Computing Global Dimensions of Endomorphism Rings of Modules Over Rings of Finite Cohen-Macaulay Type

by

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Abstract

In this thesis we describe a method for computing the global dimensions of endomorphism rings of finitely generated maximal Cohen-Macaulay modules over complete local Cohen-Macaulay rings. (Note that where rings and modules are discussed in this thesis, the reference is to Cohen-Macaulay rings and finitely generated maximal Cohen-Macaulay modules, unless otherwise noted). The global dimension of a ring R is defined to be the supremum of the lengths of minimal projective resolutions of modules over R. Modules whose endomorphism rings have finite global dimension have applications in algebraic geometry, as every module over a singularity of finite type and dimension less than or equal to 2 whose endomorphism ring has finite global dimension provides a different non-commutative resolution of the singularity, as discussed in [8]. As such, it is of interest to determine the global spectra of such singularities (the set of finite global dimensions which endomorphism rings of MCM modules over a given ring can have), as well as the total number of MCM modules with each global dimension over a given ring, up to Morita-equivalence.

To perform these computations, we made use of an algorithm developed by Iyama and Wemyss, which we have implemented in the software package Sage. We present the results obtained from using this program, and the underlying algorithm, to compute the global dimensions of endomorphism rings of all finitely generated MCM modules over various curve and surface singularities. We also use the algorithm to prove a theorem which guarantees the existence of finitely generated MCM modules over a ring whose endomorphism rings have particular global dimensions, based on certain conditions which the ring's Auslander-Reiten quiver must satisfy.

1 Background

We will begin by explaining some background material.

1.1 Cohen-Macaulay Rings and Modules

The following definitions and results are well-known. For further discussion, such sources as [1], [3], [5], [11], and [19] may be consulted.

Definition 1.1. A commutative ring A is said to be local if it contains exactly one maximal ideal; that is, there exists exactly one ideal $I \neq A$ such that, if J is an ideal such that $I \subseteq J \subseteq A$, then $I = J$ or $J = A$.

Definition 1.2. A simple module is one which has no submodules other than the zero module and itself.

Definition 1.3. The Jacobson radical J_A of a ring A is the intersection of all maximal left ideals of A (or equivalently, all maximal right ideals of A).

Definition 1.4. A prime ideal of a commutative ring A is a proper ideal I such that $ab \in I$ implies $a \in I$ or $b \in I$.

Definition 1.5. A complete local ring is a local ring A with maximal ideal m such that every Cauchy sequence in A has a limit in A, under the m-adic metric. (See [17, Chapter 5] for a description of this metric). An example of a complete local ring is the ring $k[[x, y]]$ consisting of formal power series in x and y over a field k.

Definition 1.6. A ring A is said to be Noetherian if it satisfies the ascending chain condition; that is, any ascending chain of ideals $I_1 \subset I_2 \subset ... \subset I_k \subset ...$ eventually terminates.

Definition 1.7. The Krull dimension (or simply dimension) of a ring is the supremum of the lengths of ascending chains of prime ideals in the ring.

Definition 1.8. The dimension of a nonzero module M over a ring R is the Krull dimension of $R/ann(M)$, where $ann(M)$ is the annihilator of M, the ideal consisting of elements a in R such that $am = 0$ for all m in M. The dimension of the zero module is defined to be -1.

Definition 1.9. Let A be a local, Noetherian ring, and M a finitely generated module over A. Let $a = a_1, ..., a_n$ be a sequence in A. Then a is called an M-sequence (or M-regular sequence) if:

- $a_1 x \neq 0$ for all nonzero $x \in M$ (that is, a_1 is not a zerodivisor of M).
- a_i is not a zerodivisor of $M/(a_1, ..., a_{i-1})M$ for all i such that $2 \leq i \leq n$
- $(a_1, ..., a_n)M \neq M$

Proposition 1.1. If x is an M-sequence, then $(x_1) \subset (x_1, x_2) \subset ... \subset (x_1, x_2, ..., x_n)$ is an ascending chain of ideals in A. Because A is Noetherian, any such chain must eventually terminate, so any M-sequence can be extended to a maximal M-sequence. The assumption that M is finitely generated guarantees that all such maximal sequences have the same length; see [3, 7.4] for a proof. This length is called the depth of M.

Definition 1.10. If the depth of M is equal to its dimension, then M is called a Cohen-Macaulay module, or CM module. If the dimension of M is equal to the dimension of R, then M is called a maximal Cohen-Macaulay (MCM) module. A ring which is an MCM module over itself is called a Cohen-Macaulay ring. All Cohen-Macaulay modules discussed in this thesis should be assumed to be maximal Cohen-Macaulay unless otherwise specified.

Definition 1.11: If there are only finitely many indecomposable MCM modules over a ring R (up to isomorphism), then R is said to be of finite Cohen-Macaulay type. See [13] for further discussion of finite Cohen-Macaulay type.

1.2 Morita-equivalence

The following category-theoretic concepts are described in [4, 1.5 and Chapter 7].

Definition 1.12. Let $f : A \to B$ be a morphism in a category C. f is said to be an isomorphism if it has an inverse; that is, there exists a morphism $f^{-1} : B \to A$ such that $f \circ f^{-1} = id_B, f^{-1} \circ f = id_A$.

Definition 1.13. Given two functors F and G from category C to category D, a natural transformation from F to G is a family of arrows $a_I : F(I) \to G(I)$ in D (indexed by the objects of C), such that for any arrow $f : A \to B$ in C, $a_B \circ F(f) = G(f) \circ a_A$.

Essentially, the arrows a_I transform the image of F to the image of G. We can visualize this using the following commutative diagram:

$$\begin{tikzcd} F(A) \arrow[r, "a_A"] \arrow[d, "F(f)"] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "a_B"] & G(B) \end{tikzcd}$$

The arrow a_A of a natural transformation N is referred to as the component of N at A.

Definition 1.14. A natural isomorphism is a natural transformation whose components are all isomorphisms.

Definition 1.15. Two categories C and D are said to be equivalent if there exist functors $F : C \to D$, $G : D \to C$ such that $F \circ G$ is naturally isomorphic to the identity functor on D, and $G \circ F$ is naturally isomorphic to the identity functor on C.

Definition 1.16.The category of modules over a commutative ring R is the category whose objects are modules over R and whose arrows are homomorphisms of those modules. Two rings are said to be Morita-equivalent if their categories of modules are equivalent. See [1, Chapter 6] for further discussion of Morita-equivalence.

Definition 1.17. Where M is a module over R, $\mathrm{Hom}_R(M, -)$ is a functor from the category of modules over R to the category of modules over $\mathrm{End}_R(M)$. Where N is a module over R, $\mathrm{Hom}_R(M, N)$ is the module over $\mathrm{End}_R(M)$ consisting of homomorphisms from M to N. Where α is a homomorphism from X to Y, $\mathrm{Hom}_R(M, \alpha)$ is a homomorphism from $\mathrm{Hom}_R(M, X)$ to $\mathrm{Hom}_R(M, Y)$, which takes a mapping $x : M \to X$ to $\alpha \circ x : M \to Y$. For more general discussion of Hom functors, see [15, 2.2]

1.3 Global dimensions

In this and the following section, many concepts from Auslander-Reiten theory are defined; further discussion of these can be found in [2], [3, Chapter 7], [10], and [20].

Definition 1.18. An exact sequence is a sequence of modules and homomorphisms $... \xrightarrow{\phi_{n-1}} M_{n-1} \to M_n \xrightarrow{\phi_n} M_{n+1} \xrightarrow{\phi_{n+1}} ...$ such that $Im(\phi_i) = Ker(\phi_{i+1})$ for all homomorphisms ϕ_i in the sequence. A short exact sequence is an exact sequence of the form $0 \to A \to B \to C \to 0$.

We will now prove a well-known result concerning exact sequences:

Proposition 1.2. Given two exact sequences $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ and $0 \to C \oplus D \xrightarrow{\begin{pmatrix} \gamma & \delta \end{pmatrix}} E \xrightarrow{\epsilon} F \to 0$, we can splice the two sequences together to obtain:

$$0 \to A \xrightarrow{\alpha} B \oplus D \xrightarrow{\begin{pmatrix} \gamma \circ \beta & \delta \end{pmatrix}} E \xrightarrow{\epsilon} F \to 0$$

Proof: To show that $0 \to A \xrightarrow{\alpha} B \oplus D \xrightarrow{\begin{pmatrix} \gamma \circ \beta & \delta \end{pmatrix}} D \xrightarrow{\epsilon} E \to 0$ is indeed an exact sequence, it suffices to show that $Im(\alpha) = Ker(\begin{pmatrix} \gamma \circ \beta & \delta \end{pmatrix})$ and $Im(\begin{pmatrix} \gamma \circ \beta & \delta \end{pmatrix}) = Ker(\epsilon)$. First, note that $Ker(\alpha) = Im(0) = 0$, so α is injective (as are γ and δ). Similarly, note that $Im(\beta) = Ker(0) = C$, so β is surjective (as is ϵ).

Thus $\gamma(x) = 0$ if and only if $x = 0$, and so $(\gamma \circ \beta)(x) = 0$ if and only if $\beta(x) = 0$, i.e. $x \in Ker(\beta)$. By exactness, $Ker(\beta) = Im(\alpha)$. Thus we see that

$$Ker(\begin{pmatrix} \gamma \circ \beta & \delta \end{pmatrix}) = Ker(\gamma \circ \beta) \oplus Ker(\delta) = Ker(\gamma \circ \beta) \oplus 0 = Ker(\beta) = Im(\alpha)$$

Similarly, note that by the surjectivity of β,

$$Im(\begin{pmatrix} \gamma \circ \beta & \delta \end{pmatrix}) = Im(\begin{pmatrix} \gamma & \delta \end{pmatrix}) = Ker(\epsilon)$$

Thus the sequence is exact.

In a similar fashion, we can splice together exact sequences of any length, provided there are zeros at both ends.

Definition 1.19. Given a module B, a split monomorphism is an injective homomorphism (i.e., a monomorphism) $f : A \to B$ for which there exists $r : B \to A$ such that $r \circ f = Id_A$.

Definition 1.20. Given a module B, a split epimorphism is a surjective homomorphism (i.e., an epimorphism) $g : A \to B$ for which there exists $s : B \to A$ such that g $\circ s = Id_B$.

Definition 1.21. If every surjective homomorphism $g : X \to B$ is split, where X is any module, then B is said to be a projective module.

Definition 1.22. A submodule S of a module M is said to be superfluous if, for any submodule X of M, $X + S = M$ implies $X = M$.

Definition 1.23. Given a module M, a homomorphism $f : P \to M$ is said to be a projective cover of M if P is projective and the kernel of f is a superfluous submodule of P.

Definition 1.24. A semi-perfect ring is a ring R such that every finitely generated module over R has a projective cover.

Definition 1.25. A projective resolution of a module M is an exact sequence

$$... \to P_n \to ... \to P_1 \to P_0 \to M \to 0$$

where all the P_i terms are projective.

Definition 1.26. The projective dimension of a module is the minimum length of a projective resolution of that module. The global dimension of a ring is the maximum projective dimension of all modules over that ring.

Proposition 1.3. If two rings are Morita-equivalent, then their global dimensions are equal. See [16, 3.5] for a proof.

1.4 Almost-split sequences and Auslander-Reiten quivers

Definition 1.27. A homomorphism of modules $f : A \to B$ is called right almost split if the following conditions are satisfied:

- f is not a split epimorphism
- for any homomorphism h from some module X into B which is not a split epimorphsim, there is a homomorphism $k : X \to A$ such that $f \circ k = h$

Dually, such a homomorphism is called left almost split if the following conditions are satisfied:

- f is not a split monomorphism
- For any homomorphsim h from A into some module X which is not a split monomorphism, there is a homomorphism $k : X \to B$ such that $k \circ f = h$.

Definition 1.28. A short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called an *almost-split sequence* or *Auslander-Reiten (AR) sequence* if f is left almost split and g is right almost split.

Proposition 1.4. For every indecomposable Cohen-Macaulay module C over a ring R, there exists an AR sequence $0 \to A \to B \to C \to 0$, which is unique up to isomorphism. (For a proof, see [20, Chapter 2]). We will call this the Auslander-Reiten sequence of C. If a Cohen-Macaulay module M is not indecomposable, we can obtain the AR sequence of M by taking the direct sum of the AR sequences of its indecomposable direct summands.

Definition 1.29. An irreducible homomorphsim is a homomorphism of modules $f : A \to B$ which satisfies the following properties:

- f is not a split monomorphism or epimorphism.
- If there exist homomorphisms $\alpha : A \to X, \beta : X \to B$ such that $\beta \circ \alpha = f$, then either α is a split monomorphism or β is a split epimorphism.

Definition 1.30. A right-minimal homomorphism is a homomorphism f such that, if there is a homomorphism α such that $f \circ \alpha = f$, then α is an isomorphism.

Definition 1.31. A right τ-sequence for a module X over a ring R is a sequence $\tau X \xrightarrow{\alpha} \theta X \xrightarrow{\beta} X$ such that:

- α and β are right minimal.
- For any module M over R, the following sequences are exact, where $A = \text{End}_R(M)$:

$$\text{Hom}_R(M, \tau X) \xrightarrow{\text{Hom}_R(M,\alpha)} \text{Hom}_R(M, \theta X) \xrightarrow{\text{Hom}_R(M,\beta)} J_A \cap \text{Hom}_R(M, X) \to 0$$

$$\text{Hom}_R(\theta X, M) \xrightarrow{\text{Hom}_R(M,\alpha)} J_A \cap \text{Hom}_R(\tau X, M) \to 0$$

Proposition 1.5. The Auslander-Reiten sequence of an indecomposable CM module over an isolated singularity of dimension less than or equal to 2 is a τ-sequence. This result can be found in [10, Example 3.3].

Definition 1.31. The Auslander-Reiten Quiver of a ring R is a directed graph whose vertices correspond to isomorphism classes of finitely generated indecomposable Cohen-Macaulay modules over R, with an arrow from M to N for each irreducible homomorphism from M to N. This quiver has a partially defined relation τ, known as the Auslander-Reiten translate, which has the following properties:

- τ is an automorphism on the non-projective modules of the quiver.
- For each arrow $M \to N$ there is an arrow $\tau N \to M$.
- The sequence $\tau M \to \theta M \to M$, where θM designates the direct sum of all predecessors of M, is a right τ-sequence.
- If R is an isolated singularity of dimension less than or equal to 2, the sequence $0 \to \tau M \to \theta M \to M \to 0$ is the AR sequence of M.

2 The Ladder Method

We will now discuss the theory involved in computing global dimensions via the method of ladders, covered in more detail in [10] and [12].

2.1 Projective resolutions of simple modules

The material in this section can be found in [10, section 2], in which much of it is phrased in more general category-theoretic terms.

Proposition 2.1. For a finitely generated semi-perfect algebra A over a Noetherian ring, the global dimension of A is the supremum of projective dimensions of simple modules over A; see [8, Section 5].

Proposition 2.2. If R is a complete local ring and M is a module over R, then $\text{End}_R(M)$ is semi-perfect, thus the condition above is satisfied.

Proposition 2.3. For an endomorphism ring $A = \text{End}_R(M)$, the simple modules are of the form $S_i = P_i/(J_A \cap P_i)$, where the P_i are projective and J_A is the Jacobson radical of A, and there is one such simple for each projective module. Furthermore, the indecomposable projective modules over A are all of the form $P_i = \text{Hom}_R(M, M_i)$, where the M_i are the indecomposable summands of M.

Proposition 2.4. $\text{End}_R(\bigoplus M_i^{a_i})$, where $a_i \geq 1$, is Morita-equivalent to $\text{End}_R(\bigoplus M_i)$, and so their global dimensions are equal. Thus we need only consider the case where the multiplicity of each indecomposable summand of M is 1.

Thus, by finding a projective resolution of each simple module over A, we can compute their projective dimensions, and therefore, the global dimension of A.

Definition 2.1. The category $\text{add}\,(M)$ is the category of finite direct sums of indecomposable summands of M.

Definition 2.2. If B is a module over R, then a right $\text{add}\,(M)$-approximation of B is a homomorphism $f : Y \to B$, where $Y \in \text{add}\,(M)$, such that any homomorphism $g : X \to B$, where $X \in \text{add}\,(M)$, factors through f. In other words, f is a right $\text{add}\,(M)$-approximation of B if, for any $X \in \text{add}\,(M)$, $g : X \to B$, there exists $h : X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} Y \arrow[rr, "f"] & & B \\ & X \arrow[ul, dashed, "h"] \arrow[ur, "g"'] & \end{tikzcd}$$

We can write this approximation in the form of a short exact sequence, as follows:

$$0 \to K \to Y \xrightarrow{f} B$$

where K is the kernel of f. This sequence has the property that

$$0 \to \text{Hom}_R(M, K) \to \text{Hom}_R(M, Y) \xrightarrow{\text{Hom}_R(M,f)} \text{Hom}_R(M, B) \to 0$$

is exact. For reasons of brevity, we will often refer to right add (M)-approximations as simply add (M)-approximations.

Definition 2.3. If B is a module in add (M), then a right add (M)-almost split map is a homomorphism $f : Y \to B$ with kernel K, where $Y \in$ add (M), such that the sequence

$$0 \to \operatorname{Hom}_R(M, K) \to \operatorname{Hom}_R(M, Y) \xrightarrow{\operatorname{Hom}_R(M,f)} J_A \cap \operatorname{Hom}_R(M, M_i) \to 0$$

where $A = \operatorname{End}_R(M)$, is exact. As in the case of right add (M)-approximations, we often represent these maps by the exact sequence $0 \to K \to Y \xrightarrow{f} B$ (a right add (M)-almost split sequence).

Consider a right add (M)-almost split map f_0 for the indecomposable summand M_i:

$$0 \to X \to \bigoplus M_a \xrightarrow{f_0} M_i$$

By the definition of a right add (M)-almost split map, the sequence

$$0 \to \operatorname{Hom}_R(M, X) \to \bigoplus \operatorname{Hom}_R(M, M_a) \xrightarrow{\operatorname{Hom}_R(M,f_0)} J_A \cap \operatorname{Hom}_R(M, M_i) \to 0$$

is exact. Since $S_i = \operatorname{Hom}_R(M, M_i)/(J_A \cap \operatorname{Hom}_R(M, M_i))$, the sequence

$$0 \to J_A \cap \operatorname{Hom}_R(M, M_i) \to \operatorname{Hom}_R(M, M_i) \to S_i \to 0$$

is exact as well. Splicing these two sequences together yields the beginning of a minimal projective resolution of S_i:

$$0 \to \operatorname{Hom}_R(M, X) \to \bigoplus \operatorname{Hom}_R(M, M_a) \xrightarrow{\operatorname{Hom}_R(M,f_0)} \operatorname{Hom}_R(M, M_i) \to S_i \to 0$$

Now consider a right add (M)-approximation f_1 of the kernel X:

$$0 \to Y \to \bigoplus M_b \xrightarrow{f_1} X$$

Applying the Hom functor to this sequence gives:

$$0 \to \operatorname{Hom}_R(M, Y) \to \bigoplus \operatorname{Hom}_R(M, M_b) \xrightarrow{f_1} \operatorname{Hom}_R(M, X) \to 0$$

This sequence can be spliced together with the one above to continue constructing the minimal projective resolution of S_i:

$$0 \to \operatorname{Hom}_R(M, Y) \to \bigoplus \operatorname{Hom}_R(M, M_b) \to \bigoplus \operatorname{Hom}_R(M, M_a) \to \operatorname{Hom}_R(M, M_i) \to S_i \to 0$$

By repeating this process of approximating the kernels of sequences, applying the Hom functor, and then splicing the sequences together, we can obtain a minimal projective resolution of S_i, which may be infinite:

$$\ldots \to \bigoplus \operatorname{Hom}_R(M, M_b) \to \bigoplus \operatorname{Hom}_R(M, M_a) \to \operatorname{Hom}_R(M, M_i) \to S_i \to 0$$

For convenience, we often write this sequence as a "pre-resolution", without the Hom functor applied:

$$\dots \to \bigoplus M_b \to \bigoplus M_a \to M_i$$

So by finding add (M)-almost split maps for the indecomposable summands of M and add (M)-approximations for the other finitely generated indecomposable MCM modules over R, we can compute the global dimension of End $_R(M)$.

2.2 Ladders

To construct these resolutions using the method of ladders, as explained by Iyama and Wemyss [12], we must define for each indecomposable module a right τ-sequence. For modules M for which τM is defined, the AR sequence can be used. For the case where R is an isolated singularity of dimension less than or equal to 2, R is an indecomposable projective module over itself. Thus it has no AR sequence, because any surjective homomorphism into R is split, thus there can be none which is almost split. However, we are able to define τR in the following ways:

- When R is a simple curve singularity, we have the fundamental sequence $0 \to 0 \to m \to R \to k \to 0$, from which we can derive the τ-sequence $0 \to m \to R$, so that $\tau R = 0$. We thus add a zero module to the AR quiver to serve as τR.
- When R is an isolated surface singularity, there exists a finitely generated indecomposable MCM module ω_R such that $\omega_R \to \theta R \to R$ is a right τ-sequence; thus we can define $\tau R = \omega_R$ without adding extra modules to the quiver.

To compute the global dimension of End $_R(M)$, where $M = \bigoplus M_i$, we must compute right add (M)-almost split maps for all indecomposable summands M_i, and right add (M)-approximations of the kernels of the sequences we obtain. To compute the right add (M)-almost split sequence for a summand M_i, we will construct a right ladder for M_i, a diagram:

$$\begin{tikzcd}
\dots \arrow[r, "g_3"] & Z_3 \arrow[r, "g_2"] \arrow[d, "a_3"] & Z_2 \arrow[r, "g_1"] \arrow[d, "a_2"] & Z_1 \arrow[r, "g_0"] \arrow[d, "a_1"] & Z_0 = 0 \arrow[d, "a_0"] \\
\dots \arrow[r, "f_3"] & Y_3 \arrow[r, "f_2"] & Y_2 \arrow[r, "f_1"] & Y_1 \arrow[r, "f_0"] & Y_0 = M_i
\end{tikzcd}$$

such that each square commutes, and for each m there exist modules U_{m+1} and morphisms h_m such that

$$Z_{m+1} \oplus U_{m+1} \xrightarrow{\begin{pmatrix} -g_m & h_m \\ a_{m+1} & 0 \end{pmatrix}} Z_m \oplus Y_{m+1} \xrightarrow{\begin{pmatrix} a_m & f_m \end{pmatrix}} Y_m$$

is a right τ-sequence.

A ladder is said to terminate if there is some n such that $Y_n = 0$.

To construct the ladder, we begin with the τ-sequence of M_i: $\tau M_i \to \theta M_i \to M_i$. Let $Z_1 = \tau M_i$, $Y_1 = \theta M_i$. Now we construct the rest of the ladder according to the following recursion formula: decompose Y_m into $Y_m = A_m \oplus B_m$, where A_m is a sum of modules in $\mathrm{add}\,(M)$ and B_m is a sum of modules not in $\mathrm{add}\,(M)$. Decompose Z_m into $Z_m = C_m \oplus U_m$, where C_m is a direct summand of θB_m and U_m has no indecomposable direct summand in common with θB_m. Now let $Z_{m+1} = \tau B_m$, and choose Y_{m+1} such that $Y_{m+1} \oplus C_m = \theta B_m$. That is, we obtain Y_{m+1} from θB_m by removing all of its direct summands which it has in common with C_m.

Eventually we obtain some Y_n such that $Y_n = A_n$, that is, $B_n = 0$. Therefore, θB_n will also be zero, and so $Y_n = 0$, thus the ladder terminates. We can then obtain the exact sequence:

$$0 \to Z_n \oplus \bigoplus_{i=1}^{n} U_i \to Y_n \oplus \bigoplus_{i=1}^{n} B_i \to M_i$$

which is a right $\mathrm{add}\,(M)$-almost split sequence. The method can also be applied to modules not in $\mathrm{add}\,(M)$, in which case it will produce an $\mathrm{add}\,(M)$-approximation rather than a right τ-sequence. The sequences thus produced can then be spliced together to obtain an $\mathrm{add}\,(M)$-resolution of M_i, which then gives a projective resolution of S_i, as outlined in the previous section.

For further discussion of this method, and a proof that the sequences it produces have the desired properties, see [10, Section 3].

3 The Iyama-Wemyss Algorithm

In order to compute these approximations efficiently using a computer, we have an algorithm, explained by Iyama and Wemyss in [12, Section 4], which uses a numbering scheme on the vertices of the universal cover of a ring's AR quiver to compute a ladder in $\mathrm{add}\,(M)$ for a finitely generated indecomposable CM module over R. This algorithm reduces the problem to one of combinatorics and graph theory.

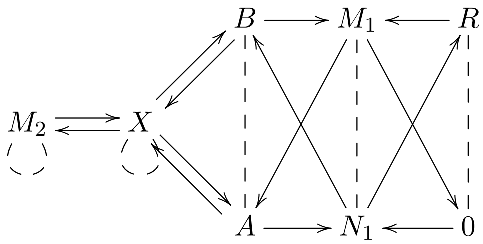
3.1 The universal cover

First we must construct the universal cover of the AR quiver of R. Note that this does not refer to the universal cover in the usual graph-theoretic sense, but rather to the universal cover of the AR quiver as a translation quiver - a graph which preserves the AR translate τ, and covers every cover of the quiver which preserves this relation. In practice, we construct only a finite part of the universal cover when executing the algorithm. We construct it according to the following process:

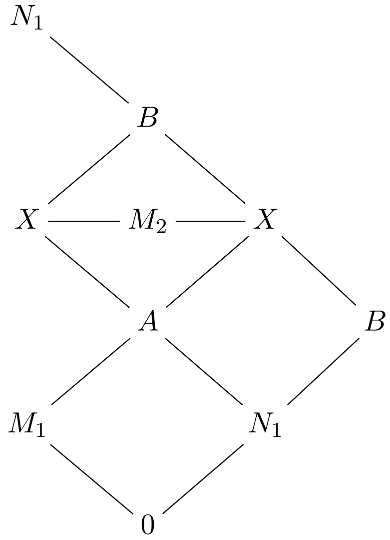
- Begin with one starting vertex, which we may call v, identified with some module A in the AR quiver. We may group the vertices of the graph into levels based on their distance from v; v itself is at level 0.
- Add a vertex w, with an arrow from w to v, for each predecessor of A in the AR quiver. These vertices are at level 1.
- For each vertex w at level a identified with module M, add a vertex x at level $a+1$, with an arrow from x to w, corresponding to each predecessor of M. If the modules corresponding to two or more vertices at the same level have a predecessor in common, only one vertex corresponding to this predecessor is added, with arrows to the vertices corresponding to all its successors (this is where this construction differs from the construction of a purely graph-theoretic universal cover). This process may be continued until the graph is as large as is necessary.

In this way, we construct a segment of the universal cover containing only vertices from which there is a path to our starting vertex. To construct the full universal cover, we would add vertices corresponding to successors of those already added, as well as their predecessors; however, this is not necessary for the purpose of this algorithm. The AR translate is defined as follows: where vertex w, at level a, is identified with module M, we define τw to be the vertex at level $a+2$ identified with τM.

Example 3.1. For example, here is the AR quiver of the ring $R = k[[x,y]]/(x^3+y^4)$, where k is an algebraically closed field of characteristic zero. Here, and elsewhere, the AR translate is indicated by a dashed line. This is a simple curve singularity, known as the E_6 singularity. (The diagram below is adapted from [20, Chapter 9]).



(Note that a zero module is included, to serve as τR, so that τ will be defined for every module in the quiver). A finite part of the universal cover of this quiver, constructed with B as the starting vertex, is:



3.2 Vertex numbering

To compute the global dimension of $\operatorname{End}_R(M)$, where $M = \bigoplus M_i$, we must find pre-resolutions of the indecomposable summands M_i. We will do this by numbering the vertices of the universal cover according to the following scheme.

To compute a right add(M)-almost split sequence for M_i, we first select a starting vertex, v, which is identified with M_i (in the implementation of this algorithm in Sage, this is the same starting vertex which is used to construct the finite part of the graph that we work with, but there is no theoretical reason why this must be so). We assign to this vertex the number 1, and to all vertices from which there is no path to v, we assign the number 0 (note that this means all vertices not in the part of the graph which we explicitly construct will be numbered 0). We then circle all vertices which are identified with summands of M, or with the zero module. Next, we assign numbers to the rest of the vertices in the graph, working backwards from v, as follows: the number of vertex w is equal to the sum of the numbers of all successors of w, minus the number of $\tau^{-1}w$.

For the purpose of this numbering, all vertices which are circled, as well as those with negative numbers, are treated as if their numbers were zero. When we reach a point where all vertices at two consecutive levels of the graph are numbered zero, or satisfy one of the conditions to have their numbers treated as zero, we stop numbering vertices. The reason for this is as follows:

Suppose that all vertices at levels a and $a+1$ are either circled, or have numbers less than or equal to zero, and consider a vertex w at level $a+2$. By the construction of the graph and the definition of τ, all successors of w are at level $a+1$, and $\tau^{-1}w$ is at level a. Therefore, all of these vertices will have their numbers treated as zero, and so the number assigned to w will be $0-0=0$. Thus all vertices at level $a+2$ will be numbered 0; by induction, we can see that the same will be true for all levels greater than $a+2$. Thus we know what numbers will be assigned to all vertices in the graph, so we can terminate the numbering process.

Now we must use the numbered graph to construct a right $\operatorname{add}(M)$-almost split sequence for M_i. The sequence constructed is $0 \to K_0 \to P_0 \to M_i$, where the terms K_0 and P_0 are determined as follows:

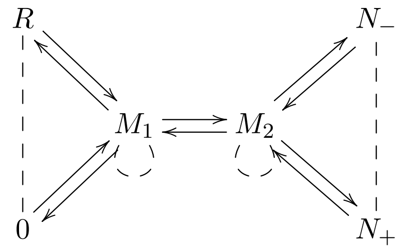
- K_0 is the direct sum of the modules corresponding to all negative-numbered vertices, with multiplicities given by the vertices' numbers. For instance, if there are two vertices numbered -1 identified with module A and one vertex numbered -2 identified with module B, then $K_0 = A \oplus A \oplus B^2 = A^2 \oplus B^2$.

- P_0 is the direct sum of the modules corresponding to circled vertices (i.e., either indecomposable summands of M or the zero module) with positive numbers, with multiplicities again given by the numbers of the vertices. For instance, if there are two vertices numbered 1 identified with module A and one vertex numbered 2 identified with module B, where both A and B are circled, then $K_0 = A \oplus A \oplus B^2 = A^2 \oplus B^2$.

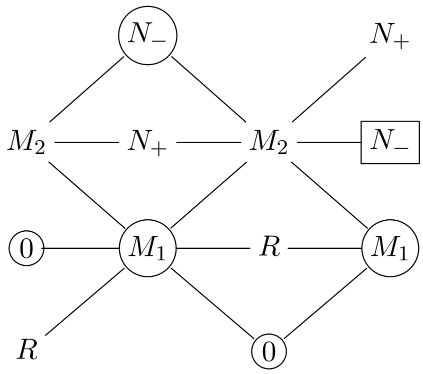
Of course, we can apply this algorithm in the same way to construct an $\operatorname{add}(M)$-approximation of a finitely generated indecomposable MCM module which is not a summand of M, since it is equivalent to the ladder method described in the previous section.

3.3 Example

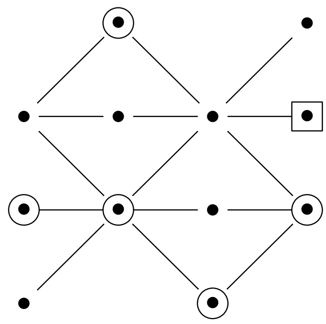
Example 3.2. Let R be the A_5 singularity: $R = k[[x,y]]/(x^6 + y^2)$. The AR quiver of this ring, with the zero module included, is shown below. (Once again, the diagram is adapted from [20, Chapter 9]).



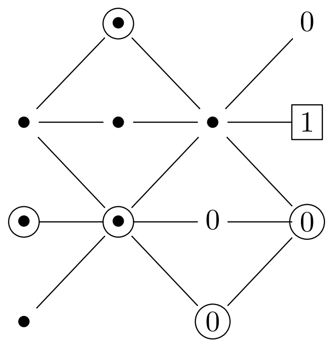
We will calculate the global dimension of $M = M_1 \oplus N_-$. A finite portion of the universal cover of the AR quiver of R is shown below. Note that arrowheads have been omitted for clarity, but arrows are understood to run from left to right, and the AR translate of a vertex w can be seen directly across from w, two stages to the left. The initial vertex has been boxed, and other vertices identified with modules in the set $\{M_1, N_-, 0\}$, the indecomposable summands of M and the zero module, have been circled.



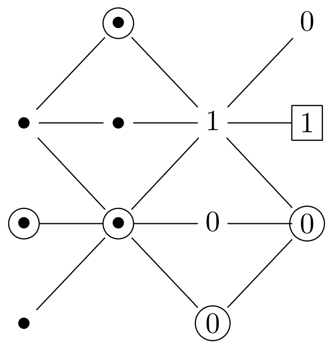
We will replace the vertex labels with bullets:



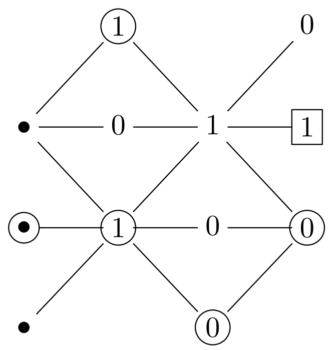
We will begin by computing the right $\mathrm{add}\,(M)$-almost split sequence for N_-. We begin numbering the vertices, according to the algorithm described in the previous subsection. First, the starting vertex, identified with N_-, is numbered 1, and all vertices from which there is no path to the initial vertex (including those outside the portion of the graph shown below) are numbered 0:



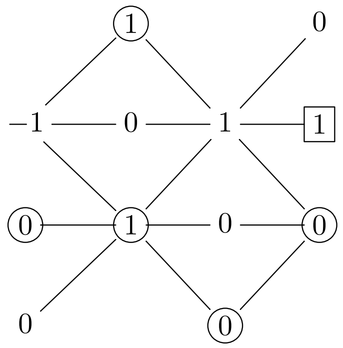
Now we number the predecessor of the initial vertex. As its only successor is numbered 1, and its inverse AR translate is numbered 0, it is numbered 1 - 0 = 1:



Next we number the predecessors of this predecessor, according to the same formula:



Now we number the vertices at the next level of the graph, treating the two circled vertices as if they were numbered zero:



Now we have two consecutive levels of the graph at which every vertex is either numbered zero, circled, or has a negative number; thus we can now read off the right add (M)-almost split sequence from this graph. We see that the vertex numbered -1 is identified with module M_2, while the two circled vertices numbered 1 are identified with modules N_- and M_1. So the right add (M)-almost split sequence is:

$$0 \to M_2 \to N_- \oplus M_1 \to N_-$$

Now we must approximate M_2, the kernel of the approximation. Using the same method, we obtain:

$$0 \to N_- \to N_- \oplus M_1 \to M_2$$

Splicing the two sequences together, we get:

$$0 \to N_- \to N_- \oplus M_1 \to N_- \oplus M_1 \to N_-$$

By the same method, we obtain the following add (M)-resolution for M_1:

$$0 \to N_- \to N_- \oplus M_1 \to M_1$$

So we can see, by examining the lengths of the sequences, that the global dimension of End $_R(M)$ is 3, as this will be length of the longest projective resolution of a simple module obtained from the sequences above.

4 Implementation of the algorithm in Sage

Note that the description in this section also appears in [10, 4.3.1].

4.1 Computing approximations

We will now give a brief description of the Sage program which we have written, which uses this algorithm to compute global dimensions of endomorphism rings. First we will describe the routine used to compute a ladder for an indecomposable module. The program takes as input:

- The graph of the AR quiver, including a zero module which is treated as the AR translate of the ring R as a module over itself. When the "number" of a module is referred to, the reference is to the number of the corresponding vertex in this graph.
- An adjacency matrix identifying the AR translate of each module.
- The number of the module whose right $\operatorname{add}(M)$-approximation or almost split sequence is to be computed (we will refer to either of these simply as the module's "sequence" for brevity).
- A set containing the numbers of the indecomposable summands of M, called S.

We define the following variables:

- Various lists to keep track of data concerning the vertices of the graph, such as their associated numbers and the level of the graph at which they are found.
- A number to keep track of the current level of the graph (the level whose vertices are being numbered).
- A vector representing the kernel of the module's sequence, with the multiplicity in the kernel of the module numbered n in the n^{th} position (initially these are all set to zero).
- A vector representing the middle term of the module's sequence, similar to the above.

The program associates numbers to the vertices of the universal cover by the following process. To start with, a graph UC is constructed, containing only the initial vertex v, which is numbered 1 (i.e., the number in the first position of the appropriate list is set to 1). A new level is then added to the graph, consisting of the predecessors of v, according to the construction described in the previous section.

Whenever a new level is added, it becomes the "current" level, and all vertices at that level are numbered. The number of a vertex w is calculated according to the formula in the previous section, by adding up the numbers of the successors of w in the universal cover, and subtracting the number of $\tau^{-1}w$. If any of these vertices are not in the finite portion of the graph which is constructed by the program, their numbers are assumed to be zero, since the only vertices that can have nonzero numbers are those from which there exists a path to the initial vertex, and these are also the only vertices that can be included in the finite graph. For the purpose of this computation, vertices corresponding to modules in the set S or the zero module (i.e., vertices which would be circled when carrying out the algorithm by hand) and vertices whose numbers are negative are treated as if their numbers were zero.

After a vertex is assigned the number n, we first check to see if n is negative; if so, the multiplicity of the corresponding module in the kernel is increased by $-n$ (i.e., the entry in the corresponding position of the vector representing the kernel is increased by $-n$). If n is non-negative, we then check to see if w corresponds to a module in S; if so, the multiplicity of the corresponding module in the middle term is increased by n.

After all vertices at the current level are numbered, the program checks whether all vertices at both the current and previous levels are either numbered zero, or satisfy the criteria to have their numbers treated as zero when numbering other vertices. If this is the case, the algorithm is finished (since any further vertices would always be numbered zero) and the program returns a list consisting of the vector representing the kernel, followed by the vector representing the middle term. If not, a new level is added to the graph, consisting of all predecessors of vertices at the current level. This new level then becomes the current level, and we begin the numbering process again.

4.2 Computing global dimensions

After a sequence is computed, the kernel is examined. The algorithm may terminate under two circumstances:

- If all of its summands are in S, then the pre-resolution of M_i is complete, and a list of vectors representing all terms in the pre-resolution is returned.
- If the set of its summands not in S is identical to that of the kernel of a previously-computed sequence (up to multiplicity), then it is clear that the resolution is infinite, since computing further approximations will eventually return the same kernel again, leading to an infinite loop. In this case, a null value is returned.

Otherwise, the following procedure is applied: each summand of the kernel which is not in S is removed from the kernel (i.e. has its multiplicity changed to zero) and approximated. These approximations are then added together, with the same multiplicities as the corresponding modules had in the kernel, to produce an approximation for the part of the kernel which is not in S (i.e., the vectors representing the kernels of the individual approximations are added, and likewise for the middle terms). We then splice this approximation sequence together with the partially-complete pre-resolution which was obtained by splicing together all the previous approximations. This is done by appending its kernel to the front of the list representing the partially complete pre-resolution, and adding its middle term to the kernel of that sequence (which has had all its modules not in S removed). This increases the length of the list representing the partially complete pre-resolution by one. We then examine the kernel of this new approximation, and repeat the procedure until one of the two stopping conditions listed above is satisfied.

To compute the global dimension of the endomorphism ring of a module, we simply compute pre-resolutions for each of the module's summands, with S being the set of its summands. The global dimension of the endomorphism ring is equal to the length of the longest such sequence (which may be infinite). By applying this procedure to all modules over a ring whose indecomposable summands have multiplicity 1 (i.e., letting S run through all subsets of modules in the ring's AR quiver), we can obtain the ring's global spectrum.

5 Results for simple curve singularities

In [20, Chapters 5 and 9], Yoshino describes the following types of simple curve singularities:

- $A_n : x^{n+1} + y^2 = 0, n \geq 1$
- $D_n : x^2y + y^{n-1} = 0, n \geq 4$
- $E_6 : x^4 + y^3 = 0$
- $E_7 : x^3y + y^3 = 0$
- $E_8 : x^5 + y^3 = 0$

We applied the implementation of the Iyama-Wemyss algorithm described above to rings of the form $R = k[[x, y]]/(p)$, where k is an algebraically closed field and p is a polynomial defining one of the singularities above. The AR quivers for these rings can be found in [20, Chapters 5 and 9]. Our goal was to determine the set of finite global dimensions which MCM modules over each ring can have; that is, we wished to determine each ring's global spectrum. By computing the global dimensions of endomorphism rings of all finitely generated indecomposable MCM modules (up to multiplicities of indecomposable summands) over the A_n, E_n, and $D_n, n \leq 13$ singularities, we obtained the following global spectra:

- A_n, n even: $\{1, 2\}$
- A_n, n odd: $\{1, 2, 3\}$ (This and the previous result were previously computed using other methods, in [7].)
- $E_6, D_4, D_5 : \{1, 2, 3, 4\}$
- $D_n, 6 \leq n \leq 13 : \{1, 2, 3, 4, 5\}$
- $E_7, E_8 : \{1, 2, 3, 4, 5, 6\}$

The exact numbers of modules with endomorphism rings of each global dimension over each ring we examined are shown in the following table, taken from [10, 4.3.1]:

Singularity	# of subsets with GLDIM						Finite GLDIM	Infinite GLDIM	Total
	1	2	3	4	5	6			
E_6	1	13	34	4	0	0	52	75	$2^7 - 1$
E_7	3	80	$7,638$	$6,933$	486	8	$15,148$	$17,619$	$2^{15} - 1$
E_8	1	94	$24,614$	$26,479$	$2,500$	48	$53,736$	$77,335$	$2^{17} - 1$
D_n, n even									
D_4	7	28	207	90	0	0	332	179	$2^9 - 1$
D_6	7	73	$2,416$	$1,713$	66	0	$4,275$	$3,916$	$2^{13} - 1$
D_8	7	146	$25,601$	$26,743$	$1,458$	0	$53,955$	$77,116$	$2^{17} - 1$
D_{10}	7	253	$265,602$	$389,942$	$23,422$	0	$679,226$	$1,417,925$	$2^{21} - 1$
D_{12}	7	400	$2,745,634$	$5,449,152$	$353,644$	0	$8,548,837$	$25,005,594$	$2^{25} - 1$
D_n, n odd									
D_5	3	20	95	26	0	0	144	111	$2^8 - 1$
D_7	3	58	$1,164$	555	16	0	$1,796$	$2,299$	$2^{12} - 1$
D_9	3	122	$12,541$	$9,527$	382	0	$22,575$	$42,960$	$2^{16} - 1$
D_{11}	3	218	$130,672$	$146,418$	$6,778$	0	$284,089$	$764,486$	$2^{20} - 1$
D_{13}	3	352	$1,352,109$	$2,113,324$	$109,690$	0	$3,575,478$	$13,201,737$	$2^{24} - 1$
A_{2k+1}	3	$k+1$	k^2+3k+2	0	0	0	k^2+4k+6		$2^{k+3} - 1$
A_{2k}	1	k	0	0	0	0	$k+1$		$2^{k+1} - 1$

Based on these results, we have conjectured that the global spectrum of D_n is $\{1, 2, 3, 4, 5\}$ for all $n \geq 6$; however, this has not been proven.

6 Results for isolated surface singularities

The following background information can be found in [20, Chapter 10].

Theorem 6.1. Let $G \subseteq GL(V)$ be a finite matrix group and ρ a representation of G. The McKay graph $McK(G,\rho)$ is a directed graph whose vertices correspond to the indecomposable representations of G, with n arrows from A to B whenever $\mathrm{Hom}_{kG}(B, \rho\otimes A)$, the set of homomorphisms from B to $\rho \otimes A$ is an n-dimensional vector space over k, the underlying field of the vector space V.

Definition 6.2. A pseudoreflection is an element of $GL(V)$, where V is an n-dimensional vector space, which fixes a subset of V of dimension $n-1$.

Theorem 6.2. If the group G contains no pseudoreflections, then $McK(G,\rho)$, where ρ is the representation of G given by its inclusion in $GL(V)$, is isomorphic to the AR quiver of $k[V]^G$, the invariant ring of G over a field of characteristic zero. The AR translate is given by: $\tau N = N \otimes Det$, where Det is the determinant representation of G, and is defined for all modules in the AR quiver (as described in section 2.2). In these cases, these invariant rings correspond to isolated surface singularities.

We used a program in the GAP software package to compute the McKay graphs for four finite matrix groups, and thus obtain the AR quivers for their invariant rings and the corresponding curve singularities. The singularities in question, using the notation of Riemenschneider (see [18]), are given below, along with the global spectra obtained:

Singularity	Generators of matirx group	Generators of invariant ring	Global spectrum
$C_{8,5}$	$\begin{bmatrix} \zeta_8 & 0 \\ 0 & -\zeta_8 \end{bmatrix}$	x^8, x^3y, xy^3, y^8	$\{2,3,4,5\}$
$C_{16,9}$	$\begin{bmatrix} \zeta_{16} & 0 \\ 0 & -\zeta_{16} \end{bmatrix}$	$x^{16}, x^7y, x^5y^3, x^3y^5, xy^7, y^{16}$	$\{2,3,4,5,6,7,8,9\}$
$D_{5,3}$	$\begin{bmatrix} \zeta_3^2 & 0 \\ 0 & \zeta_3 \end{bmatrix}, \begin{bmatrix} 0 & \zeta_8 \\ \zeta_8 & 0 \end{bmatrix}$	$x^4y^4, x^{12}-y^{12}, x^7y+xy^7, x^9y^3-x^3y^9$	$\{2,3,4,5\}$
$D_{7,5}$	$\begin{bmatrix} \zeta_5 & 0 \\ 0 & \zeta_5^{-1} \end{bmatrix}, \begin{bmatrix} 0 & \zeta_8 \\ \zeta_8 & 0 \end{bmatrix}$	$x^4y^4, x^{20}-y^{20}, x^{11}y-xy^{11}, x^{13}y^3+x^3y^{13}$	$\{2,3,4,5\}$

More detailed results are found in the table below, taken from [10, 4.4]:

Singularity	# of subsets with gl. dim									Finite (total)	Infinite	Total
	1	2	3	4	5	6	7	8	9			
$C_{8,5}$	0	1	72	8	8	0	0	0	0	89	166	2^8-1
$C_{16,9}$	0	1	10,488	23,032	10,144	2,304	336	16	16	46,337	19,198	$2^{16}-1$
$D_{5,3}$	0	1	732	340	280	0	0	0	0	1,353	2,742	$2^{12}-1$
$D_{7,5}$	0	1	7,568	5,968	3,548	0	0	0	0	17,085	48,450	$2^{16}-1$

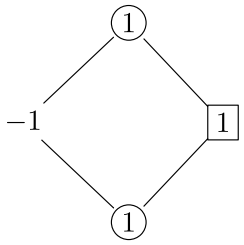
7 Orbit theorems

7.1 Zero-predecessor case

Theorem 7.1. Let R be a complete local Cohen-Macaulay ring of formal power series over an isolated singularity of dimension less than or equal to 2 and finite Cohen-Macaulay type (i.e., a ring of the kind studied in the previous sections) and let Q be its AR quiver. Suppose there is an orbit J of the AR translate τ having cardinality n, such that no module in J has a predecessor in J; then there exists a finitely generated MCM module over R having global dimension k for all $2 \leq k \leq n+1$.

Proof: The method used in this proof is adapted from that used by Dlab and Ringel in [9, Proposition 2]. Let C be the direct sum of all modules in Q which are not in J, and let N be any module in J. Let $M = C \oplus N$; so all modules not in J, and N, are summands of M. We will show that the global dimension of $\text{End}_R(M)$ is $n+1$.

First, we will use the Iyama-Wemyss algorithm to compute the pre-resolution of N. Following the usual numbering scheme, N and its predecessors in the universal cover of Q will all be numbered 1. By assumption, all predecessors of N are outside the orbit J, so they will all be summands of M, and therefore circled. Thus they will be counted as zero in future numbering computations. So τN will be numbered $0-1=-1$, while any other vertices at its level will be numbered $0-0=0$.



Let P_i designate the direct sum of all predecessors of $\tau^i N$. By the result above, we see that the first sequence obtained is the AR sequence of N:

$$0 \to \tau N \to P_0 \to N \to 0$$

Now we must approximate τN (assuming $\tau N \neq N$). Again, by assumption, all of its predecessors will be outside J, and so we obtain its AR sequence as an approximation:

$$0 \to \tau^2 N \to P_1 \to \tau N \to 0$$

This pattern continues until we obtain the sequence:

$$0 \to \tau^n N \to P_{n-1} \to \tau^{n-1} N \to 0$$

Since $\tau^n N = N$, the kernel of this sequence is an indecomposable summand of M, and so we have obtained our final approximation. Splicing all these approximations together, we get the pre-resolution:

$$0 \to N \to P_{n-1} \to ... \to P_1 \to P_0 \to N \to 0$$

We can see that the length of the projective resolution of a simple module which this will provide is $n+1$.

Now consider the $\operatorname{add}(M)$-almost split sequence for some module A which is not in J:

$$0 \to K \to Q \to A$$

For simplicity, assume that K is indecomposable; if not, we can simply apply the following argument to each of its indecomposable summands. First, suppose that $K = N$, or that K is not in J. Then K is a summand of M, so the sequence above, with length 2, is the pre-resolution of A.

Alternatively, suppose that $K = \tau^a N$ for some $1 \leq a \leq n-1$. Then, by the argument used in constructing the resolution of N, the pre-resolution of A will be:

$$0 \to N \to P_{n-1} \to ... \to P_a \to Q \to A$$

We can see that this sequence has length $n+2-a \leq n+1$.

So the maximum length of a pre-projective resolution of an indecomposable summand of M is $n+1$; thus the global dimension of $\operatorname{End}_R(M)$ is $n+1$.

We will now generalize this to prove the theorem as stated. Let $2 \leq k \leq n+1$, and let N be some module in J, as before. Now let M be the direct sum of all modules not in the orbit J, plus $\bigoplus_{i=0}^{n+1-k} \tau^i N$.

As before, the $\operatorname{add}(M)$-almost split sequence for any module $\tau^i N \in J$ is its AR sequence:

$$0 \to \tau^{i+1} N \to P_i \to \tau^i N \to 0$$

For $0 \leq i \leq n-k$, $\tau^{i+1}N$ is a summand of M, so this is the pre-resolution, of length 2. For $i = n+1-k$, we approximate $\tau^{i+1}N$, obtaining its AR sequence, and proceed as before until we obtain a sequence having N as its kernel. So the pre-resolution of $\tau^{n-k+1}N$ is:

$$0 \to N \to P_{n-1} \to ... \to P_{n-k+1} \to \tau^{n-k+1} N \to 0$$

We can see that this sequence has length k. As in the previous case, a pre-resolution of a module not in J must have length less than or equal to k, thus the maximum length of these pre-resolutions, and therefore the global dimension of $\operatorname{End}_R(M)$, is k.

7.2 One-predecessor case

7.2.1 Background

Next we will present a partial result which extends this theorem to the case where each module in J has exactly one predecessor in J. To begin with, we will provide some background information which is used in the proof of the theorem; see [6, 0.6] for further discussion of these concepts.

Definition 7.1. The rank of a module M over an integral domain R is the dimension of the tensor product $k \otimes M$ as a vector space over k, where k is a skew field in which

R can be embedded. If R is not an integral domain, then it must instead be embedded in a product of skew fields, $k_1 \times k_2 \times ... \times k_n$, in which case the rank of M is a vector $(r_1, ..., r_n)$, where r_i is the dimension of $M \otimes k_i$.

Lemma 7.1. If $0 \to X \to Y \to Z \to 0$ is a short exact sequence, then $rank(Y) = rank(X) + rank(Z)$; this result can be found in [6, 0.6].

Definition 7.2. The torsion submodule of a module M over a ring A is the submodule tM consisting of elements m in M such that $am = 0$ for some a in A which in not a zerodivisor of A.

Definition 7.3. If $tM = M$, then M is called a torsion module.

Definition 7.4. If $tM = 0$, then M is called a torsion free module.

Proposition 7.1. If M is an MCM module of rank 0 over a CM ring, then $M = 0$.

Proof. As described in [6, 0.6], any module of rank 0 is a torsion module. As described in [14, Chapter 4], any MCM module over a CM ring is torsion free. So M must be both torsion and torsion free; from Definitions 7.3 and 7.4 we can see that this implies $M = 0$.

7.2.2 Orbit theorem in the one-predecessor case

Theorem 7.2. As before, let R be a complete local Cohen-Macaulay ring of formal power series over an isolated singularity of dimension less than or equal to 2 and finite Cohen-Macaulay type, and let Q be its AR quiver. Suppose there is an orbit J of the AR translate τ having cardinality n, such that each module in J has exactly one predecessor in J (note that by this, we also mean there are no multiple arrows, so each module A in J has one predecessor B in J, with a single arrow from B to A. The reason for this restriction is that the Iyama-Wemyss algorithm as described in section 3 would need to be modified to account for multiple arrows). Let M be the direct sum of all finitely generated indecomposable MCM modules over R (up to isomorphism) which are not in J, plus one module $N \in J$. Then the global dimension of M is:

- Infinite, if $n = 0 \bmod 3$.
- $2(\frac{n+2}{3})$, if $n = 1 \bmod 3$.
- Infinite, if $n = 2 \bmod 3$.

Proof: Say there is an arrow from $\tau^k N$ to N for each N in J. Because all predecessors of N have τN as a predecessor, there is an arrow from τN to $\tau^k N$. By assumption, the only predecessor in J of $\tau^k N$ is $\tau^{2k} N$, so $\tau^{2k} N = \tau N$. Thus $2k = 1 \bmod n$. From this we can see that n must be odd, and we can solve this equation to get $k = \frac{n+1}{2}$. Note that $gcd(n, k) = 1$.

Now we will determine the global dimension of a module M. For simplicity, we will show only those vertices which are in J, since those outside of it will all be circled, and thus will always be treated as zero, so they will not affect which vertex is the kernel of any sequence.

Let N be the one circled vertex in J. We will write $\tau^a N$ as a. Suppose we compute a sequence for some a, not having N as a predecessor; then the universal cover of Q is numbered as follows:

$$-1 \longrightarrow 0 \longrightarrow 1 \longrightarrow \boxed{1}$$

$$a+3k \qquad a+2k \qquad a+k \qquad a$$

So the sequence is $0 \to a+3k \to P \to a$. Note that when a module identified as P, Q, L, or P_i or Q_i for some i appears from here on, it is simply the middle term of some approximation, whatever that should happen to be.

If we approximate the succesor of N in J, that is $-k$, then we get:

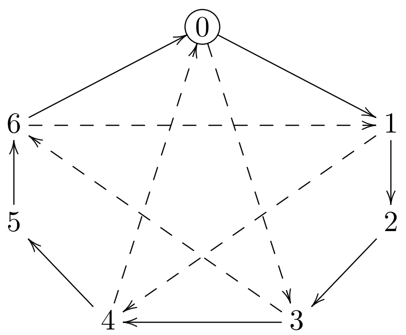
$$-1 \longrightarrow 1 \longrightarrow \boxed{1}$$

$$k \qquad 0 \qquad -k$$

So the sequence is $0 \to k \to P \to -k$.

Let $\mu = \tau^k$. We can reorganize J into a μ-orbit, and we will now write a for $\mu^a N$ (all modules in J are in the same μ-orbit since $gcd(n,k) = 1$). We can see that the kernel of the sequence for a is $a+3$ mod n, unless $a = n-1$, in which case it is 1. We can think of this as "jumping forward" by three positions in the μ-orbit to reach the next kernel, unless this would cause us to pass over N, in which case we land at 1 regardless of where we began; using this rule we can identify the kernels of right add(M)-approximations and almost split sequences, and thus find the lengths of pre-resolutions of summands of M.

7.2.3 Example: n = 7



In computing the pre-resolution of N in the μ-orbit shown above, we get the following sequences (the zero modules are omitted for clarity; 0 here means $\mu^0 N$, that is, N):

$3 \to P_1 \to 0$

$6 \to P_2 \to 3$

$1 \to P_3 \to 6$

$4 \to P_4 \to 1$

$0 \to P_5 \to 4$.

The dashed arrows run from each module we approximate to the kernel of its approximation. From this we can see that the resolution of N is

$$0 \to N \to P_5 \to P_4 \to P_3 \to P_2 \to P_1 \to N \to 0$$

Using this method, we can determine the global dimension of $\operatorname{End}_R(M)$. We will begin by demonstrating that J cannot be the entire quiver, or an isolated component of the quiver.

Lemma 7.2. J must be a proper subgraph of G, connected to the rest of the quiver; that is, there must be some modules $A \in J$, $B \in G \setminus J$ with an arrow from A to B.

First, suppose J forms an isolated component of the AR quiver (i.e. no module not in J has a predecessor in J), and let A be a module in J. Then the AR sequence of A is:

$$0 \to \tau A \to \mu A \to A \to 0$$

By Lemma 7.1, we see that $rank(\tau A) + rank(A) = rank(\mu A)$.

Now consider the AR sequence of $\mu^{-1}A$, the successor of A. Since A is a predecessor of $\mu^{-1}A$, $\tau\mu^{-1}A$ must be a predecessor of A. Since A has only one predecessor, it must therefore be the case that $\tau\mu^{-1}A = \mu A$. So the AR sequence of $\mu^{-1}A$ is:

$$\mu A \to A \to \mu^{-1}A$$

Thus $rank(\mu A) + rank(\mu^{-1}A) = rank(A)$. Substituting this into the previously obtained rank equation, we see that

$$rank(\tau A) + rank(\mu A) + rank(\mu^{-1}A) = rank(\mu A)$$

Simplifying and rearranging this, we get:

$$rank(\tau A) = -rank(\mu^{-1}A)$$

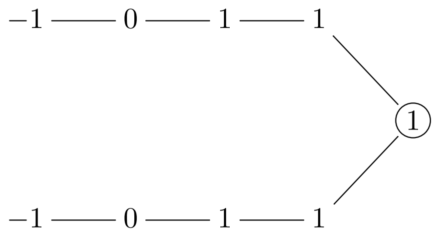
Since the rank of a module cannot be negative, this implies that the ranks of both τA and $\mu^{-1}A$ are zero. Thus we see that every module in J has rank zero; by Proposition 7.1, this implies that they are all equal to zero. This, however, contradicts the requirement that distinct modules in an AR quiver cannot be isomorphic, except in the trivial case where the cardinality of J is 1.

Therefore, it must be the case that there for some module A in J, there exists some module B not in J with an arrow from B to A.

7.3 Procedure for determining the global dimension of $\operatorname{End}_R(M)$

Lemma 7.3. Now suppose there is an arrow from some arbitrary vertex $\tau^b N$ to $A \in G/J$. Then for all $0 \le a \le n-1$; then there is an arrow from $\tau^{b+a}N$ to $\tau^a N$. So because at least one module in J has a successor outside J, so does every module in J. Likewise, if $\tau^b N$ has a predecessor A outside J, then A is a successor of $\tau^{b-1}N$, so once again, each module in J will have a successor outside J.

Lemma 7.4. When we compute a sequence for a module A outside J having $\tau^a N$ as a predecessor, $\tau^a N$ gets numbered 1. Its predecessors will then be numbered the same as they would be when computing the sequence for $\tau^a N$, since all their predecessors not in J will be circled. Thus the kernel of the sequence for A will be the sum of the kernels of the sequences for all its predecessors in J. For instance:



A couple of technical notes:

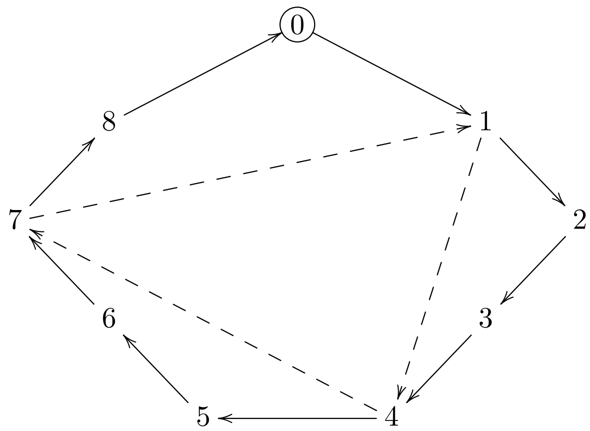
- The exception to this is when N is a predecessor of A, since it is circled, so its predecessors will all just be numbered zero.
- The "lines" corresponding to different predecessors of A will remain separate, because if $B \neq C$, then $\mu B \neq \mu C$, etc., thus the predecessors at each level of the graph of each of the predecessors of A in J will remain distinct.

So to compute the global dimension of $\mathrm{End}_R(M)$, we must compute resolutions of all modules in J. We will now break the problem into three distinct cases.

7.3.1 Case 1: n = 0 mod 3

Consider the pre-resolution of $\mu N = 1$. Because $n = 0 \bmod 3$, $n - 2 = 1 \bmod 3$. So when we begin at 1 and jump forward by three steps in the μ-orbit, we eventually get to $n - 2$, and then back to 1. Thus the pre-resolution never terminates, so the global dimension of $\mathrm{End}_R(M)$ is infinite.

Illustration, for $n = 9$:



7.3.2 Case 2: n = 1 mod 3

As in the zero-predecessor case, we begin by computing the pre-resolution of N. Since $n = 1 \bmod 3$, $0 = n - 1 \bmod 3$. So adding 3 to the kernels eventually gives a sequence with $n-1$ as its kernel. The kernel of the next sequence will then be 1. We then continue to add 3 until we finally get a sequence with the kernel $n = N$. So the sequences we get are:

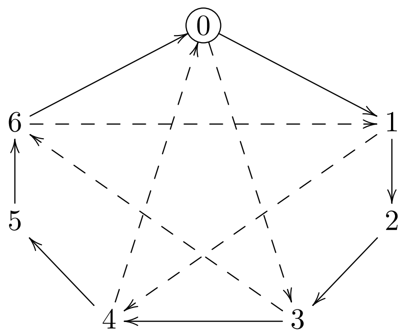
$3 \to P_1 \to N$
$6 \to P_2 \to 3$
...
$n - 1 \to P_{\frac{n-1}{3}} \to n - 4$
$1 \to L \to n - 1$
$4 \to Q_1 \to 1$
...
$n = N \to Q_{\frac{n-1}{3}} \to n - 3$

So the pre-resolution will be

$$N \to Q_{\frac{n-1}{3}} \to ... \to Q_1 \to L \to P_{\frac{n-1}{3}} \to ... \to P_1 \to N$$

having length $2(\frac{n-1}{3}) + 2 = 2(\frac{n+2}{3})$.

Illustration, for $n = 7$:

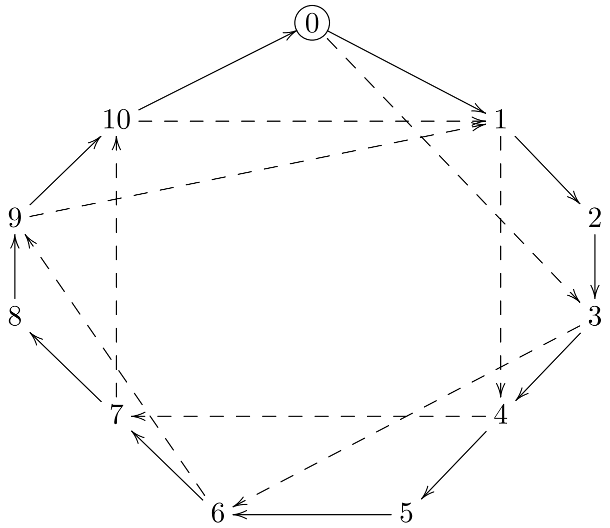


It is relatively easy to see that no pre-resolution can be longer than this: In the process of moving around the μ-orbit, we must eventually either land on n, or pass over it and land on $n - 1$, then jump to 1 and continue until we get to n. In approximating $N = 0$, we start as far away from $n - 1$ as possible, so we land on the greatest possible number of modules on the way to $n - 1$. Thus the global dimension of $\mathrm{End}_R(M)$ is $2(\frac{n+2}{3})$.

7.3.3 Case 3: n = 2 mod 3

Consider the pre-resolution of N. Since $n = 2 \bmod 3$, $n - 2 = 0 \bmod 3$, so we eventually get $n - 2$ as a kernel, then add 3 to get 1. We see that $n - 1 = 1 \bmod 3$, so we continue to add 3 until we get $n - 1$ as a kernel. Following the usual procedure, we then jump to 1, and so we are stuck in an infinite loop. Thus the global dimension of $\mathrm{End}_R(M)$ is infinite.

Illustration, for $n = 11$:



Thus the theorem is proven.

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