Hochschild Cohomology and The Theory of Algebraic Deformations

by

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Abstract

Homological algebra is a tool with myriad applications. In particular Hochschild cohomology is useful when constructing algebraic deformations of an associative algebra. The ideas and tools required for computing the Hochschild cohomology of associative algebras are presented, as well as some worked examples. Then the theory of algebraic deformations shows how to build new algebras form existing ones. The Hochschild cohomology of the underlying algebra determines if the new algebraic operations are associative and gives a meaningful idea of equivalence between deformations. Some examples of algebraic deformations are given showing the link to Hochschild cohomology directly in a very computable way. The theory of algebraic deformations fits nicely into algebraic geometry, giving some new insights into studying the moduli spaces of algebras. A recent conjecture due to Deligne even gives algebraic deformations a link to string theory. In short the theory of algebraic deformations is promising field which gives new methods of computing various useful mathematical quantities.

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Introduction

In the first chapter, a necessary background is developed in order to define and compute Hochschild cohomology. The definitions are presented following chapters 1 and 9 from Weibel [5]. The definition of the bar resolution and the reduced bar definition is taken from Chapter 9 section 6 from Cartan and Elienberg [2] as well as chapter 10 from MacLane [4]. After the definitions are in order, meaning is given to the cohomology modules by interpreting the modules abstractly in some low degrees. The end of the chapter gives examples of computing the cohomology for some specific algebras. (This is my own contribution)

The second chapter presents the theory of algebraic deformations and a relation to cohomology is established. The bulk of the definitions and proofs come from chapter 4 of Witherspoon [6]. Here more details are added to the proofs and examples are given of various deformations. Motivation to study algebraic deformations from the perspective of algebraic geometry is shown. Then the chapter closes with a brief discussion of the Kodaira Spencer map. The Kodaira Spencer map for algebras is a very new area of study and cannot easily be found in literature.

Chapter 1

Homological Algebra

Homological algebra arose form the following problem in the late 1800s [5, Section 1.1]. Suppose we have two matrices f and g, such that $gf = 0$. If $g * v = 0$ for some column vector of length n, it is not always possible to write $v = f * u$. The failure is measured by the *defect*

$$d = n - \mathrm{rank}(f) - \mathrm{rank}(g).$$

A modern way of representing this is with linear maps

$$U \xrightarrow{\mathrm{f}} V \xrightarrow{\mathrm{g}} W$$

where $gf = 0$. Then the homology module is

$$H = \mathrm{Ker}(g)/f(U).$$

Poincaré and others used these ideas to describe n-dimensional holes in simplicial complexes. Homological algebra became a discipline in its own right around the second world war. Eilenberg and various others found that the homological methods could be applied to algebraic systems. For example, such techniques show how two groups can be 'combined' into a larger 'extended' group (see [5, Chapter 6]). Since Cartan and Eilenberg's text [2], many other authors have written about the subject. In particular Weibel's text [5] gives a more modern treatment of the subject.

1.1 Homology of Chain Complexes

Fix an associative ring R with identity and let A, B and C belong to the category of right R-modules. Given R-module homomorphisms $f : A \to B$ and $g : B \to C$, we obtain a sequence

$$A \xrightarrow{\mathrm{f}} B \xrightarrow{\mathrm{g}} C$$

A sequence of this form is *exact at B* if $\mathrm{Ker}(g) = \mathrm{Im}(f)$. This implies that the composite mapping $gf : A \to C$ is zero. An infinite sequence of modules is also possible:

$$\ldots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \ldots \tag{1.1}$$

If such a sequence is exact at every module C_i then it is called a *long exact sequence.*

Definition 1.1.1. A *chain complex* (C_*, d_*) of R-modules is a sequence of R-modules $\{C_n\}_{n\in\mathbb{Z}}$ connected by R-module homomorphisms $d_n : C_n \to C_{n-1}$ such that the composition of any two consecutive mappings $d_n d_{n+1} : C_{n+1} \to C_{n-1}$ is zero. A chain complex is usually written as in Equation 1.1. The mappings d_n are called the *differentials* of the chain complex. For convenience, d is used to refer to them collectively. When it is not ambiguous (C_*, d_*) is written simply as C_*. The kernel of d_n is denoted $Z_n(C_*)$, or Z_n for brevity, and is called the module of *n-cycles* of C_*. Similarly the image of d_{n+1} is denoted $B_n(C_*)$ or just B_n and is called the module of *n-boundaries* of C_*. Note that

$$0 \subseteq B_n \subseteq Z_n \subseteq C_n$$

for all n. The n^{th} *homology module* of C_* is defined as the quotient module $H_n(C_*) = Z_n/B_n$.

If a long exact sequence has zero homology for all n, then it is called *acylic.* Dualizing the construction yields a *cochain complex* (C^*, d^*) with modules $\{C^n\}_{n\in\mathbb{Z}}$ and differentials $d^n : C^n \to C^{n+1}$. It is thus defined for any sequence

$$\dots \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \dots$$

with $d^{n+1}d^n = 0$. When it is not ambiguous the notation C^* is used. Then

$Z^n(C^*) = \mathrm{Ker}(d^n)$ is the module of *n-cocycles* of C^* and $B^n(C^*) = \mathrm{Im}(d^{n-1})$ is the module of *n-coboundaries* of C^*. The *cohomology* module is then given by $H^n(C^*) = Z^n/B^n$.

1.2 Chain Homotopy

A morphism $u : C_* \to D_*$ between two chain complexes (C_*, d_*^C) and (D_*, d_*^D) is a collection of module homomorphisms with the relation $d_n^D u_n = u_{n-1} d_n^C$. Equivalently the following diagram commutes:

$$\begin{tikzcd}
\ldots \arrow[r, "d_{d_n+2}"] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "u_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "u_n"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "u_{n-1}"] & \ldots \\
\ldots \arrow[r, "d_{n+2}"] & D_{n+1} \arrow[r, "d_{n+1}"] & D_n \arrow[r, "d_n"] & D_{n-1} \arrow[r, "d_{n-1}"] & \ldots
\end{tikzcd}$$

Proposition 1. The condition here forces morphisms to send boundaries to boundaries and cycles to cycles. This induces maps $H_n(C_*) \to H_n(D_*)$, as described in the following.

Proof. Firstly for cycles let $a \in \mathrm{Ker}(d_n^C)$ then

$$d_n^D(u(a)) = u(d_n^C(a)) = u(0) = 0.$$

Thus $u(a) \in \mathrm{Ker}(d_n^D)$ so u sends cycles to cycles.

For boundaries let $b \in \mathrm{Im}(D_{n+1}^C)$, so $b = d_{n+1}^C(b')$ for some $b' \in C_{n+1}$, so

$$d_{n+1}^D(u(b')) = u(d_{n+1}^C(b')) = u(b).$$

Thus $u(b) \in \operatorname{Im}(d^D_{n+1})$ so u also sends boundaries to boundaries.

It follows that homology modules are sent to homology modules. Abusing the notation write $u : H_n(C_*) \to H_n(D_*)$. ■

Definition 1.2.1. In the special case that the maps $H_n(C_*) \to H_n(D_*)$ are all isomorphisms, then u is called a *quasi-isomorphism*.

Definition 1.2.2. Another mapping $f : C_* \to D_*$ between two chain complexes can be defined from arbitrary maps $s_n : C_n \to D_{n+1}$ by setting $f_n = d_{n+1}s_n + s_{n-1}d_n$.

$$\begin{tikzcd}
\dots \arrow[r, "d_{n+2}"] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] \arrow[dl, "s_n"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"] \arrow[dl, "s_{n-1}"] & \dots \\
\dots \arrow[r, "d_{n+2}"] & D_{n+1} \arrow[r, "d_{n+1}"] & D_n \arrow[r, "d_n"] & D_{n-1} \arrow[r, "d_{n-1}"] & \dots
\end{tikzcd}$$

This is indeed a chain map, since

$$df = d(ds + sd) = dsd = (ds + sd)d = fd.$$

Here the indices are dropped for simplicity and the property $dd = 0$ is used. We say f is *null homotopic*. If the difference between two chain maps $f : C_* \to D_*$ and $g_* : C \to D$ is null homotopic then f and g are *chain homotopic*. This defines an equivalence relation $f \sim g$ on chain maps. Lastly a chain map $f : C_* \to D_*$ is a *chain homotopy equivalence* if there is a map $g : D_* \to C_*$ such that fg and gf are chain homotopic to the identity maps on C_* and D_*.

Lemma 1.2.1. If $f : C_* \to D_*$ is a null homotopic map then the maps

$$f_n : H_n(C_*) \to H_n(D_*)$$

are zero for all n.

Proof. Let $f : C_* \to D_*$ be a null homotopic map and $x \in H_n(C_*)$ be some n-cycle. Thus $f(x) = d(s(x)) + s(d(x)) = d(s(x))$, since the later term is a boundary of C_*. But $d(s(x))$ is a boundary of D_* so that $f(x) = 0$. ■

An immediate consequence of this is that if $f : C_* \to D_*$ and $g : C_* \to D_*$ are chain homotopic then they induce the same map on homology
$H_n(C_*) \to H_n(D_*)$.
All of the ideas in this section apply also to cochains and cohomology, with little change other than for indices.

1.3 Resolutions of Modules

Definition 1.3.1. Given a module M over a ring R and a chain complex C_*, we say C_* is *augmented* if there is a map $\epsilon : C_0 \to M$ such that $\epsilon \circ d_1 = 0$. If a sequence formed by an augmented chain complex

$$\cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\epsilon} M \to 0$$

is exact then it is called a *left resolution* of M.

There is a dual construction on a cochain complex C^*. If the sequence

$$0 \to M \xrightarrow{\epsilon} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \to \cdots$$

is exact then it is called a *right resolution* or *coresolution* of M. In the literature, both are sometimes referred to as resolutions.

1.4 Tensor products of modules

We summarize here the construction of tensor products of R-modules [1, pp. 24-31]. Let A and B be two R-modules, and let F be the free module over their Cartesian product $A \times B$. Thus F is the R-module of formal R-linear combinations of pairs like (a, b).

Definition 1.4.1. The *tensor product* of two modules A,B over a commutative ring R is defined to be the quotient module

$$A \otimes_R B = F/G.$$

Here G is the submodule of F generated by all formal linear combinations $(a, b) + (a', b) - (a + a', b)$, $(a, b) + (a, b') - (a, b + b')$, $r(a, b) - (ra, b)$ and $r(a, b) - (a, rb)$, for all $a, a' \in A$, $b, b' \in B$ and $r \in R$.

The image of (a, b) in F/G is usually denoted $a \otimes b$, so we get the usual rules for tensor calculations, such as $a \otimes b + a' \otimes b = (a + a') \otimes b$, $r(a \otimes b) = a \otimes rb$, etc. We do not make explicit use of the universal property of the tensor product found in [1, Proposition 2.12].

When R is not ambiguous $\otimes$ is written instead of $\otimes_R$. $A^{\otimes n}$ denotes the n-fold tensor product of A with itself.

1.5 Bar Resolution

To build a resolution for algebras, take R to be an associative algebra over some field k. Define the *standard complex* or *bar resolution*,

$$... \xrightarrow{d} R \otimes R \otimes R \xrightarrow{d} R \otimes R \xrightarrow{d} R \to 0$$

as follows. Here $\otimes$ denotes the tensor product over k. The differential d for the bar resolution can be obtained from subsidiary maps

$$\begin{aligned} s_n : R \otimes \cdots \otimes R &\to R \otimes R \otimes \cdots \otimes R \\ r_0 \otimes \cdots \otimes r_{n+1} &\mapsto 1 \otimes r_0 \otimes \cdots \otimes r_{n+1}. \end{aligned}$$

First define $d_0 : R \otimes R \to R$ by setting

$$d_0(r_0 \otimes r_1) = r_0 r_1.$$

Next impose the relations

$$(r_0 \otimes \cdots \otimes r_{n+1}) = d_{n+1}s_n(r_0 \otimes \cdots \otimes r_{n+1}) + s_{n-1}d_n(r_0 \otimes \cdots \otimes r_{n+1})$$

for $n \geq 0$.

These relations determine d_n by induction. In fact, d_n can be expressed in a closed form as

$$d_n(r_0 \otimes \cdots \otimes r_{n+1}) = \sum_{0 \leq i \leq n} (-1)^i r_0 \otimes \cdots \otimes (r_i r_{i+1}) \otimes \cdots \otimes r_{n+1}.$$

The property $d_{n-1}d_n = 0$ follows from induction as well. For $n = 1$, we have

$$\begin{aligned} d_0(d_1(r_0 \otimes r_1 \otimes r_2)) &= d_0(r_0r_1 \otimes r_2 - r_0 \otimes r_1r_2) \\ &= (r_0r_1)r_2 - r_0(r_1r_2) \\ &= 0, \end{aligned}$$

which is true by the associativity of R. For $n > 1$ the defining relations give

$$d_nd_{n+1}s_n = d_n - d_ns_{n-1}d_n = s_{n-2}d_{n-1}d_n = 0.$$

Since the image of s_n generates $R^{\otimes n+1}$ in the chain complex we conclude $d_nd_{n+1} = 0$.

A computationally useful variant of this complex called the *normalized standard complex.* It replaces $R \otimes R \otimes \cdots \otimes R$ with $R \otimes (R/k) \otimes \cdots \otimes (R/k)$. Here R/k

is $R/(k1_R)$, the cokernal of the k-module map $I : k \to R$ defined by $k \mapsto k1_R$. The elements of R/k are the cosets $\lambda + k$ of R. If $r_0 \otimes ... \otimes r_n$ is some element in the un-normalized standard complex, we denote by $r_0[r_1|...|r_n]$ the corresponding element in the normalized standard complex. The normalization comes from the fact that $[r_1|...|r_n] = 0$ if any $r_i \in k$.
The un-normalized and normalized standard complex are equivalent up to homotopy. A proof of this is found in [4, p. 282].

1.6 Hochschild Homology of Algebras

For computing the homology of algebras suppose R is an associative algebra over a field k and let M be a R-R-bimodule. To get more algebraic information about R, it is useful to first remove the last R in the standard resolution then apply the functor $M \otimes_R -$. This gives a new chain complex

$$... \xrightarrow{d} M \otimes R \otimes R \xrightarrow{d} M \otimes R \xrightarrow{d} M \to 0.$$

The elements of the chain complex will then be the k-modules

$$C_n(R, M) := M \otimes_R R^{\otimes n}$$

with differentials given by

$$d = \sum (-1)^i \partial_i$$

where

$$\partial_i(m \otimes r_1 \otimes ... \otimes r_n) = \begin{cases} mr_1 \otimes r_2 \otimes ... \otimes r_n & \text{if } i = 0 \\ m \otimes r_1 \otimes ... \otimes r_i r_{i+1} \otimes ... \otimes r_n & \text{if } 0 < i < n \\ r_n m \otimes r_1 \otimes ... \otimes r_{n-1} & \text{if } i = n \end{cases}$$

Definition 1.6.1. The homology of this chain complex is called the *Hochschild homology* $H_*(R, M)$ of R with coefficients in M. The homology is the set of k-modules

$$H_n(R, M) = H_n C(M \otimes R^{\otimes *})$$

The dualization of this construction gives a cochain complex

$$0 \to M \xrightarrow{\partial^0 - \partial^1} \mathrm{Hom}_k(R, M) \xrightarrow{d} \mathrm{Hom}_k(R \otimes R, M) \xrightarrow{d} ...$$

The elements are the k-modules of the k-multilinear maps $f : R^n \to M$

$$C^n(R, M) := \mathrm{Hom}_k(R^{\otimes n}, M).$$

The differentials are given by

$$d = \sum (-1)^i \partial^i$$

where

$$(\partial^i f)(r_0, \ldots, r_n) = \begin{cases} r_0 f(r_1, \ldots, r_n) & \text{if } i = 0 \\ f(r_0, \ldots, r_{i-2}, r_{i-1}r_i, r_{i+1}, \ldots r_n) & \text{if } 0 < i < n \\ f(r_0, \ldots, r_{n-1})r_n & \text{if } i = n \end{cases}$$

Definition 1.6.2. The cohomology of this cochain complex is called the *Hochschild cohomology* $H^*(R, M)$ of R with coefficients in M. The cohomology is the set of k-modules

$$H^n(R, M) = H^n C(\mathrm{Hom}_k(R^{\otimes *}, M)).$$

1.7 Interpretations at Low Degrees

1.7.1 $H^0(R, M)$

From the definition of $H^0(R, M) = \mathrm{Ker}(d^1)$, so elements of $m \in M$ in the kernel of $\partial^0 - \partial^1$ satisfy

$$0 = (\partial^0 - \partial^1)(m)(r) = \partial^0 m(r) - \partial^1 m(r) = mr - rm$$

for all $r \in R$. Thus

$$H^0(R, M) \cong \{m \in M \mid rm = mr, \forall r \in R\}$$

In the case $M = R$ this is the centre of R:

$$H^0(R) \cong Z(R)$$

1.7.2 $H^1(R, M)$

From the definition $H^1(R, M) = \mathrm{Ker}(d^2)/\mathrm{Im}(d^1)$. For $f \in \mathrm{Hom}_k(R, M)$ to be in $\mathrm{Ker}(d^2)$ we need

$$\begin{aligned} 0 &= (\partial^0 - \partial^1 + \partial^2)(f)(r_0, r_1) \\ &= r_0 f(r_1) - f(r_0 r_1) + f(r_0) r_1 \end{aligned}$$

for all $r_1, r_0 \in R$. So

$$f(r_0 r_1) = r_0 f(r_1) + f(r_0) r_1$$

Functions f of this form are called *k-derivations* from R to M. The space of all such functions is denoted $\mathrm{Der}_k(R, M)$. The image of d^1 is given by functions $g \in \mathrm{Hom}_k(R, M)$ where

$$g(r) = rm - mr.$$

These functions are also k-derivations since

$$\begin{aligned}
r_0 g(r_1) + g(r_0) r_1 &= r_0(r_1 m - m r_1) + (r_0 m - m r_0) r_1 \\
&= r_0 r_1 m - r_0 m r_1 + r_0 m r_1 - m r_0 r_1 \\
&= (r_0 r_1) m - m(r_0 r_1) \\
&= g(r_0 r_1),
\end{aligned}$$

Such a function is called an *inner k-derivation* from R to M. The space of these is denoted $\operatorname{InnDer}_k(R, M)$. Combining all these notions gives

$$H^1(R, M) \cong \operatorname{Der}_k(R, M) / \operatorname{InnDer}_k(R, M).$$

1.7.3 $H^2(R, M)$

From the definition $H^2(R, M) = \operatorname{Ker}(d^3)/\operatorname{Im}(d^2)$. For $f \in \operatorname{Hom}_k(R \otimes R, M)$ to be in $\operatorname{Ker}(d^3)$ we must have

$$\begin{aligned}
0 &= (\partial^0 - \partial^1 + \partial^2 - \partial^3)(f)(r_0, r_1, r_2) \\
&= r_0 f(r_1, r_2) - f(r_0 r_1, r_2) + f(r_0, r_1 r_2) - f(r_0, r_1) r_2,
\end{aligned}$$

so

$$r_0 f(r_1, r_2) + f(r_0, r_1 r_2) = f(r_0 r_1, r_2) + f(r_0, r_1) r_2.$$

The image of d^2 is given by functions $f \in \mathrm{Hom}_k(R \otimes R, M)$ where

$$f(r_0, r_1) = r_0 g(r_1) - g(r_0 r_1) + g(r_0) r_1,$$

for some $g \in \mathrm{Hom}_k(R, M)$. The interpretation of $H^2(R, M)$ is not obvious. However, when $M = R$, the formulas above arise in algebraic deformation theory, as described in the next chapter.

1.8 Some Worked Examples

1.8.1 $H^*(k)$

Considering a field as an algebra gives the basic idea of computing cohomology.

$H^0(k)$

The centre of any field k is itself so $H^0(k) \cong Z(k) \cong k$

$H^1(k)$

The 2-cocycles here are computed by calculating only $(df)(1,1)$. This is the case as df is a bilinear map so $(df)(a,b) = ab(df)(1,1)$.

$$(df)(1,1) = 1f(1) - f(1) + f(1)1 = 0$$

The cocycle condition is $f(1) = 0$ which means all cocycles are trivial and the coboundary need not be computed, so $H^1(k) \cong 0$

$H^2(k)$

The 3-cocycles here are computed by calculating $(df)(1, 1, 1)$.

$$(df)(1, 1, 1) = 1f(1, 1) - f(1, 1) + f(1, 1) - f(1, 1)1 = 0$$

The cocycle condition is trivial so all mappings are cocycles. In this case define the general cocycle $f(1, 1) = \lambda$ for $\lambda \in k$. The coboundaries are easily computed. We have

$$(dg)(1, 1) = 1g(1) - g(1) + g(1)1 = g(1)$$

so the general coboundary is $(dg)(1, 1) = \mu$ for $\mu \in k$. The resulting quotient gives the trivial group again, so $H^2(k) \cong 0$.
The cohomology groups in higher odd degrees are computationally similar to H^1 and the even like H^2, so the full picture is

$$H^n(k) \cong \begin{cases} k & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

1.8.2 $H^*(k[x]/x^2)$

The ring of dual numbers $k[x]/x^2$ as an associative algebra is of dimension 2 over k. This algebra is isomorphic to a subalgebra of the 2 by 2 matrices over k, generated by $1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $x = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. A general element is $a + bx = \left[\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right]$ and multiplication is done in the usual sense. Note the element x is nilpotent since $x^2 = 0$.

$H^0(k[x]/x^2)$

The algebra is commutative so, $H^0(k[x]/x^2) \cong k[x]/x^2$.

$H^1(k[x]/x^2)$

Suppose f is a 2-cocycle. Then evaluating $df = 0$ on all pairs of basis elements 1 and x gives all cocycle conditions. Thus

$$(df)(1, 1) = 1f(1) - f(1) + f(1)1 = 0$$

which means $f(1) = 0$. Next we have

$$(df)(1, x) = 1f(x) - f(x) + f(1)x = 0$$

giving no new condition. Likewise

$$(df)(x, 1) = xf(1) - f(x) + f(x)1 = 0$$

does not contribute anything. Lastly

$$(df)(x,x) = xf(x) - f(0) + f(x)x = 0$$

meaning $2xf(x) = 0$. Thus $f(1) = 0$ and $f(x) \in \mathrm{Ann}_{k[x]}(2x) := \{a \in k[x] | 2xa = 0\}$. This implies $f(x) = \lambda x$, for some $\lambda \in k$. The coboundaries here are functions where $g(a) = am - ma$, and since the algebra is commutative $g(a) = 0$ for all a, so the coboundary is trivial. Thus $H^1(k[x]/x^2) \cong k$.

$H^2(k[x]/x^2)$

The 3-cocycle conditions are computed in the same fashion as the 2-cocycle conditions.

$$(df)(1,1,1) = 1f(1,1) - f(1,1) + f(1,1) - f(1,1)1 = 0$$

contributes nothing.

$$(df)(1,1,x) = 1f(1,x) - f(1,x) + f(1,x) - f(1,1)x = 0$$

gives the condition $f(1,x) = f(1,1)x$.

$$(df)(1,x,1) = 1f(x,1) - f(x,1) + f(1,x) - f(1,x)1 = 0$$

also gives no new conditions.

$$(df)(1,x,x) = 1f(x,x) - f(x,x) + f(1,0) - f(1,x)x = 0$$

gives a new condition $f(1,x)x = f(1,1)x^2 = 0$

$$(df)(x,1,1) = xf(1,1) - f(x,1) + f(x,1) - f(x,1)1 = 0$$

gives a final condition $xf(1,1) = f(x,1)$. At the same time

$$(df)(x,1,x) = xf(1,x) - f(x,x) + f(x,x) - f(x,1)x = 0$$
$$(df)(x,x,1) = xf(x,1) - f(0,1) + f(x,x) - f(x,1) = 0$$
$$(df)(x,x,x) = xf(x,x) - f(0,x) + f(x,0) - f(x,x)x = 0$$

all vanish. Combing all this gives the cocycle conditions

$$f(1,x) = f(1,1)x$$
$$f(1,x) = f(x,1).$$

All the cocycles are of the form

$$f(1,1) = \lambda_1 + \lambda_2 x$$
$$f(1,x) = \lambda_1 x$$
$$f(x,1) = \lambda_1 x$$
$$f(x,x) = \lambda_3 + \lambda_4 x$$

where $\lambda_i \in k$. To calculate the coboundaries look at the general functions

$$g(1) = \mu_1 + \mu_2 x$$
$$g(x) = \mu_3 + \mu_4 x$$

where $\mu_i \in k$. Then

$$\begin{aligned}
(dg)(1,1) &= 1g(1) - g(1) + g(1)1 \\
&= \mu_1 + \mu_2 x \\
(dg)(1,x) &= 1g(x) - g(x) + g(1)x \\
&= \mu_1 x \\
(dg)(x,1) &= xg(1) - g(x) + g(x)1 \\
&= \mu_1 x \\
(dg)(x,x) &= xg(x) - g(0) + g(x)x \\
&= 2x\mu_3
\end{aligned}$$

the coboundaries then cancel every cocycle except $f(x,x) = \lambda_3$. Thus $H^2(k[x]/x^2) \cong k$.

1.8.3 $H^*(k[\mathbb{Z}/2\mathbb{Z}])$

The group algebra for a finite multiplicative group W over a field k consists of all k-linear combinations of group elements, with the algebraic operations defined in a natural way. The group algebra of $\mathbb{Z}/2\mathbb{Z}$ is of dimension 2. Similarly to the previous example, an isomorphic algebra is generated by the matrices $1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $x = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. A general element is then $a + bx = \left[\begin{smallmatrix} a+b & 0 \\ 0 & a-b \end{smallmatrix}\right]$. This algebra differs from the previous example since it lacks a nilpotent element.

$H^0(k[\mathbb{Z}/2\mathbb{Z}])$

Since $k[\mathbb{Z}/2\mathbb{Z}]$ is a commutative algebra $H^0(k[\mathbb{Z}/2\mathbb{Z}]) \cong k[\mathbb{Z}/2\mathbb{Z}]$.

$H^1(k[\mathbb{Z}/2\mathbb{Z}])$

The cocycle conditions are the same as in the previous example, so $f(1) = 0$ and $f(x) = \mathrm{Ann}(2x)$, where x is the non-identity element in $\mathbb{Z}/2\mathbb{Z}$. The difference is that $\mathrm{Ann}(2x)$ contains only 0, so the cocycles and the cohomology group are trivial. Thus $H^1(k[\mathbb{Z}/2\mathbb{Z}]) \cong 0$.

$H^2(k[\mathbb{Z}/2\mathbb{Z}])$

The cocycle conditions are also the same as in the previous example. Thus

$$f(1,x) = f(1,1)x$$
$$f(1,x) = f(x,1).$$

The general cocycles differ from those in the previous case, as follows:

$$f(1,1) = \lambda_1 + \lambda_2 x$$
$$f(1,x) = \lambda_2 + \lambda_1 x$$
$$f(x,1) = \lambda_2 + \lambda_1 x$$
$$f(x,x) = \lambda_3 + \lambda_4 x.$$

The coboundaries are again calculated by general functions

$$g(1) = \mu_1 + \mu_2 x$$
$$g(x) = \mu_3 + \mu_4 x$$

where $\mu_i \in k$. Then

$$\begin{aligned}
(dg)(1,1) &= 1g(1) - g(1) + g(1)1 \\
&= \mu_1 + \mu_2 x \\
(dg)(1,x) &= 1g(x) - g(x) + g(1)x \\
&= \mu_2 + \mu_1 x \\
(dg)(x,1) &= xg(1) - g(x) + g(x)1 \\
&= \mu_2 + \mu_1 x \\
(dg)(x,x) &= xg(x) - g(1) + g(x)x \\
&= (2\mu_4 - \mu_1) + (2\mu_3 - \mu_2)x
\end{aligned}$$

The coboundaries also differ from the previous example as now they cancel every cocycle. Then $H^2(k[x]/x^2) \cong 0$.

1.8.4 $H^*(\left[\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}\right])$

The algebra of upper triangular matrices is of dimension 3, with a basis $e_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $e_2 = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $e_3 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$.

$H^0(\left[\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}\right])$

This algebra is not commutative since $e_2 e_3 \neq e_3 e_2$. The centre is generated by only the identity matrix. Thus $H^0(\left[\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}\right]) \cong k$.

$H^1([\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}])$

The cocycle conditions are

$$\begin{aligned} f(e_1) &= 0 \\ e_2 f(e_2) &= -f(e_2)e_2 \\ e_3 f(e_3) &= -f(e_3)e_3 \\ e_2 f(e_3) + e_3 f(e_2) &= -(f(e_2)e_3 + f(e_3)e_2). \end{aligned}$$

The cocycles satisfy

$$\begin{aligned} f(e_1) &= 0 \\ f(e_2) &= \lambda_1 e_3 \\ f(e_3) &= \lambda_2 e_3 \end{aligned}$$

The coboundaries are the functions where $g(a) = am - ma$. Since e_2 and e_3 do not commute. There are non-trivial functions in the coboundary:

$$\begin{aligned} &g(e_1) = 0 \\ &g(e_2) = e_2(\mu_1 e_3) - (\mu_1 e_3)e_2 = -2\mu_1 e_3 \\ &g(e_3) = e_3(\mu_2 e_2) - (\mu_2 e_2)e_3 = 2\mu_2 e_3 \end{aligned}$$

The coboundaries then cancel the cocycles and so $H^1([\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}]) \cong 0$.

$H^2(\left[\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}\right])$

The cocycle conditions are

$$
\begin{aligned}
f(e_1, e_2) &= f(e_1, e_1)e_2 \\
f(e_2, e_1) &= e_2 f(e_1, e_1) \\
f(e_1, e_3) &= f(e_1, e_1)e_3 \\
f(e_3, e_1) &= e_3 f(e_1, e_1) \\
e_2 f(e_2, e_2) + f(e_2, e_1) &= f(e_1, e_2) + f(e_2, e_2)e_2 \\
e_3 f(e_3, e_3) &= f(e_3, e_3)e_3 \\
(e_2 - e_1)f(e_2, e_2) &= (f(e_1, e_1) + f(e_2, e_2))e_3 \\
f(e_2, e_3)e_3 &= (e_2 + e_1)f(e_3, e_3) \\
f(e_3, e_2)(e_2 + e_1) &= e_3(f(e_1, e_1) + f(e_2, e_2)) \\
e_3 f(e_3, e_2) &= f(e_3, e_3)(e_2 - e_1) \\
(e_2 + e_1)f(e_3, e_2) &= f(e_2, e_3)(e_2 - e_1) \\
e_3 f(e_2, e_3) - f(e_3, e_2)e_3 &= 2f(e_3, e_3).
\end{aligned}
$$

These boil down to

$$f(e_1,e_1)=\left[\begin{smallmatrix}\lambda_1 & \lambda_2\\ 0 & \lambda_3\end{smallmatrix}\right] \qquad f(e_1,e_2)=\left[\begin{smallmatrix}-\lambda_1 & \lambda_2\\ 0 & \lambda_3\end{smallmatrix}\right] \qquad f(e_1,e_3)=\left[\begin{smallmatrix}0 & \lambda_1\\ 0 & 0\end{smallmatrix}\right]$$

$$f(e_2,e_1)=\left[\begin{smallmatrix}-\lambda_1 & -\lambda_2\\ 0 & \lambda_3\end{smallmatrix}\right] \qquad f(e_2,e_2)=\left[\begin{smallmatrix}-\lambda_4 & \lambda_2\\ 0 & \lambda_5\end{smallmatrix}\right] \qquad f(e_2,e_3)=\left[\begin{smallmatrix}0 & -\frac{\lambda_1+\lambda_2}{2}\\ 0 & 2\lambda_6+\lambda_7\end{smallmatrix}\right]$$

$$f(e_3,e_1)=\left[\begin{smallmatrix}0 & \lambda_3\\ 0 & 0\end{smallmatrix}\right] \qquad f(e_3,e_2)=\left[\begin{smallmatrix}\lambda_7 & \frac{\lambda_3+\lambda_5}{2}\\ 0 & 0\end{smallmatrix}\right] \qquad f(e_3,e_3)=\left[\begin{smallmatrix}0 & \lambda_6\\ 0 & 0\end{smallmatrix}\right]$$

For the coboundaries let

$$g(e_1)=\left[\begin{smallmatrix}\mu_1 & \mu_2\\ 0 & \mu_3\end{smallmatrix}\right] \qquad g(e_2)=\left[\begin{smallmatrix}\mu_4 & \mu_5\\ 0 & \mu_6\end{smallmatrix}\right] \qquad g(e_3)=\left[\begin{smallmatrix}\mu_7 & \mu_8\\ 0 & \mu_9\end{smallmatrix}\right]$$

and thus

$$g(e_1,e_1)=\left[\begin{smallmatrix}\mu_1 & \mu_2\\ 0 & \mu_3\end{smallmatrix}\right] \qquad g(e_1,e_2)=\left[\begin{smallmatrix}-\mu_1 & \mu_2\\ 0 & \mu_3\end{smallmatrix}\right] \qquad g(e_1,e_3)=\left[\begin{smallmatrix}0 & \mu_1\\ 0 & 0\end{smallmatrix}\right]$$

$$g(e_2,e_1)=\left[\begin{smallmatrix}-\mu_1 & -\mu_2\\ 0 & \mu_3\end{smallmatrix}\right] \qquad g(e_2,e_2)=\left[\begin{smallmatrix}-2\mu_4-\mu_1 & -\mu_2\\ 0 & 2\mu_6+\mu_3\end{smallmatrix}\right] \qquad g(e_2,e_3)=\left[\begin{smallmatrix}0 & \mu_4\\ 0 & 2\mu_9\end{smallmatrix}\right]$$

$$g(e_3,e_1)=\left[\begin{smallmatrix}0 & \mu_3\\ 0 & 0\end{smallmatrix}\right] \qquad g(e_3,e_2)=\left[\begin{smallmatrix}-2\mu_7 & \mu_6\\ 0 & 0\end{smallmatrix}\right] \qquad g(e_3,e_3)=\left[\begin{smallmatrix}0 & \mu_7+\mu_9\\ 0 & 0\end{smallmatrix}\right]$$

Again the coboundaries cancel all of the cocycles and so $H^2(\left[\begin{smallmatrix}k & k\\ 0 & k\end{smallmatrix}\right]) \cong 0$.

Chapter 2

Deformations of Algebras

Formal deformations of associative algebras will be introduced and some concrete examples will be worked out. Of particular interest is that cohomology gives modern insights into deformations. Throughout this chapter A is an associative algebra over a field k, and t will be an indeterminate.

2.1 Formal deformations

The goal here is to construct a new algebra from A by using a formal power series. A *formal power series* in t with coefficients in A is an infinite polynomial $\sum_{i\geq 0} a_i t^i$ where $a_i \in A$. The space of all such series is denoted $A[[t]]$. An algebraic structure is formed by defining an addition on $A[[t]]$ by

$$\sum_{i\geq 0} a_i t^i + \sum_{i\geq 0} b_i t^i = \sum_{i\geq 0} (a_i + b_i) t^i.$$

Next, multiplication is given by the Cauchy product

$$\sum_{i\geq 0} a_i t^i \sum_{j\geq 0} b_j t^j = \sum_{l\geq 0}\left(\sum_{i+j=l} a_i b_j\right) t^l.$$

(This definition is motivated by the way we multiply polynomials in $\mathbb{R}[t]$.) In this way, the formal power series ring $A[[t]]$ is an associative algebra over A. A new multiplication $*$ on A within the algebra $A[[t]]$ can be defined as follows:

$$a * b = ab + \mu_1(a,b)t + \mu_2(a,b)t^2 + \dots$$

Here $a, b \in A$, ab is the usual multiplication in A; and $\mu_i \in \mathrm{Hom}_k(A \otimes A, A)$ are arbitrary k-linear maps. The maps μ_i give the coefficients for powers of t. For convenience define $\mu_0(a,b) = ab$. This condenses the previous formula to

$$a * b = \sum_{i\geq 0} \mu_i(a,b)t^i.$$

The multiplication can also be written as a function

$$\mu_t = \mu_0 + \mu_1 t + \mu_2 t^2 + \dots$$

where $\mu_t : A \otimes A \mapsto A[[t]]$ maps $a \otimes b$ to $a * b$. The multiplication is extended to all of $A[[t]]$ by the Cauchy product

$$\sum_{i\geq 0} a_i t^i * \sum_{j\geq 0} b_j t^j = \sum_{l\geq 0}\sum_{i\geq 0}\sum_{j\geq 0} \mu_l(a_i, b_j) t^{i+j+l}.$$

This new multiplication agrees with A in the sense that when $t = 0$, A is retrieved. Equivalently, the multiplication μ_t modulo the ideal generated by t gives the original action. This follows from

$$\begin{aligned}[a * b] &\cong [ab + \mu_1(a,b)t + \mu_2(a,b)t^2 + \dots] \pmod t \\ &\cong ab \pmod t \\ &= ab \in A.\end{aligned}$$

This multiplication is called a *formal deformation of* A and is denoted $(A_t, *)$ or equivalently (A_t, μ_t).
The same ideas can be applied to other algebraic structures like $k[t]$ or $k[t]/(t^n)$ to give other types of deformations.

2.2 Associative deformations

In order for the multiplication to be associative we require $(a*b)*c = a*(b*c)$, for all $a, b, c \in A$. Now expand both sides to get

$$\begin{aligned}(a * b) * c &= ab * c + \mu_1(a,b) * ct + \mu_2(a,b) * ct^2 + \dots \\ &= abc + \mu_1(ab,c)t + \mu_2(ab,c)t^2 + \mu_1(a,b)ct + \mu_1(\mu_1(a,b),c)t^2 + \mu_2(a,b)ct^2 + \dots\end{aligned}$$

Similarly,

$$a * (b * c) = a * bc + a * \mu_1(b,c)t + a * \mu_2(b,c)t^2 + \dots$$
$$= abc + \mu_1(a,bc)t + \mu_2(a,bc)t^2 + a\mu_1(b,c)t + \mu_1(a,\mu_1(b,c))t^2 + a\mu_2(b,c)t^2 + \dots$$

Equating the coefficients for only the t^1 terms gives

$$\mu_1(ab,c) + \mu_1(a,b)c = \mu_1(a,bc) + a\mu_1(b,c) \tag{2.1}$$

for all $a, b, c \in A$. The connection to cohomology is seen here as, this implies μ_1 is a Hochschild 2-cocycle. This is not the whole picture, however, since there are infinitely many more terms to balance. Equating the t^2 terms gives

$$\mu_2(ab,c) + \mu_1(\mu_1(a,b),c) + \mu_2(a,b)c = \mu_2(a,bc) + \mu_1(a,\mu_1(b,c)) + a\mu_2(b,c).$$

Rearranging, we get

$$\mu_1(\mu_1(a,b),c) - \mu_1(a,\mu_1(b,c)) = a\mu_2(b,c) - \mu_2(ab,c) + \mu_2(a,bc) - \mu_2(a,b)c.$$

The right side is expressed nicely using a differential from the bar resolution

$$\mu_1(\mu_1(a,b),c) - \mu_1(a,\mu_1(b,c)) = d^3(\mu_2)(a,b,c)$$

This is called the *2nd obstruction*. Working out higher powers of t, we obtain the $(i-1)$*st obstruction*

$$\sum_{j=1}^{i-1} \mu_j(\mu_{i-j}(a,b),c) - \mu_j(a,\mu_{i-j}(b,c)) = d^3(\mu_i)(a,b,c).$$

2.3 Equivalence of deformations

Two formal deformations (A_t, μ_t), (A'_t, μ'_t) are *equivalent* if there is a $k[[t]]$-linear function $\phi_t : A_t \mapsto A'_t$ of the form

$$\phi_t(a) = a + \phi_1(a)t + \phi_2(a)t^2 + \dots,$$

where $\phi_i : A \mapsto A$ is an algebra automorphism satisfying

$$\phi_t \mu_t(a,b) = \mu'_t(\phi_t(a), \phi_t(b)) \tag{2.2}$$

for all $a, b \in A$. A formal deformation is *trivial* if it is equivalent to $A[[t]]$. This means $\phi_t \mu_t(a,b) = \phi(a)\phi(b)$. The multiplication μ_t can be pulled back to the usual multiplication in $A[[t]]$ where $\mu_0(a,b) = ab$ and $\mu_i(a,b) = 0$ for all $i > 0$. The following lemma gives a further connection to cohomology by showing that equivalence of formal deformations is related to Hochschild 2-coboundaries.

Lemma 2.3.1. If (A_t, μ_t) and (A'_t, μ'_t) are equivalent by a mapping ϕ_t, then $\mu'_1 = \mu_1 - d\phi_1$. Moreover if (A_t, μ_t) is equivalent to a trivial deformation, then μ_1 is a coboundary.

Proof. The proof follows by expanding Equation 2.2:

$$
\begin{aligned}
\phi_t\mu_t(a,b) &= \mu'_t(\phi_t(a),\phi_t(b)) \\
\phi_t(ab+\mu_1(a,b)t+\dots) &= \phi_t(a)\phi_t(b)+\mu'_1(\phi_t(a),\phi_t(b))t\dots \\
\phi_t(ab)+\phi_t(\mu_1(a,b)t)+\dots &= (a+\phi_1(a)t+\dots)(b+\phi_1(b)t+\dots)+\mu'_1(a,b)t+\dots \\
ab+\phi_1(ab)t+\mu_1(a,b)t+\dots &= ab+\phi_1(a)bt+a\phi_1(b)t+\mu'_1(a,b)t\dots
\end{aligned}
$$

Equating the t terms we get

$$
\begin{aligned}
\phi_1(ab)+\mu_1(a,b) &= \phi_1(a)b+a\phi_1(b)+\mu'_1(a,b) \\
\mu'_1(a,b) &= \mu_1(a,b)-(a\phi_1(b)-\phi_1(ab)+\phi_1(a)b) \\
\mu'_1(a,b) &= \mu_1(a,b)-d\phi_1(a,b).
\end{aligned}
$$

If (A'_t,μ'_t) is trivial then $\mu'_1=0$ and $\mu_1=d\phi_1$. ■

Lemma 2.3.2. A non-trivial formal deformation (A_t,μ_t) is equivalent to another formal deformation (A'_t,μ'_t) such that the first non zero cochain μ'_n is a 2-cocycle that is not a coboundary.

Proof. Suppose the contrary. Let (A_t,μ_t) be some formal deformation where the first non-zero cochain is a coboundary. Write out the deformation as

$$\mu_t(a,b)=ab+\mu_n(a,b)t^n+\mu_{n+1}(a,b)t^{n+1}+\dots$$

where $\mu_n = d\beta$ for some $\beta \in \mathrm{Hom}_k(A, A)$. Now define a $k[[t]]$-linear function ϕ where

$$\phi_t(a) = a + \beta(a)t^n$$

for all $a \in A$. The inverse of this function can be computed:

$$\begin{aligned}
\phi_t^{-1}(\phi(a)) &= a \\
\phi_t^{-1}(a + \beta(a)t^n) &= a \\
\phi_t^{-1}(a) &= a - \phi_t^{-1}(\beta(a)t^n) \\
&= a - (\beta(a)t^n - \phi_t^{-1}(\beta(\beta(a))t^n)t^n) \\
&= a - \beta(a)t^n + \phi_t^{-1}(\beta(\beta(a)))t^{2n} \\
&= a - \beta(a)t^n + \beta(\beta(a))t^{2n} - \beta(\beta(\beta(a)))t^{3n} + \dots
\end{aligned}$$

Now define a new multiplication $\mu'_t(a, b) = \phi_t(\mu_t(\phi_t^{-1}a, \phi_t^{-1}b))$.

$$\begin{aligned}
\mu'_t(a, b) &= \phi_t(\mu_t((a - \beta(a)t^n + \dots), (b - \beta(b)t^n + \dots))) \\
&= \phi(ab - a\beta(b)t^n - b\beta(a)t^n + \mu_n(a, b)t^n + \mu_{n+1}(a, b)t^{n+1} + \dots) \\
&= ab - a\beta(b)t^n - b\beta(a)t^n + \mu_n(a, b)t^n + \beta(ab)t^n + \mu_{n+1}(a, b)t^{n+1} + \dots \\
&= ab + (\mu_n(a, b) - (a\beta(b) - \beta(ab) + b\beta(a)))t^n + \mu_{n+1}(a, b)t^{n+1} + \dots \\
&= ab + (\mu_n(a, b) - (d^*\beta)(a, b))t^n + \mu_{n+1}(a, b)t^{n+1} + \dots \\
&= ab + \mu_{n+1}(a, b)t^{n+1} + \dots
\end{aligned}$$

If μ'_{n+1} is also a coboundary then by the same reasoning (A'_t, μ'_t) is equivalent to another deformation (A''_t, μ''_t), where

$$\mu''_t(a,b) = ab + \mu''_{n+2}(a,b)t^{n+2} + \dots,$$

for another $k[[t]]$-linear function ϕ'_t such that $\phi'_t(a) = a + \beta'(a)t^{n+1}$. Doing this as many times as necessary, a function Φ_t can be defined as the composition of all $\dots \phi''_t \phi'_t \phi_t$. Such a composition is reasonable since the coefficient for each power of t is a finite sum of finite compositions of β, β', β''. For example, if $n = 1$ then

$$a + \beta(a)t + \beta'(a)t^2 + (\beta''(a) + \beta'(\beta(a)))t^3 + (\beta'''(a) + \beta''(\beta(a)))t^4 + \dots$$

Applying Φ_t to (A_t, μ_t) shows that the deformation is indeed trivial. ■

The immediate consequence of the lemma is that if $H^2(A) \cong 0$, then A has no deformations up to isomorphism. Such an algebra is called *rigid*.

2.4 First order deformations

A deformation over $k[t]/(t^2)$ is called a *first order deformation*. The associativity condition is manageable since there are no obstructions other than the first (given by Equation 2.1). Equivalent deformations can be interpreted as cocycles that differ by a coboundary. The class of first order deformations is thus computed by the second Hochschild cohomology group. For an example of a first order deformation, consider $k[x]/(x^2)$ and define a new multiplication by

$$1 * 1 = 1$$
$$1 * x = x$$
$$x * 1 = x$$
$$x * x = t$$

The usual multiplication has been perturbed only slightly. Setting t to 0 retrieves the original multiplication. The resulting algebra is $k[x]/(x^2 - t)$. In the previous chapter it was shown that $H^2(k[x]/x^2) \cong k$, and the only cocycle that was not a coboundary was $f(x, x) = \lambda$. This means that the new multiplication is the only associative first order deformation on $k[x]/(x^2)$ up to equivalence.

Now consider $k[\mathbb{Z}/2\mathbb{Z}]$. It has been shown $H^2(k[\mathbb{Z}/2\mathbb{Z}])$ is trivial. Thus any deformation will be isomorphic to the original multiplication. For example, consider the deformation

$$1 * 1 = 1$$
$$1 * x = x$$
$$x * 1 = x$$
$$x * x = 1 + t$$

Now define a $k[t]$ linear map ϕ with values $\phi(1) = 1$ and $\phi(x) = x + 2^{-1}xt$. This ϕ transforms the trivial multiplication into $*$. Since the algebra only has two generators, checking is simple:

$$\begin{aligned}
\phi(1 * 1) &= 1 \\
\phi(1)\phi(1) &= 1 \\
\phi(1 * x) &= \phi(x) = x + 2^{-1}xt \\
\phi(1)\phi(x) &= x + 2^{-1}xt \\
\phi(x * x) &= \phi(1 + t) = \phi(1) + \phi(1)t = 1 + t \\
\phi(x)\phi(x) &= (x + 2^{-1}xt)(x + 2^{-1}xt) = 1 + t
\end{aligned}$$

The cohomology of $k[\mathbb{Z}/2\mathbb{Z}]$ shows that for any associative deformation there will be some linear mapping transforming the original multiplication of $k[\mathbb{Z}/2\mathbb{Z}]$ into the new multiplication.

2.5 Associative formal deformations

In some cases an associative formal deformation can be built up using the multiplication from a first order deformation. For example define a deformation of $A = k[x]$ with the new multiplication $\mu = \mu_0 + \mu_1 t$ where $\mu_1(x^n, x^m) = nx^{n-1}mx^{m-1}$. This deformation is associative modulo t^2 but not modulo t^3. Here is a check:

$$\begin{aligned}
\mu(\mu(x^2, x), x) &= \mu(x^3 + 2xt, x) \\
&= x^4 + 2x^2t + 3x^2t + 2t^2 \\
&= x^4 + 5x^2t + 2t^2 \\
\mu(x^2, \mu(x, x)) &= \mu(x^2, x^2 + t) \\
&= x^4 + x^2t + 4x^2t \\
&= x^4 + 5x^2t
\end{aligned}$$

The multiplication can be made associative modulo t^3 by extending $\mu = \mu_0 + \mu_1 t + \mu_2 t^2$, where

$$\mu_2(x^n, x^m) = (n)(n-1)x^{n-2}(m)(m-1)x^{m-2}$$

The previous calculation is now repeated.

$$\begin{aligned}
\mu(\mu(x^2, x), x) &= \mu(x^3 + 2xt, x) \\
&= x^4 + 2x^2t + 3x^2t + 2t^2 \\
&= x^4 + 5x^2t + 2t^2 \\
\mu(x^2, \mu(x, x)) &= \mu(x^2, x^2 + t) \\
&= x^4 + x^2t + 4x^2t + \frac{4t^2}{2} \\
&= x^4 + 5x^2t + 2t^2.
\end{aligned}$$

The new deformation μ is not associative modulo t^4. The deformation can be extended indefinitely to give an associative formal deformation with a multiplication defined by $\mu = \sum_{i=0} \mu_i t^i$, where $\mu_i(a,b) = D^i a D^i b/i!$ and $Dx^n = nx^{n-1}$ [3].

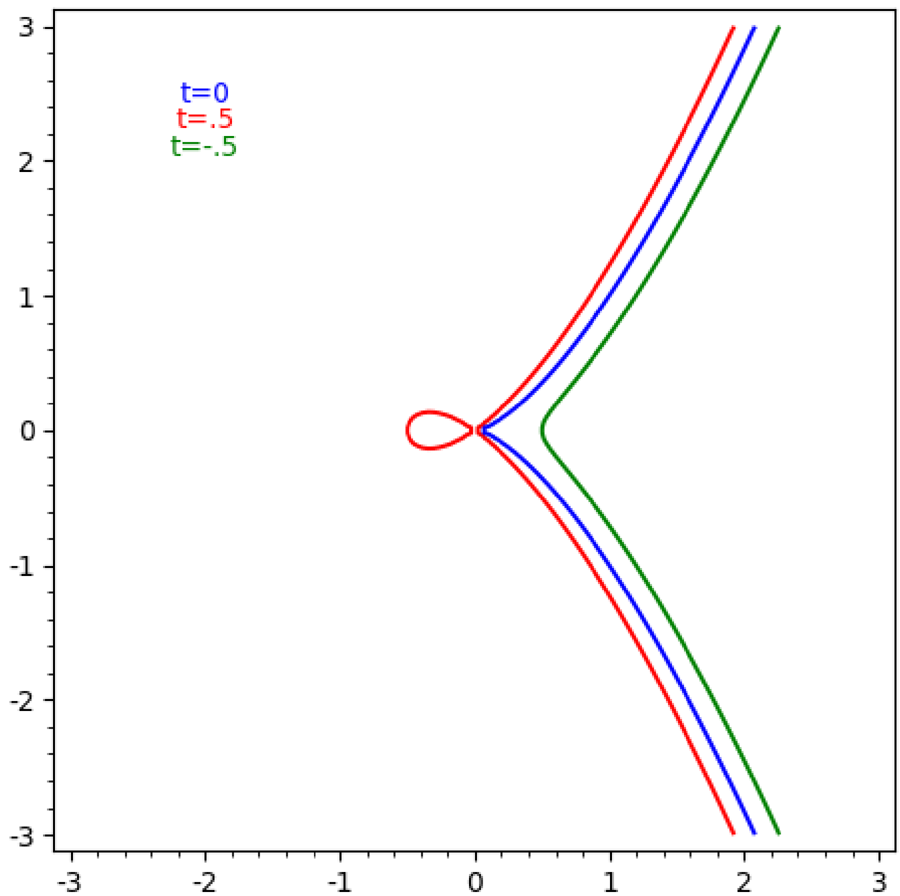
2.6 Algebraic deformations in algebraic geometry

The theory of algebraic deformations fits nicely into algebraic geometry. For a motivating example, look at a deformation of the algebraic curve $y^2 = x^3$. We may instead examine its coordinate ring, say $k[x,y]/(y^2 - x^3)$, but with a deformed multiplication

$$
\begin{aligned}
x^n * x^m &= x^{n+m} \\
y * x^n &= yx^n \\
x^n * y &= yx^n \\
y * y &= x^3 + tx^2.
\end{aligned}
$$

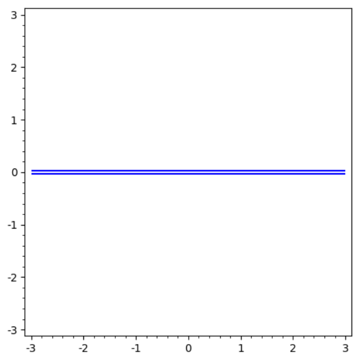
This deformation is the algebra $k[t][x,y]/(y^2 - x^3 - tx^2)$, which is the coordinate ring for a family of curves parameterized by t. Geometrically the curves looked deformed, as seen in the figure bellow. The original curve is also retrieved when t is localized at 0.

$k[t][x, y]/(y^2 - x^3 - tx^2)$

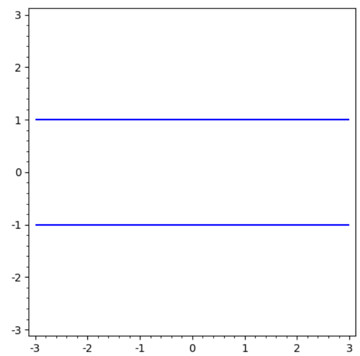


Curves for some values of t

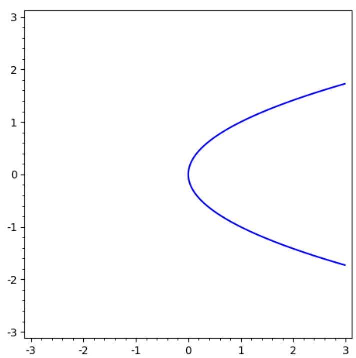
Likewise some other families of plane curves can be thought of as deformations of the double line $k[x, y]/(y^2)$. For simplicity only $y * y$ is given in the next figure.



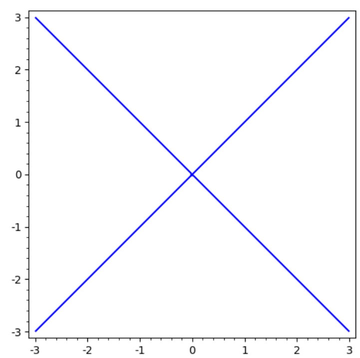
$y * y = 0$



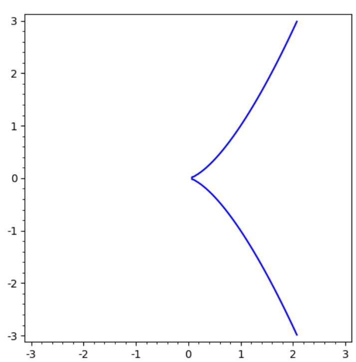
$y * y = t$



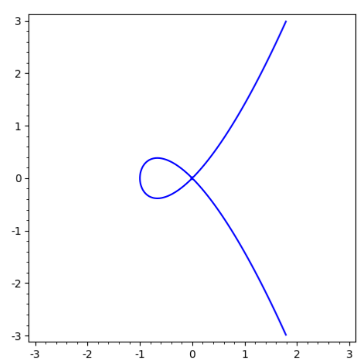
$y * y = xt$



$y * y = x^2 t$



$y * y = x^3 t$



$y * y = (x^3 + x^2)t$

2.7 Kodaira Spencer map

A modern interest in this area comes from studying the *Kodaira Spencer map*. Suppose we have a ring $B = k[t_1, \ldots, t_n]$ and some algebra A over that with relations depending on t_i, so

$$A = B[x_1, \ldots, x_m]/(f_1, \ldots, f_l)$$

where the f_i depend on t_i and x_i, and each t_i commutes with everything in the algebra. Now we pick a maximal ideal $\mathfrak{m} \in \mathrm{Spec}(B)$. We take k to be algebraically closed. By Hilbert's Nullstellensatz $\mathfrak{m}$ corresponds to a point $(t_1 - a_1, \ldots, t_n - a_n)$ where $a_i \in k$. Let A_a be the algebra A specialized at $t_i = a_i$. In other words

$$A_a = A/mA.$$

The *Zariski tangent* space to $\mathrm{Spec}(B)$ at the point a corresponding to $\mathfrak{m}$ is defined to be $(\mathfrak{m}/\mathfrak{m}^2)^*$ where * is the k-linear dual. The *Kodaira Spencer map* takes a from the Zariski tangent space to the second Hochschild Cohomology module of A_a.

$$(m/m^2)^* \to H^2(A_a, A_a).$$

This says a tangent vector in the Zariski tangent space gives a first order deformation of A. The map gives insights into the moduli space $\mathcal{M}_d$ which parameterize all algebras of dimension d up to isomorphism. If the mapping is surjective then the family of algebras parameterized by $\mathrm{Spec}(B)$ maps onto an open dense subset of an irreducible component of $\mathcal{M}_d$. Conversely if it is not surjective, then there are algebras of dimension d which are not parameterized.

Conclusion

The theory of algebraic deformations is a relatively new area of study. Continuing the development this field can lead to new tools in algebraic geometry and even string theory. The Kodaira Spencer map gives a way to study the moduli space $\mathcal{M}_d$ aiding in the classification of algebraic varieties. The Deligne conjecture gives a connection to string theory. The relation to Hochschild cohomology is enticing as computers can compute the dimension of modules quickly. In closing the theory of algebraic deformations is a rich subject that can potentially solve unanswered questions by computation.

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