders, not only the ramification indices for branch points are different, but also the types of singularities for branch points are more various.

In the following diagrams we classify all ramification configurations for discriminants of degree 4. In 4 we saw the lists for terminal del Pezzo orders so here we give the classification for canonical del Pezzo orders. Note that in all the configurations the ramification degrees are $e = 2$ and the ramification indices for each branch point are written near the curves intersecting at the point. The numbers are all in the modular group $\mathbb{Z}_2$.

Theorem A.0.10. *The diagrams in the following give a complete classification of quartic discriminant of canonical del Pezzo orders.*

Before proving the theorem let us give a short description of some of the singularities which may not be familiar to reader. We write a possible equation for these singularities, however, there are equations in degree 4 with the same types of singularities. An e_6-singularity is the singularity of the curve $y^3 = x^4$ at the origin, a rhamphoid cusp is a double cusp which can be written as $y^2 = x^5$, and an oscular rhamphoid singularity is the singularity of the curve $y^2 = x^7$ at the origin.

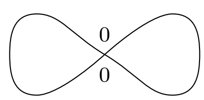
Proof. See [7, p:449,450] to find a complete classification of plane quartic curves with simple (canonical) singularities. The following list follows the same classification but it is missing some of them because of more restrictions on the ramification curves of canonical del Pezzo orders. It is easy to see that all the diagrams satisfy the hypothesis of Definition 3.3.1 and the types of canonical singularities in Table 5.1. All we need to show is to prove that the missing curves can not occur in our case. Comparing the following list with the list in [7, p:449,450] we see that the missing ones are the following.

- a cuspidal cubic and its cuspidal tangent;
- a conic with a tangent line and another line through the point of contanct;
- four lines with three concurrent;
- a cuspidal cubic and an inflection tangent;
- a nodal cubic and its inflection tangent;
- a nodal cubic and a line tangent at one branch;
- a cubic and its inflection tangent;
- an irreducible quartic with one e_6-singularity;
- a three cuspidal quartic;
- one oscular rhamphoid cusp;
- one rhamphoid cusp and a cusp;
- two conics intersecting at one point.

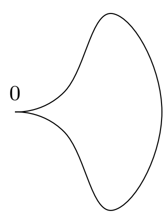
We can see that the first 7 ones which contain line(s) are against the rule that the cover of any line has to ramify at some point and also sum of all the ramification indices of the line is zero. The last 4 cases fail for the same reasoning as they involve rational curves with geometric genus 0. The remaining one, i.e. e_6-singularity is not a type of canonical singularities of orders listed in Table 5.1. $\square$

Note that the names of the singularities are chosen to agree with [8, p:260-263] and [7, p:449,450].

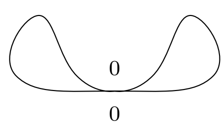
Irreducible plane quartic curves.



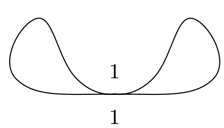
One node



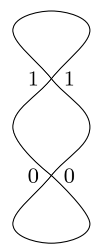
One cusp



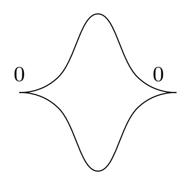
One tacnode



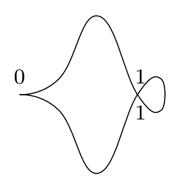
One tacnode



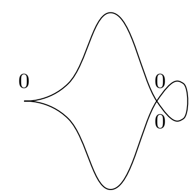
Two nodes



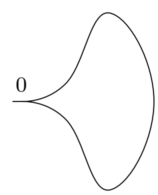
Two cusps



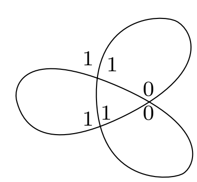
One cusp and One node



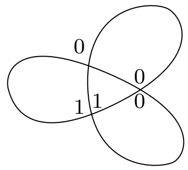
One cusp and One node



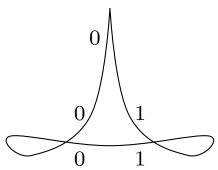
One ramphoid cusp



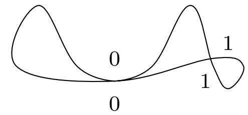
Three nodes



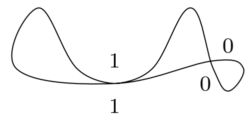
Three nodes



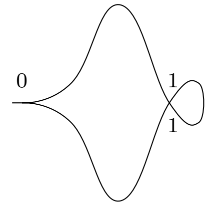
One cusp and two nodes



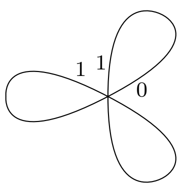
One node and One taconode



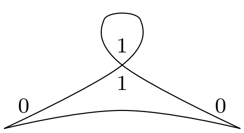
One node and One taconode



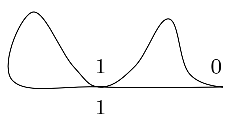
One ramiphoid cusp and One node



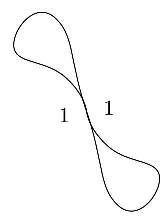
One ordinary triple point



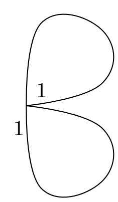
One node and two cusps



One cusp and One taconode

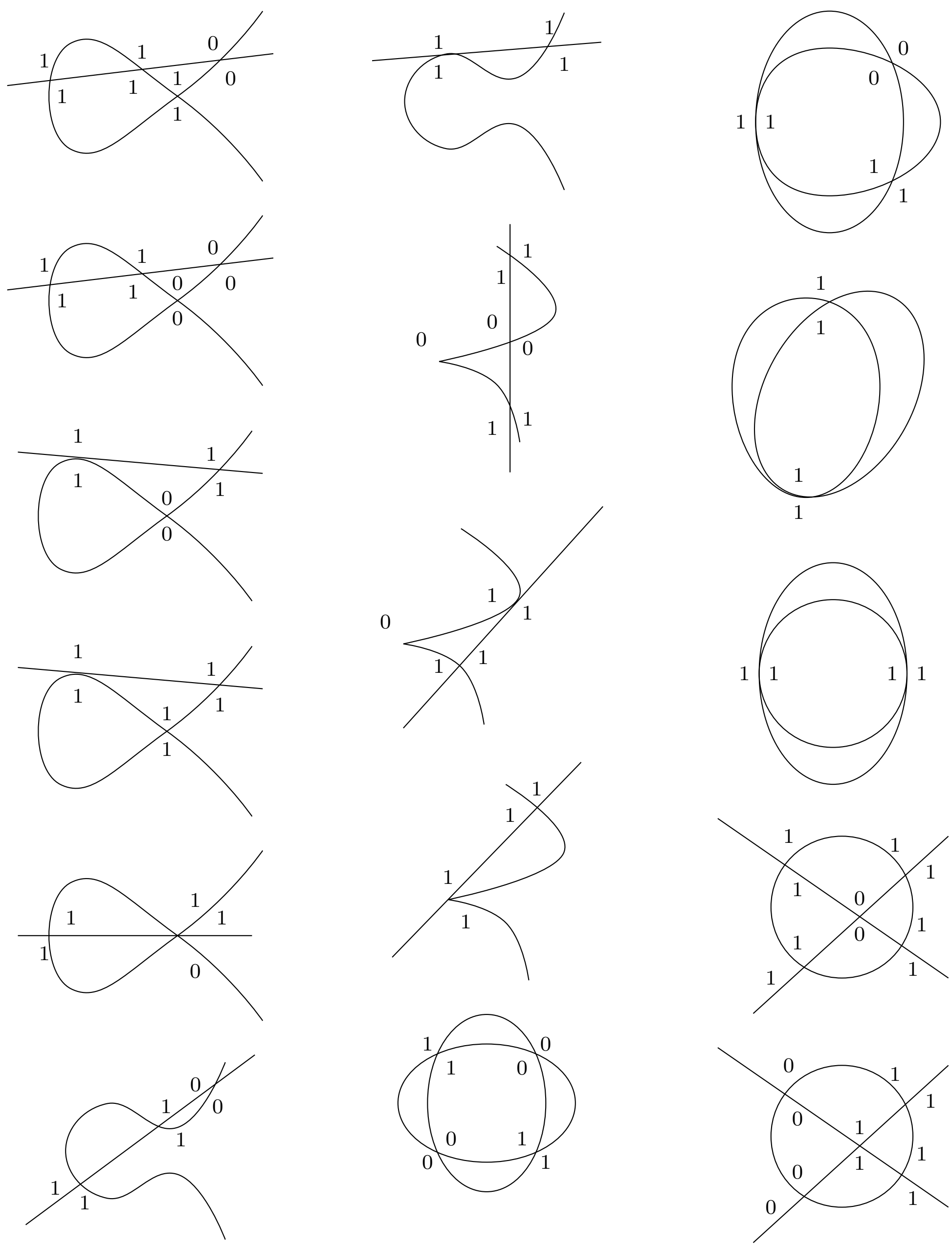


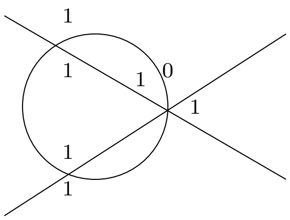
One oscnode



Triple point with one cuspidal branch

For the next list which is the classification of reducible plane quartic curves we skip writing the names of the singularities as they are no worse than the singularities in above list.

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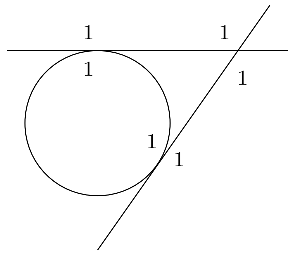
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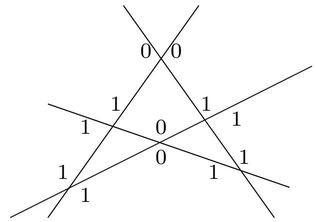
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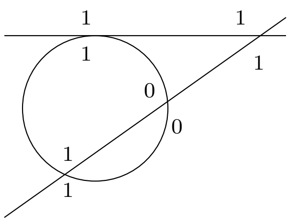
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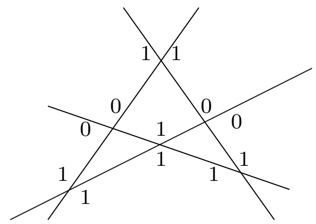
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Bibliography

[1] *Sulle superficie dell’nmo ordine immerse nello spazio din dimensioni*, Rendiconti del Circolo Matematico di Palermo (1884-1940) **1** (1887), no. 1, 241–271.

[2] Pieter Belmans and Dennis Presotto, *Construction of noncommutative surfaces with exceptional collections of length 4*, arXiv preprint arXiv:1705.06943 (2017).

[3] Daniel Chan, *Lectures on orders*, 2011.

[4] Daniel Chan, Paul Hacking, and Colin Ingalls, *Canonical singularities of orders over surfaces*, Proceedings of the London Mathematical Society **98** (2008), no. 1, 83–115.

[5] Daniel Chan and Colin Ingalls, *The minimal model program for orders over surfaces*, Invent. Math. **161** (2005), no. 2, 427–452. MR 2180454 (2008b:16043)

[6] Daniel Chan and Rajesh S Kulkarni, *Del pezzo orders on projective surfaces*, Advances in Mathematics **173** (2003), no. 1, 144–177.

[7] Igor V Dolgachev, *Classical algebraic geometry: a modern view*, Cambridge University Press, 2012.

[8] Joe Harris, *Algebraic geometry: a first course*, vol. 133, Springer Science & Business Media, 2013.

[9] Robin Hartshorne, *Algebraic geometry*, vol. 52, Springer Science & Business Media, 2013.

[10] Fumio Hidaka, Kei-ichi Watanabe, et al., *Normal gorenstein surfaces with ample anti-canonical divisor*, Tokyo Journal of Mathematics **4** (1981), no. 2, 319–330.

[11] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, vol. 134, Cambridge university press, 2008.

[12] Kenji Matsuki, *Introduction to the mori program*, Springer Science & Business Media, 2013.

[13] Miles Reid, *The du val singularities an, dn, e6, e7, e8*, 2012.