

## **Abstract**

Infinitesimal deformations of maximal orders over smooth algebraic surfaces are studied. We classify which maximal orders may admit deformations that are not finite over their centres. The classification is in terms of the algebraic surface that corresponds to the centre of the order and the ramification divisor of the order.

# Quantizable Orders over Surfaces

Colin Ingalls

February 9, 2012

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Hochschild Cohomology</b>	<b>7</b>
<b>3</b>	<b>Gröbner Bases and Deformations</b>	<b>13</b>
<b>4</b>	<b>The Five Term Exact Sequence</b>	<b>22</b>
<b>5</b>	<b>Deformations at Height One Primes</b>	<b>26</b>
<b>6</b>	<b>Deformations at Singularities of the Discriminant</b>	<b>33</b>
<b>7</b>	<b>Deformations of Orders over Surfaces</b>	<b>39</b>

## 1 Introduction

Let  $X$  be a smooth projective surface over  $k$ , an algebraically closed field of characteristic zero. Let  $K$  be the field of rational functions on  $X$ , and  $A$  be an  $\mathcal{O}_X$ -order in a central simple  $K$ -algebra. We study deformations of  $A$  as an algebra over the base field  $k$ . We determine conditions necessary for the existence of an infinitesimal deformation not finite over its centre. The main result classifies which maximal orders allow such deformations. The proof has an affine case which determines the deformations of an order, and a global geometric part which classifies the possible projective surfaces and ramification divisors associated to an order that admits deformations

not finite over its centre. The main result is the following combination of propositions 7.10, 7.5 and 7.16.

**Theorem 1.1** *Let  $A$  be a maximal order over a smooth projective surface  $X$ . If  $A$  admits a deformation that is not finite over its centre then  $X$  has an effective anti-canonical divisor  $D$ , and one of the following is true*

1.  *$X$  is rational or birationally ruled and  $A \simeq \underline{\text{End}}(V)$  for some vector bundle  $V$ ,*
2.  *$X$  is rational,  $A$  is ramified on  $D$ , and  $D$  is one of the following:*
  - (a) *a smooth elliptic curve,*
  - (b) *a rational curve with one node,*
  - (c) *a polygon of smooth rational curves,*
3.  *$X$  is birationally ruled over an elliptic curve  $E$ ,  $A$  is ramified on  $D$ , and  $D = E_0 + E'_0$  where  $E_0, E'_0$  are disjoint sections of  $E$ ,*
4.  *$X$  is a minimal K3 surface or an abelian surface and  $A$  is an Azumaya algebra.*

The affine part of the proof uses Hochschild cohomology and Gröbner bases to compute the deformations using the known local structures of the order. Let  $R$  be a two dimensional  $k$ -algebra and let  $X = \text{Spec } R$ . Let  $A$  be a maximal order in a central simple  $K(R)$ -algebra. We will write  $\Theta = \text{Der}_k(R, R)$  and  $\Theta^2 = \wedge^2 \text{Der}_k(R, R)$ . Let  $D$  be the curve that is the locus of the reduced discriminant of  $A$ . We will denote the singular locus of  $D$  by  $\text{Sing } D$ . Let  $I$  be the ideal sheaf of  $D$ . There are  $R$ -modules  $\text{AlgExt}_k(A, A)$  and  $\text{AlgExt}_R(A, A)$  parameterizing first order deformations of  $A$  as an  $k$ -algebra and as an  $R$ -algebra respectively. Let  $S(A)$  denote the image of the natural map  $\text{AlgExt}_R(A, A) \rightarrow \text{AlgExt}_k(A, A)$ . The next result follows from theorem 5.11.

**Theorem 1.2** *Let  $R$  be a smooth  $k$ -algebra of dimension two. Let  $A$  be a maximal  $R$ -order. Let  $I$  be the ideal of the reduced discriminant of  $A$ . There is a canonical exact sequence*

$$0 \rightarrow S(A) \rightarrow \text{AlgExt}_k(A, A) \rightarrow I\Theta^2,$$

and  $S(A)$  is a finite length  $R$ -module whose support contained in  $\text{Sing } D$ .

The deformations corresponding to the elements of  $S(A)$  are deformations of  $A$  as an  $R$ -algebra, so they are finite over their centres.

**Example 1.3** Let  $R = k[u, v]$ , so  $X = \text{Spec } R$  is the affine plane. Let  $A$  be the  $R$ -algebra generated by elements  $x, y$  satisfying the relations  $x^2 = u, y^2 = v, xy + yx = 0$ . Then  $A$  is a maximal order in the central simple algebra  $A \otimes k(u, v)$ . The discriminant locus of  $A$  is the union of the coordinate axes  $uv = 0$ . The order  $A$  is generated as a  $k$ -algebra by  $x, y$  satisfying the relation  $yx + xy = 0$ . Let  $k_1 = k[\varepsilon]/\varepsilon^2$  and let  $A_1$  be the  $k_1$ -algebra generated by  $x, y$  satisfying the relation  $xy + yx - \varepsilon xy = 0$ . The centre of  $A_1$  is contained in  $k + \varepsilon A_1$ , and so  $A_1$  is not finite over its centre.

**Proposition 1.4** *Let  $A_1$  be a deformation of  $A$  over  $k[\varepsilon]/\varepsilon^2$ . Then there are unique  $\alpha$  in  $k$  and  $f$  in  $R$  so that  $A_1$  is isomorphic to the  $k$ -algebra generated by elements  $x, y$  satisfying the relation  $xy + yx - \varepsilon\alpha - \varepsilon fxy$ .*

This proposition follows from 6.1 and 5.11. The term  $\alpha$  is an element of  $S(A) \simeq k$  and is torsion supported at  $\text{Sing } D$ . The term  $f$  is a commutator. If we lift  $u, v$  in  $R$  to  $\bar{u}, \bar{v}$  then  $[\bar{u}, \bar{v}] = \varepsilon 4fuv$ . The commutator vanishes on the discriminant locus  $D$ . The deformation  $A_1$  is finite over its centre if and only if  $f = 0$ .

There is an exact sequence that helps in computing the deformations of  $A$ .

**Proposition 1.5** *Let  $A$  be a maximal order over a smooth  $k$ -algebra  $R$  of dimension two. There is a canonical exact sequence*

$$0 \rightarrow \text{OutDer}_k(A, A) \rightarrow \Theta \xrightarrow{\phi_1} \text{AlgExt}_R(A, A) \rightarrow \text{AlgExt}_k(A, A) \xrightarrow{\phi_2} \Theta^2$$

In the above sequence  $\text{OutDer}_k(A, A)$  is the  $R$ -module of outer derivations on  $A$ . The proof of this proposition uses a change of base spectral sequence for Hochschild cohomology, 2.5, and the fact that  $\text{OutDer}_R(A, A) = 0$  in proposition 4.6.

An analysis of this sequence is used to compute  $\text{AlgExt}_k(A, A)$ . Let  $N_{X/D} = \text{Hom}_R(I, R/I)$ , the normal sheaf of  $D$  in  $X$ . The natural map  $\Theta \rightarrow N_{X/D}$  factors through  $\text{AlgExt}_R(A, A)$ . This gives the following six term exact sequence of kernels and cokernels.

$$0 \rightarrow \text{OutDer}_k(A, A) \rightarrow \Theta(-\log D) \xrightarrow{\theta} \ker \psi \rightarrow S(A) \rightarrow T_1 \rightarrow \text{cok } \psi \rightarrow 0$$

In the above sequence  $T_1$  is the module of deformations of the curve  $D$  and  $\Theta(-\log D)$  is module of derivations preserving  $I$ .

Local computations using Gröbner bases show that  $\theta = 0$  in codimension one and that the final four terms of the sequence are all torsion supported in  $\text{Sing } D$ . Using the result of these computations it is determined that  $\text{Im}(\phi_2) \subseteq \wedge^2 \Theta(-\log D) \simeq I \wedge^2 \Theta$ . This completes the algebraic part of the proof.

The structure of the affine deformations is analyzed in three cases. These cases correspond to the possible étale local structures of the order. The next proposition follows from [Reiner] and [Artin82a], [Artin86]. The arguments of proposition 3.4 and theorem 3.8 (a) of [Artin82a] apply without change to prove part 2.

**Proposition 1.6** *Let  $A$  be a maximal order on a smooth surface  $X$  ramified on the discriminant curve  $D$ . Then*

1. *if  $p$  is not in  $D$  then  $A$  is étale locally isomorphic to a matrix algebra at  $p$ .*
2. *if  $p$  is a smooth point of  $D$  then  $A$  is étale locally isomorphic to a standard order at  $p$ .*
3. *if  $p$  is a singular point of  $D$  and  $A$  has global dimension two then there is a list of possible complete local structures of  $A$  at  $p$  in [Artin86].*

Standard orders are defined in 4.1. The deformations in the first case are described in 5.1. The deformations in second case are analyzed in section 5. These deformations together with reflexivity are used to prove the main algebraic theorem 5.11. The third case is analyzed in section 6. In this case a stronger version of the algebraic theorem is shown.

It follows from 1.2 that if a maximal order  $A$  over a projective surface  $X$  has a deformation not finite over its centre then  $H^0(X, \mathcal{O}_X(-D - K)) \neq 0$ . Hence  $H^0(X, \mathcal{O}_X(-K)) \neq 0$ . This leads to the geometric part of the proof. The possible surfaces  $X$  and ramification curves  $D$  so that  $H^0(X, \mathcal{O}_X(-D - K)) \neq 0$  are described. This is done using the classification of surfaces and the Brauer group. Since  $-K$  is effective,  $X$  must be rational, birationally ruled, minimal K3 or abelian by [BPV]. The possible effective anti-canonical divisors in a rational or birationally ruled surface are listed. Exact sequences involving the Brauer group of a surface limit the possible ramification curves.

There is an exact sequence in [AM] calculating the Brauer group of a simply connected surface. It follows from the proof in [AM] that this sequence is a complex for an arbitrary projective surface. The complex restricts the possible rational subcurves of  $D$ , and allows one to conclude the following fact about the ramification divisor  $D$  of a maximal order 7.12. If there is a non-trivial map  $f : \mathbb{P}^1 \rightarrow D$  then  $f^{-1}(\text{Sing } D)$  contains at least two points. This restricts the possible ramification curves in a rational surface to the ones listed in theorem 1.1. In the case of a birationally ruled surface, this allows us to reduce to the possibility that  $D$  is a smooth subcurve of  $-K$ . The possible ramification curves are reduced further by an exact sequence involving the Brauer group of  $U = X - D$ , where  $D$  is a smooth curve in  $X$ . The following is proposition 7.17.

**Proposition 1.7** *Let  $D$  be a smooth curve in a smooth projective surface  $X$ , then there is a canonical exact sequence*

$$0 \rightarrow \text{Br}(X) \rightarrow \text{Br}(U) \rightarrow H^1(D, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\gamma} H^3(X, \mu) \rightarrow H^3(U, \mu)$$

where the map  $\gamma$  is Poincaré dual to the restriction map  $H^1(X, \mu) \rightarrow H^1(D, \mu)$ .

The map  $\gamma$  is an isomorphism if  $D$  is a section of birationally ruled surface  $X$ . Using this fact we show that the possible ramification divisors contained in  $-K$  is as in the statement of the theorem 1.1.

The paper is organized as follows. Section 2 recalls some facts about Hochschild cohomology. Section 3 shows how to compute deformations and Hochschild cohomology using Gröbner bases 3.12. The method is based on the method in [Anick]. The last part of section 3 applies this method to compute the first order deformations of a certain class of orders which includes standard orders. In section 4 it is proved that  $\text{OutDer}_R(A, A) = 0$ . This fact is used to refine the 5-term exact sequence of the change of base spectral sequence of Hochschild cohomology. Section 5 computes the deformations at codimension one primes and uses reflexivity to obtain the main algebraic result. Section 6 computes the module  $S(A)$  for most orders of global dimension two and proves a stronger algebraic result for these orders. Section 7 generalizes the algebraic results to the global case and computes the possible surfaces and ramification divisors of maximal orders that may allow deformations not finite over their centres.

**Acknowledgments** This paper is based on the author's Ph.D. thesis. I would like to thank my advisor Michael Artin for many suggestions and invaluable help.

## 2 Hochschild Cohomology

In this section some known facts about Hochschild cohomology are reviewed. Most of the following is in [Wb] and [Grst]. The possibly new results are the change of base spectral sequence for Hochschild cohomology 2.5, and the resulting invariance under étale base change 2.15.

Let  $R$  be a commutative ring and let  $A$  be an  $R$ -algebra. By this it is meant that there is a map from  $R$  to the centre of  $A$ . We say that  $A$  is a projective  $R$ -algebra if  $A$  is projective as an  $R$ -module.

**Definition 2.1** We define  $R_1$  to be a *nilpotent extension* of  $R$  if there is a nilpotent ideal  $N$  of  $R_1$  so that  $R_1/N = R$ . Suppose  $A$  is a flat  $R$ -algebra. Given a nilpotent extension  $R_1$  of  $R$ , a *deformation* of  $A$  over  $R_1$  is a flat  $R_1$ -algebra  $A_1$  with a given isomorphism  $A_1 \otimes_{R_1} R \simeq A$ . Two deformations  $A_1$  and  $A'_1$  are isomorphic if there is an  $R_1$ -algebra isomorphism  $A_1 \rightarrow A'_1$  that induces the identity map on  $A$  modulo  $N$ .

Let  $R$  be a commutative ring and  $A$  be an  $R$ -algebra. The enveloping algebra of  $A$  over  $R$  is defined to be  $A_R^e = A \otimes_R A^\circ$ . A left  $A_R^e$  module  $M$  is an  $A$ -bimodule with a symmetric  $R$ -action.

The Hochschild cohomology of  $A$  with coefficients in  $M$  is defined to be

$$H_R^i(A, M) = \text{Ext}_{A_R^e}^i(A, M).$$

The low dimensional groups have useful interpretations. Define the centralizer of  $A$  in  $M$  to be

$$M^A = \{m \in M \text{ such that } [A, m] = 0\}.$$

Note that  $A^A$  is the centre of  $A$ . There is the natural map  $M \rightarrow \text{Der}_R(A, M)$  defined by  $m \mapsto [m, -]$ , the derivation given by the commutator with  $m$ . Define the outer  $R$ -derivations from  $A$  to  $M$  to be the cokernel of this map, giving the exact sequence

$$0 \rightarrow M^A \rightarrow M \rightarrow \text{Der}_R(A, M) \rightarrow \text{OutDer}_R(A, M) \rightarrow 0.$$

Let  $A$  be a projective  $R$ -algebra. Consider algebra extensions of  $A$  by  $M$

$$0 \rightarrow M \rightarrow A_1 \rightarrow A \rightarrow 0$$

with  $M^2 = 0$ . Let  $\text{AlgExt}_R(A, M)$  be the set of isomorphism classes of these extensions.

The proof of the following facts is in [Wb] 9.1.1, 9.2.1, 9.3.1, and [Grst] section 3.

**Proposition 2.2** *Let  $R$  be a commutative algebra and let  $A$  be an  $R$ -algebra, then*

$$\begin{aligned} H_R^0(A, M) &\simeq M^A, \\ H_R^1(A, M) &\simeq \text{OutDer}_R(A, M). \end{aligned}$$

If  $A$  is a projective  $R$ -algebra then

$$H_R^2(A, M) \simeq \text{AlgExt}_R(A, M)$$

and  $\text{AlgExt}_R(A, A)$  is the set of isomorphism classes of deformations of  $A$  over  $R_1 = R[\varepsilon]/\varepsilon^2$ . If we let

$$H_R^3(A, A) = \text{Obs}_R(A, A)$$

then there is a (non-linear) map  $\text{AlgExt}_R(A, A) \rightarrow \text{Obs}_R(A, A)$  that gives the obstruction to extending a first order deformation to second order.

In the case that  $A$  is a projective  $R$ -algebra, the Hochschild cohomology can be calculated by a canonical, but unfortunately large complex known as the bar complex. Let  $C_R^n(A, M) = \text{Hom}_R(A^{\otimes n}, M)$  be the bar cochains. Let  $\delta : C_R^n(A, M) \rightarrow C_R^{n+1}(A, M)$  be the boundary map defined by

$$\begin{aligned} \delta f(a_1 \otimes \cdots \otimes a_n) &= a_1 f(a_2 \otimes \cdots \otimes a_n) \\ &\quad - f(a_1 a_2 \otimes a_3 \otimes \cdots \otimes a_n) \\ &\quad + f(a_1 \otimes a_2 a_3 \otimes \cdots \otimes a_n) \\ &\quad + \cdots \\ &\quad + (-1)^n f(a_1 \otimes \cdots \otimes a_{n-2} \otimes a_{n-1}) a_n. \end{aligned}$$

It is well known that this complex computes the Hochschild cohomology [Wb] 9.1.1, [Grst] section 3. The following result is proved in [Grst] section 10.

**Proposition 2.3** *The Hochschild cohomology  $H_R^*(A, A)$  is a graded skew commutative algebra over the centre of  $A$ . The multiplication is induced by the following definition on bar cochains.*

$$(f \smile g)(a_1 \otimes \cdots \otimes a_{n+m}) = f(a_1 \otimes \cdots \otimes a_n)g(a_{n+1} \otimes \cdots \otimes a_{n+m}). \quad (2.4)$$

**Proposition 2.5** *Let  $k$  be a commutative ring and let  $R$  be a commutative  $k$ -algebra. Let  $A$  be a flat  $R$ -algebra and let  $M$  be an  $A$ -bimodule with a symmetric  $R$ -action. Then there is a spectral sequence*

$$H_k^{p+q}(A, M) \Leftarrow H_R^p(A, H_k^q(R, M)) = E_2^{pq}.$$

**Proof.** Given an arbitrary ring map  $S \rightarrow T$ , a left  $S$ -module  $M$  and a left  $T$ -module  $N$ , there are the standard change of base spectral sequences, [Wb] 5.6.3,

$$\mathrm{Ext}_S^{p+q}(M, N) \Leftarrow \mathrm{Ext}_T^p(\mathrm{Tor}_q^S(M, T), N) = E_2^{pq}, \quad (2.6)$$

$$\mathrm{Ext}_S^{p+q}(N, M) \Leftarrow \mathrm{Ext}_T^p(N, \mathrm{Ext}_S^q(T, M)) = E_2^{pq}, \quad (2.7)$$

Using the spectral sequence 2.6 for the map  $R_k^e \rightarrow A_k^e$  yields

$$\mathrm{Ext}_{R_k^e}^{p+q}(R, M) \Leftarrow \mathrm{Ext}_{A_k^e}^p(\mathrm{Tor}_q^{R_k^e}(R, A_k^e), M) = E_2^{pq}$$

Consider extending an  $R$ -bimodule  $M$  to an  $A$ -bimodule. This operation may be written in two ways by considering  $M$  as left  $R_k^e$ -module or as an  $R - R$ -bimodule,

$$M \otimes_{R_k^e} A_k^e \simeq A \otimes_R M \otimes_R A.$$

Since  $A$  is flat over  $R$ ,  $A_k^e$  is flat over  $R_k^e$ . Also  $A_R^e \simeq R \otimes_{R_k^e} A_k^e$ . So the above spectral sequence collapses to give

$$\mathrm{Ext}_{R_k^e}^q(R, M) \simeq \mathrm{Ext}_{A_k^e}^q(A_R^e, M). \quad (2.8)$$

The change of base spectral sequence 2.7, applied to the map  $A_k^e \rightarrow A_R^e$ , gives us

$$\mathrm{Ext}_{A_k^e}^{p+q}(A, M) \Leftarrow \mathrm{Ext}_{A_R^e}^p(A, \mathrm{Ext}_{A_k^e}^q(A_R^e, M)) = E_2^{pq}. \quad (2.9)$$

Combine 2.8 and 2.9.  $\square$

**Corollary 2.10** *Let  $k$  be a commutative ring,  $R$  a  $k$ -algebra. Let  $A$  be a projective  $R$ -algebra. The edge map of the above spectral sequence*

$$H_k^*(A, A) \rightarrow H_R^0(A, H_k^*(R, A))$$

*is a map of graded skew commutative algebras.*

**Proof.** The edge map is given by restriction of bar cocycles. The definition of the cup product 2.3 is compatible with restriction. Let  $a$  be in  $A$ , and  $f$  be in  $C_k^n(A, A)$ . We interpret  $a$  as being in  $C_k^0(A, A)$ . Note that  $(\delta a)(x) = [x, a]$ . Using the notation of [Grst] section 10 p. 85, we define the Hochschild cochain  $f \circ \delta a \in C_k^n(A, A)$  by

$$\begin{aligned} (f \circ \delta a)(x_1 \otimes \cdots \otimes x_n) &= f([x_1, a] \otimes x_2 \otimes \cdots \otimes x_n) \\ &\quad + f(x_1 \otimes [x_2, a] \otimes x_3 \otimes \cdots \otimes x_n) \\ &\quad + f(x_1 \otimes x_2 \otimes [x_3, a] \otimes \cdots \otimes x_n) \\ &\quad + \cdots \\ &\quad + f(x_1 \otimes \cdots \otimes x_{n-1} \otimes [x_n, a]). \end{aligned}$$

We also define  $(\delta a \circ f) \in C_k^n(A, A)$  by

$$(\delta a \circ f)(x_1 \otimes \cdots \otimes x_n) = f(x_1 \otimes \cdots \otimes x_n)a - af(x_1 \otimes \cdots \otimes x_n).$$

Gerstenhaber [Grst] section 10 p. 85, shows that the graded commutator of the product defined by  $\circ$  defines a graded lie bracket on the Hochschild cohomology  $H_k^*(A, A)$ . In particular he shows that  $f \circ \delta a - \delta a \circ f$  is a coboundary. Let  $i : R \rightarrow A$  be the algebra structure map. When we restrict to  $R$  we see that  $i^*(f \circ \delta a) = 0$ . Hence  $i^*(\delta a \circ f) = [a, i^*f]$  is a coboundary.  $\square$

**Lemma 2.11** *Let  $k$  be a commutative ring,  $R$  a Noetherian ring.  $M$  a finitely generated  $R$ -module, and  $N$  an  $R - k$ -bimodule. Let  $V$  be a flat  $k$ -module. Then*

$$\mathrm{Ext}_R^p(M, N \otimes_k V) \simeq \mathrm{Ext}_R^p(M, N) \otimes_k V.$$

**Proof.** Let  $F^* \rightarrow M$  be a resolution of  $M$  by finite free  $R$ -modules. The statement is true if  $M$  is a finite free  $R$ -module. Hence the two functors

$\text{Hom}_R(-, N \otimes_k V)$  and  $\text{Hom}_R(-, N) \otimes_k V$  give isomorphic complexes when applied to the resolution of  $M$ . Hence they have isomorphic cohomology. Since  $V$  is flat the cohomology of  $\text{Hom}_R(F^*, N) \otimes_k V$  is  $\text{Ext}_R^*(M, N) \otimes_k V$ .

□

In the case that  $R$  is smooth, the spectral sequence 2.5 has a simpler form. We define the module of dual  $q$ -forms to be  $\Theta_{R/k}^q = \wedge^q \text{Der}_k(R, R)$ .

**Proposition 2.12** *Let  $k$  be a commutative Noetherian ring, and let  $R$  be a smooth commutative  $k$ -algebra. Let  $A$  be a flat  $R$ -algebra so that  $A_R^e$  is Noetherian and let  $M$  be an  $A_R^e$ -module. Then*

$$H_k^{p+q}(A, M) \Leftarrow H_R^p(A, M) \otimes_R \Theta_{R/k}^q = E_2^{pq}$$

**Proof.** Using the fact that  $\text{Tor}_q^{R_k^e}(R, R) \simeq \Omega_{R/k}^q$  as shown in [Wb] 9.4.7 and the spectral sequence 2.6 for the map  $R_k^e \rightarrow R$ ,

$$\begin{aligned} H_k^{p+q}(R, M) &= \text{Ext}_{R_k^e}^{p+q}(R, M) \\ &\Leftarrow \text{Ext}_R^p(\text{Tor}_q^{R_k^e}(R, R), M) \\ &\simeq \text{Ext}_R^p(\Omega_{R/k}^q, M). \end{aligned}$$

Since  $\Omega_{R/k}^q$  is a projective  $R$ -module, the spectral sequence collapses,

$$\begin{aligned} H_k^q(R, M) &\simeq \text{Hom}_R(\Omega_{R/k}^q, M) \\ &\simeq \Theta_{R/k}^q \otimes_R M. \end{aligned}$$

So using lemma 2.11,

$$H_R^p(A, H_k^q(R, M)) \simeq \text{Ext}_{A_R^e}^p(A, \Theta_{R/k}^q \otimes_R M) \simeq H_R^p(A, M) \otimes_R \Theta_{R/k}^q.$$

□

**Lemma 2.13** *Let  $R$  be a commutative ring. Let  $S$  be a flat commutative  $R$ -algebra. Let  $A$  be a  $R$ -algebra so that  $A_R^e$  is Noetherian. Let  $M$  be an  $A_R^e$ -module, then*

$$H_S^i(A \otimes S, M \otimes S) \simeq H_R^i(A, M) \otimes_R S$$

**Proof.** We have that

$$(A \otimes S)_S^e \simeq (A \otimes S) \otimes_S (A \otimes S)^o \simeq A_R^e \otimes S.$$

Using flat base change for Ext, [Reiner] 2.39,

$$\mathrm{Ext}_{A_R^e \otimes S}^i(A \otimes S, M \otimes S) \simeq \mathrm{Ext}_{A_R^e}^i(A, M) \otimes S.$$

□

We will use the following statement from [Wb] 9.1.7. It follows from comparing the bar resolutions.

**Lemma 2.14** *Let  $R$  be a commutative ring. Let  $S$  be a commutative  $R$ -algebra. Let  $A$  be a projective  $R$ -algebra. Then*

$$H_R^i(A, M) \simeq H_S^i(A \otimes S, M)$$

**Proposition 2.15** *Let  $R$  be a commutative Noetherian ring. Let  $S$  be an étale  $R$ -algebra. Let  $A$  be a flat  $R$ -algebra so that  $A_R^e$  is Noetherian. Let  $M$  be a  $A_R^e$ -module. Then*

$$H_k^q(A \otimes_R S, M \otimes_R S) \simeq H_k^q(A, M) \otimes_R S.$$

**Proof.** The spectral sequence of proposition 2.12 collapses for  $R \rightarrow S$  étale giving

$$H_R^p(S, M) \simeq H_S^p(S, M).$$

Hence

$$H_R^p(S, M) \simeq \begin{cases} M & p = 0 \\ 0 & p \geq 0. \end{cases}$$

The spectral sequence 2.5 gives

$$H_k^{p+q}(S, M) \Leftarrow H_R^p(S, H_k^q(R, M)).$$

This collapses to give

$$H_k^q(S, M) \simeq H_k^q(R, M). \tag{2.16}$$

Equations 2.14 and 2.16 give natural isomorphisms of functors on  $M$ . Hence the two Grothendieck spectral sequences

$$H_k^{p+q}(A \otimes_R S, M \otimes_R S) \Leftarrow H_S^p(A \otimes_R S, H_k^q(S, M \otimes_R S)),$$

$$H_k^{p+q}(A, M \otimes_R S) \Leftarrow H_R^p(A, H_k^q(R, M \otimes_R S))$$

are isomorphic. Hence

$$H_k^p(A \otimes_R S, M \otimes_R S) \simeq H_k^p(A, M \otimes_R S).$$

Now use lemma 2.11.  $\square$

The above change of base equation does not hold for flat extensions. Étale extensions are needed to collapse the spectral sequence.

So using 2.13 and 2.15 we get that 2.5 gives a spectral sequence of étale sheaves. The following statement is due to K. R. Dennis and can be found in [Wb] 9.5.6.

**Proposition 2.17** *Let  $k$  be a commutative ring. Let  $A$  and  $B$  be Morita equivalent  $k$ -algebras, by the bimodules  ${}_A P_B$ , and  ${}_B Q_A$ . Then  $A_k^e$  and  $B_k^e$  are Morita equivalent by sending an  $A$ -bimodule  $M$  to  $Q \otimes M \otimes P$ , and*

$$H_k^i(A, M) \simeq H_k^i(B, Q \otimes_A M \otimes_A P).$$

The following Künneth formula is proved in [MacLane] X.7.4.

**Proposition 2.18** *Let  $k$  be a field. Let  $A$  and  $B$  be  $k$ -algebras. Let  $M$  be an  $A$ -bimodule and  $N$  be an  $B$ -bimodule. Then there are the following isomorphisms*

$$H_k^n(A \otimes B, M \otimes N) \simeq \bigoplus_{p+q=n} H_k^p(A, M) \otimes H_k^q(B, N) \quad (2.19)$$

### 3 Gröbner Bases and Deformations

Anick gives a complex computing the reduced Hochschild cohomology  $\text{Ext}_A^i(k, M)$  by using Gröbner bases in [Anick]. Using this method we will give a short complex for computing  $\text{AlgExt}_R(A, M)$ . Let  $R$  be a commutative ring. We recall the basic notions and definitions of Gröbner bases. Let

$\langle X \rangle$  denote the free monoid generated by the well ordered set  $X$ . We will order  $\langle X \rangle$  by the degree lexicographic order that orders by degree first and then uses lexicographic order in each degree. For  $m, n$  in  $\langle X \rangle$  we say  $m|n$  if  $n = pmq$  for some  $p, q \in \langle X \rangle$ . Suppose we have generators  $\{r_i\}_{i \in \Lambda}$  for some ideal  $I$  of  $R\langle X \rangle$ . Suppose the leading term  $m_i$  of each  $r_i$  has coefficient one. We write  $r_i = m_i - a_i$  with  $m_i \in \langle X \rangle$  and all monomials in  $a_i$  smaller than  $m_i$ . We will interpret these relations as instructions for reducing elements modulo  $I$ . For some  $f \in R\langle X \rangle$  we say  $f$  is reducible by  $\{r_i\}$  if some  $m_i$  divides some term of  $f$ . So for a reducible  $f$ , some term of  $f$  is of the form  $pm_iq$ . Replacing this term by  $pa_iq$  and obtaining  $f' = f - pr_iq$  we get a reduction of  $f$  by  $r_i$ . By continuing this process, since  $\langle X \rangle$  is well ordered, we eventually get  $\bar{f}$ , a reduction of  $f$  that is not reducible by  $\{r_i\}$ . The notion of Gröbner bases addresses the question of when this process leads to a unique reduction.

**Definition 3.1** An *order ideal of monomials* is a subset  $M$  of  $\langle X \rangle$  so that if  $m$  is in  $M$  and  $n|m$  then  $n$  is in  $M$ . Let  $R$  be a commutative ring. Let  $A$  be an  $R$ -algebra that is free as an  $R$ -module. Let  $X$  be a well ordered set of  $R$ -algebra generators of  $A$ . Let  $I$  be the ideal of relations in  $R\langle X \rangle$  giving  $A$ . Suppose we are given a set of generators  $\{r_i\}_{i \in \Lambda}$  of  $I$  so that  $m_i$ , the leading term of  $r_i$ , has coefficient one. Then  $\{r_i\}$  is a *Gröbner basis* if the order ideal of monomials  $M = \{n \in \langle X \rangle \text{ such that } m_i \nmid n \text{ for all } i\}$  is a basis of  $A$  as an  $R$ -module. We say  $\{r_i\}$  is a *reduced Gröbner basis* if in addition each  $r_i$  is not reducible by any of the  $r_j$ , for  $j \neq i$ .

Given a set of generators  $\{r_i\}$  of an ideal  $I$  there is an effective method to check that these relations form a Gröbner basis. Write  $r_i = m_i - a_i$  where  $m_i$  is the leading monomial of  $r_i$ . We must suppose that  $m_i$  has leading coefficient one. If we can write  $m_i z = x m_j = xyz$  for some  $x, y, z \in \langle X \rangle$  with  $y \neq 1$ , we call  $xyz$  an overlap of  $m_i$  and  $m_j$ . We let  $V^{(2)}$  be the set of these overlaps following the notation of [Anick]. These overlaps have potentially non-unique reductions since we may reduce  $m_i$  or  $m_j$  first. Consider reducing the difference between these two choices

$$0 = xyz - xyz = m_i z - x m_j \equiv a_i z - x a_i, \text{ modulo } I.$$

We either get zero or a new relation in  $I$ . If the reduction is zero we say the overlaps are consistent. The next proposition is one of the main reasons that Gröbner bases are a powerful tool for computation.

**Proposition 3.2** (*Bergman's Diamond Lemma*) Suppose we are given a presentation of an  $R$ -algebra

$$A = R\langle X \rangle / (r_i)_{i \in \Lambda}$$

with  $r_i = m_i - a_i$  has leading coefficient one. The generators  $\{r_i\}$  of the ideal  $I$  are a Gröbner basis if and only if the overlaps  $m_i z = x m_j = xyz$  in  $V^{(2)}$  are consistent.

The proof is in [Brg] 1.2. Note that the usual algorithm for generating a Gröbner basis resulting from this proposition does not always provide relations with leading coefficient one. So unless  $R$  is a field, we cannot generate a Gröbner basis from an arbitrary generating set  $X$  with a term order.

The following lemma is not used elsewhere, but shows what algebraic conditions are equivalent to the existence of a Gröbner basis over an arbitrary commutative ring  $R$ .

**Lemma 3.3** An  $R$ -algebra  $A$  has a Gröbner basis if and only if  $A$  is free as an  $R$ -module.

**Proof.** One direction follows immediately from the definition, so let  $A$  be an  $R$ -algebra, free as an  $R$ -module. Let  $\{1\} \cup X = \{1, x_i\}$  be a basis of  $A$  as an  $R$ -module. Consider the relations  $\{x_i x_j - \sum c_{ij}^k x_k\}$  where  $c_{ij}^k$  are the structure constants or multiplication table defining the algebra. The check of the consistency of the overlaps  $x_i x_j x_k$  exactly gives the conditions on the  $c_{ij}^k$  so that the algebra  $A$  is associative. So these relations give us a Gröbner basis.  $\square$

Since we are choosing a very large set of generators, the Gröbner basis in the above proof is usually very inefficient for calculation. If it is used for calculating  $\text{AlgExt}_R(A, A)$  using the method below, it will provide a complex isomorphic to that of the usual bar complex.

Let  $A$  be an  $R$ -algebra presented by a reduced Gröbner basis

$$A = R\langle \overline{X} \rangle / (\overline{m_i} - \overline{a_i})_{i \in \Lambda}. \quad (3.4)$$

Let  $\overline{M}$  be the resulting order ideal of monomials that forms a basis of  $A$ . Let  $R_1$  be a nilpotent extension of  $R$  by  $N$  as in definition 3.1. We will describe deformations of  $A$  by lifting the Gröbner basis.

Let  $A_1$  be a deformation of  $A$  over  $R_1$ . Recall that  $A_1$  is a flat  $R_1$ -algebra so that  $A_1 \otimes_{R_1} R \simeq A$ . Let  $X$  be a set of lifts of the elements of  $\overline{X}$  to  $A_1$ .

We lift all the monomials in  $\langle \bar{X} \rangle$  in the corresponding way. This provides lifts  $m_i$  and  $M$  of  $\bar{m}_i$  and  $\bar{M}$  and of the monomials in the  $\bar{a}_i$ . We lift the coefficients of the  $\bar{a}_i$  arbitrarily to get  $a_i$  in  $R_1\langle X \rangle$ . With these lifts we can try to form a presentation of  $A_1$ . The next lemma shows that we can present any deformation by using these lifts and making infinitesimal changes of the relations on  $A$ . The infinitesimal changes will have coefficients in  $N$  and monomials in the order ideal  $M$ .

**Lemma 3.5** *Let  $A_1$  be a deformation of  $A$  over  $R_1$ , where  $A$  is presented by 3.4, then*

$$A_1 \simeq R_1\langle X \rangle / (m_i - a_i - \alpha_i), \quad (3.6)$$

where the  $\alpha_i$  are linear combinations of the monomials in  $M$  with coefficients in  $N$ .

**Proof.** By the nilpotent version of Nakayama's lemma [E] Ex.4.6, we see that  $M$  generates  $A_1$  as an  $R_1$ -module. Since  $A_1/NA_1 \simeq A$ , the relations  $m_i - a_i$  are satisfied modulo  $N$ . So  $A_1$  has a presentation as above with  $\alpha_i$  in  $NA_1$ . We use these relations to reduce the monomials in the  $\alpha_i$  that are divisible by some  $m_j$ . Each time we carry out such a reduction, the coefficients of the reducible monomials are in higher powers of  $N$ . Since  $N$  is nilpotent we eventually get reductions of the  $\alpha_i$  with monomials in  $M$ .  $\square$  Since all deformations can be presented in the above manner, we will fix a lift of the presentation of  $A$ . We will consider whether an infinitesimal change of the relations of this fixed presentation gives a deformation. If we are given an algebra with a presentation as in 3.6, we can determine if  $A_1$  is flat over  $R_1$ .

**Lemma 3.7** *Let  $A_1$  be presented by*

$$A_1 = R_1\langle X \rangle / (m_i - a_i - \alpha_i) \quad (3.8)$$

where  $\{m_i - a_i\}$  is a reduced Gröbner basis modulo  $N$ , and the  $\alpha_i$  are linear combinations of the monomials in  $M$  with coefficients in  $N$ . Then  $A_1$  is a deformation of  $A$  over  $R_1$  if and only if  $\{m_i - a_i - \alpha_i\}$  is a Gröbner basis.

**Proof.** If  $\{m_i - a_i - \alpha_i\}$  is a Gröbner basis then  $A_1$  is a free  $R_1$ -module by definition 3.1, and so  $A_1$  is flat. Conversely, suppose  $\{m_i - a_i - \alpha_i\}$  is not a Gröbner basis. Bergman's diamond lemma 2.1 says that some overlap  $xyz = xm_i = m_jz$  gives a non-trivial relation  $f$  of  $A_1$ . Since the overlaps

are consistent modulo  $N$ , the relation  $f$  has coefficients in  $N$ . This gives us a non-zero element of  $N \otimes_{R_1} A_1$ . Consider applying the functor  $- \otimes_{R_1} A_1$  to the exact sequence

$$0 \rightarrow N \rightarrow R_1 \rightarrow R \rightarrow 0.$$

The kernel of the map  $A_1 \rightarrow R \otimes A_1$  is  $NA_1$ . So we have a nonzero element in the kernel of the map  $N \otimes_{R_1} A_1 \rightarrow NA_1$  showing that  $A_1$  is not flat.  $\square$

We define embedded deformations,  $\text{EmbDef}(A, R_1)$ , to be the set of all ideals of  $R_1\langle X \rangle$  that satisfy the conditions of the above proposition. So  $\text{EmbDef}(A, R_1)$  contains a representative of every deformation. In the case that  $N^2 = 0$  the overlap checks give  $R$ -linear conditions on the  $\alpha_i$ . We now make the assumption that  $R_1$  is an  $R$ -algebra and  $N^2 = 0$ . Let  $V = \{m_i\}$ , the set of leading monomials. Let  $RV$  be the free  $R$ -module generated by  $V$ . We write  $\alpha_i = \alpha(m_i)$  for some  $\alpha \in \text{Hom}_R(RV, N \otimes A)$ . These will be our infinitesimal changes in the relations. By lemma 3.5 we may consider the  $\alpha_i$  to give values in  $N \otimes A$ . Instead of lifting the  $\bar{a}_i$  randomly as before, we may use the algebra structure map  $R \rightarrow R_1$  to lift the  $a_i$ . We thus obtain a canonical trivial deformation of  $A$ . We write  $V^{(2)}$  for the set of overlaps.

**Corollary 3.9** *Let  $R_1$  be a nilpotent extension of  $R$  by  $N$  so that  $R_1$  is an  $R$ -algebra and  $N^2 = 0$ . Then there is a map*

$$\phi : \text{Hom}_R(RV, N \otimes A) \rightarrow \text{Hom}_R(RV^{(2)}, N \otimes A).$$

*with kernel isomorphic to  $\text{EmbDef}(A, R_1)$ . The deformation corresponding to  $\alpha$  in  $\ker \phi$  may be presented by*

$$A_1 = R_1\langle X \rangle / (m_i - a_i - \alpha(m_i)).$$

**Proof.** All flat deformations may be written in the above form by proposition 3.5. Such an algebra  $A_1$  is a deformation if and only if the relations form a Gröbner basis by lemma 3.7. Each consistent overlap on  $A$  gives a syzygy on the relations. Let  $xm_i = xyz = m_jz$  be an overlap. We have that

$$0 = xyz - xyz \equiv xa_i - a_jz, \text{ modulo } I.$$

Since  $A$  is presented by a Gröbner basis the overlap computation yields an expression in  $R\langle X \rangle$

$$\begin{aligned} \sum g_{kl}(m_k - a_k)h_{kl} &= xa_i - a_jz \\ &= xyz - xyz + xa_i - a_jz \\ &= -x(m_i - a_i) + (m_j - a_i)z. \end{aligned}$$

Now we compute the overlaps of the algebra  $A_1$  presented by 3.8. The overlaps are satisfied modulo  $N$  so we obtain the following on computing the overlaps

$$x\alpha_i - \alpha_j z + \sum g_{kl}\alpha_k h_{kl}.$$

This expression must be reducible to zero by the Gröbner basis of  $A$  in order for  $A_1$  to be flat. Hence we require that

$$x\alpha_i - \alpha_j z + \sum g_{kl}\alpha_k h_{kl} \equiv 0, \text{ modulo } I. \quad (3.10)$$

We assemble these conditions into an  $R$ -linear map  $\phi$  by sending the overlap  $xyz$  in  $V^{(2)}$  to the above equation 3.10.  $\square$

We wish to identify isomorphic deformations so we can calculate  $\text{AlgExt}_R(A, A)$ .

**Lemma 3.11** *Let  $A_1, A'_1$  be deformations of  $A$ . Let  $\psi$  be an  $R_1$ -isomorphism  $A_1 \rightarrow A'_1$  so that  $\psi$  induces the identity on  $A$ . Then  $\psi$  may be lifted to an automorphism  $\Psi$  of the deformation  $R_1\langle X \rangle$  of  $R\langle X \rangle$  so that*

$$\begin{array}{ccc} R_1\langle X \rangle & \longrightarrow & A_1 \\ \downarrow \Psi & & \downarrow \psi \\ R\langle X \rangle & \longrightarrow & A \\ \downarrow & & \downarrow \\ R_1\langle X \rangle & \longrightarrow & A_1 \end{array}$$

commutes.

**Proof.** Since  $R_1\langle X \rangle$  is free we may lift  $\psi$  to  $\Psi$  so that the diagram commutes. Since  $\Psi$  is the identity modulo the nilpotent ideal  $N$ ,  $\Psi$  is an isomorphism.  $\square$

Let  $\text{Aut}(R_1\langle X \rangle; R\langle X \rangle)$  denote the group of automorphisms of the deformation  $R_1\langle X \rangle$ . This group acts on  $\text{EmbDef}(A, R_1)$ . The quotient of this action gives the set of isomorphism classes of deformations of  $A$  over  $R_1$ .

Since  $N^2 = 0$ ,  $\text{Aut}(R_1\langle X \rangle; R\langle X \rangle) \simeq \text{Der}_R(R\langle X \rangle, N \otimes R\langle X \rangle)$ . So we may act on the kernel of the map  $\phi$  by this group. A derivation  $\delta \in \text{Der}_R(R\langle X \rangle, N \otimes R\langle X \rangle)$  acts additively on  $\alpha \in \text{Hom}_R(RV, N \otimes A)$  by

$$(\delta + \alpha)(m_i) = \delta(r_i) + \alpha(m_i).$$

So only the residue of  $\delta$  in  $N \otimes A$  affects  $\alpha$ . So the group

$$\text{Der}_R(R\langle X \rangle, N \otimes A) \simeq \text{Hom}_R(RX, N \otimes A)$$

acts additively on  $\ker \phi$ .

**Proposition 3.12** *The cohomology of the middle term of the complex*

$$0 \rightarrow \text{Hom}_R(RX, N \otimes A) \rightarrow \text{Hom}_R(RV, N \otimes A) \rightarrow \text{Hom}_R(RV^{(2)}, N \otimes A)$$

is  $\text{AlgExt}_R(A, A \otimes N)$  and classifies deformations of  $A$  over  $R_1$ . The kernel of the first map is  $\text{Der}_R(A, A \otimes N)$ .

**Proof.** The first statement follows from the above discussion. A derivation  $\delta \in \text{Der}_R(A, N \otimes A)$  may be specified by its action on the generators  $\delta(x)$  for  $x \in X$  so that it vanishes on the relations  $m_i - a_i$ . This is exactly the the kernel of the first map of the above complex.  $\square$

We will provide an application of the proceeding section. Define an  $R$ -algebra  $A$  to be diagonal if it is generated by  $x, y$  with relations  $x^n - a, y^n - b, yx - \zeta xy$ , for some  $a, b \in R$ , and  $\zeta$  a primitive  $n^{\text{th}}$  root of unity. Let  $R_1 = R[\varepsilon]/\varepsilon^2$ .

**Proposition 3.13** *Let  $R$  be a commutative  $k$ -algebra. Let  $a, b$  be regular elements in  $R$ .*

1. *Any first order deformation over  $R_1$  of the diagonal algebra  $A$  is isomorphic to an  $R_1$ -algebra  $A_1$  generated by  $x, y$  satisfying the relations  $x^n - a - \alpha\varepsilon, y^n - b - \beta\varepsilon, yx - \zeta xy - \gamma\varepsilon$ , where  $\alpha, \beta, \gamma \in R$ .*
2. *If  $A_1$  is given by  $\alpha, \beta, \gamma$  and  $A'_1$  is given by  $\alpha', \beta', \gamma'$  then  $A_1 \simeq A'_1$  iff  $\alpha \equiv \alpha'$  modulo  $aR$ ,  $\beta \equiv \beta'$  modulo  $bR$  and  $\gamma \equiv \gamma'$  modulo  $aR + bR$ . Thus*

$$\text{AlgExt}_R(A, A) \simeq \frac{R}{(a)} \oplus \frac{R}{(b)} \oplus \frac{R}{(a, b)}.$$

**Proof.** Note that the relations of the diagonal algebra  $A$  give a Gröbner basis. The resulting order ideal of monomials  $M = \{x^i y^j \text{ such that } i, j = 0 \dots n-1\}$  is an  $R$ -basis of  $A$ . The overlaps are  $x^{n+1}, y^{n+1}, y^n x, y x^n$ . Using the complex of proposition 3.12 the cohomology of

$$\text{Hom}(RX, A) \rightarrow \text{Hom}(RV, A) \rightarrow \text{Hom}(RV^{(2)}, A)$$

is  $\text{AlgExt}_R(A, A)$ . In our present case  $RX, RV, RV^{(2)}$  are free modules on the formal generators

$$\begin{aligned} RX &= Rx \oplus Ry \\ RV &= Rx^n \oplus Ry^n \oplus Ryx \\ RV^{(2)} &= Rx^{n+1} \oplus Ry^{n+1} \oplus Ry^n x \oplus Ryx^n. \end{aligned}$$

We will let  $\alpha, \beta, \gamma$  denote the coordinates of the dual generators of  $x^n, y^n, yx$  in  $\text{Hom}(RV, A)$ . They form the infinitesimal changes of the corresponding relations. We will write

$$\alpha(x, y) = \sum_{i,j=0}^{n-1} \alpha_{ij} x^i y^j$$

with  $\alpha_{ij}$  in  $R$ . The four overlap computations corresponding to the elements of  $V^{(2)}$  yield the following conditions on  $\alpha, \beta, \gamma$ .

$$[x, \alpha] = 0, [y, \beta] = 0 \quad (3.14)$$

$$[\beta, x] - \sum_{k=0}^{n-1} \zeta^k y^{n-1-k} \gamma y^k = 0 \quad (3.15)$$

$$[\alpha, y] + \sum_{k=0}^{n-1} \zeta^k x^k \gamma x^{n-k-1} = 0. \quad (3.16)$$

Observing the condition 3.14,

$$0 = [x, \alpha] = x\alpha(x, y) - \alpha(x, y)x = x\alpha(x, y) - x\alpha(x, \zeta y) = \sum (1 - \zeta^j) \alpha_{ij} x^{i+1} y^j,$$

we see that  $\alpha_{ij} = 0$  for  $j \neq 0$ . Hence we rewrite  $\alpha(x, y) = \alpha(x)$ . So

$$\alpha(x) = \sum \alpha_i x^i, \text{ with } \alpha_i \in R,$$

and similarly,

$$\beta(y) = \sum \beta_i y^i, \text{ with } \beta_i \in R.$$

We now examine the effect of the last two conditions. The condition 3.16 imposed by the overlap  $yx^n$  gives us

$$\alpha(x)y - \alpha(\zeta x)y + x^{n-1} \sum_{k=0}^{n-1} \zeta^k \gamma(x, \zeta^{n-k-1} y) =$$

$$\sum_{i,j,k=0}^{n-1} \zeta^{k+j(n-k-1)} \gamma_{ij} x^{i+n-1} y^j + \sum_{i=0}^{n-1} (1 - \zeta^i) \alpha_i x^i y = 0.$$

Since  $\sum_k \zeta^{k+j(n-k-1)} = 0$  unless  $j = 1$  we see that the above condition involves only linear terms in  $y$  and simplifies to

$$\sum_{i=0}^{n-1} n \zeta^{-1} \gamma_{i1} x^{i+n-1} y + \sum_{i=0}^{n-1} (1 - \zeta^i) \alpha_i x^i y = 0.$$

Separating this equation by the  $R$ -basis  $\{x^i y^j\}$  and solving,

$$\alpha_i = \frac{n \zeta^{-1} a}{(\zeta^i - 1)} \gamma_{i+1,1}$$

for  $i = 1, \dots, n-2$ . Also we see

$$\alpha_{n-1} = \frac{n \zeta^{-1}}{(\zeta^{-1} - 1)} \gamma_{01}$$

and  $\gamma_{11} = 0$ . We obtain similar equations relating  $\beta$  and  $\gamma$  by symmetry. So we conclude that  $\alpha_0, \beta_0$ , are arbitrary, and  $\gamma_{ij}$  are arbitrary for  $j \neq 1, i \neq 1$ , and that  $\alpha_1, \dots, \alpha_{n-1}, \beta_1, \dots, \beta_{n-1}$  are determined by  $\gamma_{i1}, \gamma_{1j}$  by the regularity hypothesis.

This completes the calculation of the kernel. We now need the image of  $\text{Hom}(RX, A)$ . We let  $d(x) = \sum d_{ij}(x) x^i y^j, d(y) = \sum d_{ij}(y) x^i y^j$  be the coordinates of the generators of  $\text{Hom}(RX, A)$  dual to  $x, y$ . They represent the infinitesimal changes in  $x$  and  $y$ . We calculate that  $\gamma$  is taken modulo

$$d(y)x + yd(x) - \zeta d(x)y - \zeta xd(y) =$$

$$\sum (\zeta^j - \zeta) d_{ij}(y) x^{i+1} y^j + \sum (\zeta^i - \zeta) d_{ij}(x) x^i y^{j+1}.$$

We also see that  $\alpha$  is taken modulo

$$\sum_{i,j,k} d_{ij}(x) \zeta^{-j(1+k)} x^{i+n-1} y^j.$$

The terms in the sum are zero unless  $j = 0$ , so it simplifies to

$$\sum d_{i1}(x) n x^{i+n-1}.$$

Each  $\alpha_i$  except  $\alpha_{n-1}, \alpha_0$  is in  $aR$ , from the above calculation. So we can kill each  $\alpha_i$  for  $i = 1, \dots, n-1$ . We see that we can take  $\alpha_0$  modulo  $d_{11}(x)x^n = d_{11}(x)a$ . Lastly, we examine how  $\gamma$  is taken modulo the above expression. By examining the effect on the basis  $\{x^i y^j\}$  we see that we can kill each  $\gamma_{ij}$  where  $i, j \neq 0, 1$  by using  $d_{ij}(x)$  or  $d_{ij}(y)$  with  $i, j = 1, \dots, n-2$ . We can also kill  $\gamma_{i1}$  by using  $d_{i0}(x)$  for  $i = 0, 2, 3, \dots, n-1$ . This is the same way we killed  $\alpha_i$  above for  $i = 1, \dots, n-1$ . By symmetry we get similar results for  $\beta$ . Finally we see  $\gamma_{00}$  can be taken modulo  $d_{0,n-1}(x)y^n$  and  $d_{n-1,0}(y)x^n$ .  $\square$

## 4 The Five Term Exact Sequence

Let  $k$  be a characteristic zero algebraically closed field. Let  $Z$  be a normal two dimensional  $k$ -algebra with field of fractions  $K$ . Let  $A$  be a maximal  $Z$ -order in a central simple  $K$ -algebra. Let  $R$  be a smooth two dimensional subalgebra of  $Z$  so that  $Z$  is a finite projective  $R$ -module. In this section we will show that  $\text{OutDer}_R(A, A) = 0$  and use this fact to get a refined five term exact sequence from the spectral sequence 2.5.

**Definition 4.1** Given a commutative domain  $R$  and ideal  $p$  we define the *standard  $R$ -order* in  $K(R)^{n \times n}$  to be

$$\text{Stan}^n(R, p) = \begin{pmatrix} R & R & R & \dots & R \\ p & R & R & \dots & R \\ p & p & R & \dots & R \\ \vdots & \vdots & & \ddots & \vdots \\ p & p & p & \dots & R \end{pmatrix}$$

The following proposition follows from [Reiner] 39.14 and [Janusz] theorem 4. Let  $\zeta$  be a primitive  $n^{th}$  root of unity. Suppose  $p$  is the principal ideal of  $R$  generated by some  $t$  in  $R$ . In this case the standard order  $\text{Stan}^n(R, tR)$  may be presented as the  $R$ -algebra generated by  $x, y$  satisfying the relations  $x^n - t, y^n - 1, yx - \zeta xy$ .

**Proposition 4.2** *Let  $k$  be a field of characteristic zero. Let  $Z$  be a normal domain over  $k$ . Let  $p$  be a height one prime of  $Z$ . Let  $A$  be a maximal  $Z$ -order that is ramified at  $p$ . There is a finite étale extension  $Z'$  of the d.v.r.*

$Z_p$  so that  $A' = A \otimes Z'$  is Morita equivalent to a standard order

$$A' \simeq (\mathrm{Stan}^n(Z', pZ'))^{m \times m}$$

We will show that  $\mathrm{OutDer}_R(A, A) = 0$  and we will calculate  $\mathrm{AlgExt}_R(A, A)$  explicitly. We first need some facts about derivations and we recall some basic facts about reflexive modules.

**Lemma 4.3** *A  $Z$ -module is reflexive if and only if it is projective as an  $R$ -module.*

**Proof.** A reflexive  $Z$ -module  $M$  has depth two since the dimension of  $Z$  is two. So  $M$  has depth two as an  $R$ -module. Since  $R$  is smooth of dimension two, the Auslander-Buchsbaum formula [E] 19.9 shows that  $M$  is a projective  $R$ -module. Conversely, if  $M$  is projective as an  $R$ -module then its depth as a  $Z$ -module is two and hence it is reflexive.  $\square$

**Lemma 4.4** *There is a canonical exact sequence*

$$0 \rightarrow \mathrm{Der}_Z(A, A) \rightarrow \mathrm{Der}_R(A, A) \rightarrow \mathrm{Der}_R(Z, Z).$$

**Proof.** The only non-trivial fact needed is that if  $\delta \in \mathrm{Der}_R(A, A)$  then  $\delta(Z) \subseteq Z$ . Let  $a \in A, z \in Z$ . We have

$$[a, \delta z] = [\delta a, z] + [a, \delta z] = \delta([a, z]) = 0.$$

Hence a derivation of  $A$  takes central elements to central elements.  $\square$

Let  $\mathrm{rtr} : A \rightarrow Z$  be the  $Z$ -linear map reduced trace as defined in section 9 of [Reiner]. We define  $sA$  to be the set of traceless elements of  $A$ .

**Lemma 4.5** *The modules  $A, \mathrm{Der}_R(Z, Z), \mathrm{Der}_R(A, A), \mathrm{Der}_Z(A, A)$  and  $sA$  are reflexive  $Z$ -modules.*

**Proof.** It is shown in [AG] 1.5 that  $A$  is a reflexive  $Z$ -module. In [EG] 3.7 it is shown that the kernel of a map of reflexive modules on a normal surface is reflexive. So proposition 3.12 shows that  $\mathrm{Der}_R(Z, Z)$  and  $\mathrm{Der}_R(A, A)$  are both reflexive. Since  $sA$  is defined to be the kernel of the reduced trace, it is reflexive. Finally the sequence of lemma 4.4 shows that  $\mathrm{Der}_Z(A, A)$  is reflexive.  $\square$

**Proposition 4.6** *Let  $k$  be a field of characteristic zero. Let  $Z$  be a normal domain of dimension two over  $k$  with field of fractions  $K$ . Let  $A$  be a maximal  $Z$ -order. Then  $\text{OutDer}_Z(A, A) = 0$ .*

**Proof.** We localize at a height one prime  $p$  of  $Z$ . Using proposition 4.2 we extend coefficients to  $Z'$  étale over  $Z$  so that  $A'$  is Morita equivalent to a standard order. Using the Morita invariance of Hochschild cohomology of proposition 2.17 and lemma 4.7,  $\text{OutDer}_{Z'}(A', A') = 0$ . By the étale base change equation 2.14, we have  $\text{OutDer}_Z(A, A) \otimes Z' = 0$ . Since  $Z'$  is finite, it is faithfully flat over  $Z_p$ . So we see that  $\text{OutDer}_Z(A, A) \otimes Z_p = 0$ .

So the support of  $\text{OutDer}_Z(A, A)$  has dimension at most zero. The reflexive module  $sA$  is isomorphic to  $A/Z$ . So  $\text{OutDer}_Z(A, A)$  is a cokernel of a map of reflexive modules

$$0 \rightarrow sA \rightarrow \text{Der}_Z(A, A) \rightarrow \text{OutDer}_Z(A, A) \rightarrow 0.$$

Let  $m$  be an associated maximal prime ideal of  $\text{OutDer}_Z(A, A)$  with residue field  $\kappa$ . Applying the functor  $\text{Ext}_Z^i(\kappa, -)$  to the above exact sequence we see that  $\text{Hom}_Z(\kappa, \text{OutDer}_Z(A, A)) = 0$ . So  $\text{OutDer}_Z(A, A) = 0$ .  $\square$

**Lemma 4.7** *Let  $Z$  be a commutative domain with field of fractions  $K$ . Let  $\{p_{ij} | i, j = 1 \dots n\}$  be a collection of non-zero ideals of  $Z$ , so that  $p_{kk} = p_{1k} = Z$  for all  $k$  and  $p_{ij}p_{jk} \subseteq p_{ik}$ . Let  $A$  be the order obtained by assembling the  $p_{ij}$  into a matrix in the natural way. Then  $\text{OutDer}_Z(A, A) = 0$ .*

**Proof.** Let  $\delta \in \text{Der}_Z(A, A)$ . Then  $\delta$  extends to a derivation in  $\text{Der}_K(A \otimes K, A \otimes K) \simeq \text{Der}_K(K^{n \times n}, K^{n \times n})$ . Since all derivations on  $K^{n \times n}$  are inner by [FD] theorem 3.22, we represent  $\delta$  by the commutator with a matrix  $d = (d_{ij})$  in  $K^{n \times n}$ . We may suppose that  $d_{11} = 0$  by subtracting  $d_{11}I$  if necessary. Since  $d$  represents a derivation on  $A$  we have that  $[d, A] \subseteq A$ . The matrix units  $e_{kk}, e_{1k}$  are in  $A$  for all  $k$  so the matrices

$$[d, e_{kk}] = \begin{pmatrix} 0 & \dots & 0 & d_{1k} & 0 & \dots & 0 \\ 0 & \dots & 0 & d_{2k} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -d_{k1} & -d_{k2} & \dots & 0 & \dots & \dots & -d_{kn} \\ 0 & \dots & 0 & \vdots & 0 & \dots & 0 \\ 0 & \dots & 0 & d_{nk} & 0 & \dots & 0 \end{pmatrix}$$

and

$$[d, e_{1k}] = \begin{pmatrix} -d_{k1} & -d_{k2} & \dots & -d_{kk} & \dots & -d_{k,n-1} & -d_{kn} \\ 0 & \dots & 0 & d_{21} & 0 & \dots & 0 \\ 0 & \dots & 0 & d_{31} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & 0 \\ 0 & \dots & 0 & d_{k1} & 0 & \dots & 0 \end{pmatrix},$$

are in  $A$ . So each  $d_{ij} \in p_{ij}$  and so  $d \in A$ .  $\square$

**Lemma 4.8** *Let  $k$  be a field of characteristic zero. Let  $Z$  be a normal domain of dimension two with field of fractions  $K$ . Let  $R$  be a smooth subring of  $Z$  so that  $Z$  is a finite projective  $R$  module. Let  $A$  be a maximal  $Z$ -order. Then  $\text{OutDer}_R(A, A) = 0$ .*

**Proof.** Since  $K$  is a finite separable field extension of the field of fractions  $K(R)$ , of  $R$ , we have that  $\text{Der}_{K(R)}(K, K) = 0$ . So the projective  $R$ -module  $\text{Der}_R(Z, Z)$  is zero at the generic point and so must be zero. So by lemma 4.4,  $\text{Der}_Z(A, A) \simeq \text{Der}_R(A, A)$ . Hence  $\text{OutDer}_Z(A, A) \simeq \text{OutDer}_R(A, A)$ . By proposition 4.6 we have  $\text{OutDer}_Z(A, A) = 0$ .  $\square$

Let  $\Theta = \text{Der}_k(R, R)$ , the module of vector fields on  $\text{Spec } R$ . Let  $\Theta^2 = \wedge^2 \Theta$ . The 5-term exact sequence from the spectral sequence 2.12 is

$$0 \rightarrow \text{OutDer}_R(A, A) \rightarrow \text{OutDer}_k(A, A) \rightarrow$$

$$\Theta_R \otimes Z \rightarrow \text{AlgExt}_R(A, A) \rightarrow \text{AlgExt}_k(A, A)$$

In the case that  $\text{OutDer}_R(A, A) = 0$ , the  $E_2^{1,-}$  column of the spectral sequence of proposition 2.12 is zero. So we get the following 5-term exact sequence using lemma 4.8.

**Proposition 4.9** *There is a canonical exact sequence*

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{OutDer}_k(A, A) & \xrightarrow{\phi_1} & \Theta_R \otimes Z & & (4.10) \\ \xrightarrow{p} & & \text{AlgExt}_R(A, A) & \xrightarrow{q} & \text{AlgExt}_k(A, A) & \xrightarrow{\phi_2} & \Theta_R^2 \otimes Z \end{array}$$

The maps of the above sequence all have useful interpretations. The map  $p$  gives a deformation of  $A$  by applying an infinitesimal automorphism.

So the multiplication changes by a derivation. Let  $\delta$  be a derivation in  $\text{Der}_k(R, Z) \simeq \Theta_R \otimes Z$ . Let  $R_1 = R \oplus \varepsilon Z$  be the trivial algebra extension of  $R$  by  $Z$ . Let  $A_1$  be the tensor product defined by the pushout of the following diagram

$$\begin{array}{ccc} A[\varepsilon]/\varepsilon^2 & \longrightarrow & A_1 \\ \uparrow & & \uparrow \\ R_1 & \xrightarrow{1+\varepsilon\delta} & R_1. \end{array}$$

The above algebra  $A_1$  will be a deformation of  $A$  over  $R$  corresponding to the derivation  $\delta$ .

The map  $\phi_2$  also has a useful interpretation. Let  $A_1$  be a deformation of  $A$  over  $k_1 = k[\varepsilon]/\varepsilon^2$ . Let  $u, v$  in  $A_1$  be lifts of  $\bar{u}, \bar{v} \in R$ . Let  $a$  be in  $A_1$ . Note that  $[a, u]$  is in  $\varepsilon A_1$ . So the Jacobi identity gives

$$[a, [u, v]] + [v, [a, u]] + [u, [v, a]] = [a, [u, v]] = 0$$

So  $[u, v] \in \varepsilon Z$  and only depends on  $\bar{u}, \bar{v}$ . So given a deformation of  $A$  we get an element of  $\text{Hom}_R(\Omega_R^2, Z) \simeq \Theta_R^2 \otimes Z$ . We define the image of this map to be the *bracket* of the deformation.  $A_1$  is finite over its centre if and only if its bracket vanishes.

The following statement is a corollary of proposition 2.10

**Corollary 4.11** *The image of the map  $\phi_2 : \text{AlgExt}_k(A, A) \rightarrow \Theta_R^2 \otimes Z$  contains  $\wedge^2 \text{Im}(\phi_1)$ .*

## 5 Deformations at Height One Primes

We will now analyze the deformations of an order in codimension one and use reflexivity to obtain the main algebraic theorem 5.11. Using the sequence 4.10 we see that we need to compute  $\text{AlgExt}_R(A, A)$  to find  $\text{AlgExt}_k(A, A)$ .

We first analyze the deformations of an order at the generic points of the surface. At the unramified points, the order  $A$  is étale locally isomorphic to a matrix algebra over its centre. The deformations of such an algebra are described by the following proposition. Since  $A$  is separable at these points, the spectral sequence 2.12 collapses to give that  $\text{AlgExt}_R(A, A) \simeq \text{AlgExt}_k(R, R)$ , but the actual correspondence of the deformations follows easily from facts about lifting idempotents.

**Proposition 5.1** *Let  $k$  be a commutative ring. Let  $R$  be a flat  $k$ -algebra. Let  $A = R^{n \times n}$ , and let  $k_1$  be a nilpotent extension of  $k$ . Let  $A_1$  be a deformation of  $A$  over  $k$ . Then  $A_1 \simeq R_1^{n \times n}$  where  $R_1$  is a deformation of  $R$  over  $k_1$ .*

**Proof.** Since matrix units lift modulo a nilpotent ideal, [Rowen] 1.1.25, 1.1.28, we lift the matrix units of  $A$  to  $A_1$ . The matrix units give an isomorphism  $A_1 \simeq R_1^{n \times n}$  for some  $R_1$ . Since  $A_1$  is flat over  $k_1$ , so is  $R_1$ . We also have  $R_1 \otimes_{k_1} k \simeq R$ .  $\square$

We now analyze the deformations along smooth points of the discriminant curve  $D$ . If  $R$  is a d.v.r. with parameter  $t$  then Sanada shows that the Hochschild cohomology of  $A = \text{Stan}^n(R, tR)$  is periodic with  $H_R^{2i}(A, A) = R/tR$  and  $H_R^{2i-1}(A, A) = 0$  for  $i > 0$ , [Sanada] Proposition 1. The next result overlaps with this statement. Let  $R_1 = R[\varepsilon]/\varepsilon^2$ . Using propositions 3.13 and 2.17, we get the following result.

**Corollary 5.2** *If  $A = \text{Stan}^n(R, tR)^{m \times m}$  then*

1.  $\text{AlgExt}_R(A, A) \simeq R/tR$ .
2. *a deformation  $A_1$  over  $R_1$  is Morita equivalent to an algebra generated over  $R_1$  by  $x, y$  satisfying the relations  $x^n - (t + \varepsilon t'), y^n - 1, yx - \zeta xy$  for some  $t' \in R$ .*
3.  $A_1 \simeq \text{Stan}^n(R_1, (t + \varepsilon t')R_1)^{m \times m}$ .

The next corollary gives an application of the above exact sequence 4.10.

**Corollary 5.3** *Let  $R = k[u]$  and let  $B = \text{Stan}^n(R, uR)$ . Let  $S = k[u, v]$  and let  $A = \text{Stan}^n(S, uS)$ . Then*

$$\begin{array}{ll} Z(B) = R & Z(A) = S \\ \text{OutDer}_k(B, B) \simeq u\Theta_R & \text{OutDer}_k(A, A) \simeq uS\frac{\partial}{\partial u} \oplus S\frac{\partial}{\partial v} \\ \text{AlgExt}_k(B, B) = 0 & \text{AlgExt}_k(A, A) \simeq u\Theta_S^2 \end{array}$$

**Proof.** It is easy to compute the centre of  $B$  and  $\text{OutDer}_k(B, B)$  directly. We let  $R = Z$  and we see that  $\text{AlgExt}_R(B, B) \simeq k$  by 5.2. We also have

that  $\Theta_R = R \frac{\partial}{\partial u}$ , and  $\Theta_R^2 = 0$ . Using the exact sequence 4.10 we get

$$0 \rightarrow uR \frac{\partial}{\partial u} \rightarrow R \frac{\partial}{\partial u} \rightarrow k \rightarrow \text{AlgExt}_k(B, B) \rightarrow 0$$

Hence we conclude that  $\text{AlgExt}_k(B, B) = 0$ . Since  $A \simeq B \otimes_k k[v]$ , the Künneth formula for Hochschild cohomology 2.18 computes the answers for  $A$ .  $\square$

Let  $R$  be a regular local ring of dimension two with field of fractions  $K$ . Let  $A$  be a maximal  $R$ -order in a central simple  $K$ -algebra of index  $n$ . Let  $A_1$  be a deformation of  $A$  over  $R_1$ . Since  $A$  is reflexive,  $A$  is a free  $R$ -module. Let  $\{x_i | i = 0 \dots n^2 - 1\}$  be a basis of  $A$ . Let  $\text{rtr} : A \rightarrow R$  be the reduced trace of  $A$  as in [Reiner] section 9. Recall that the discriminant of  $A$  is

$$\Delta = \det(\text{rtr}(x_i x_j)).$$

We factor  $\Delta = f_1^{e_1} \cdots f_n^{e_n}$  into irreducible factors. Let  $d$  be the reduced discriminant  $d = f_1 \cdots f_n$ . The reduced discriminant is defined up to multiplication by a unit. The algebra  $A[d^{-1}]$  is an Azumaya  $R[d^{-1}]$ -algebra.

**Lemma 5.4** *Let  $A$  be an Azumaya algebra. Let  $A_1$  be a deformation of  $A$  over  $R_1$ . Then  $A_1$  is the trivial deformation and  $A_1$  is Azumaya over  $R_1$ .*

**Proof.** Since  $A$  is Azumaya,  $A$  is a separable  $A_R^e$ -module. So  $\text{AlgExt}_R(A, A) = 0$ . So  $A_1$  must be the trivial deformation  $A_1 \simeq A \otimes_R R_1$ .  $\square$

**Lemma 5.5** *Let  $A_1$  be a deformation of the maximal order  $A$  over  $R_1$ . Let  $a_1 \in A_1$ . Let  $a \in A$  be the residue of  $a_1$  modulo  $\varepsilon A$ . Then there is a reduced characteristic polynomial of  $a_1$  in  $R_1[t]$ , extending the reduced characteristic polynomial of  $a$  in  $R[t]$ .*

**Proof.** Since  $A[d^{-1}]$  is Azumaya, so is  $A_1 \otimes R[d^{-1}]$ . Hence there is a reduced characteristic polynomial  $\text{rch} : A_1 \otimes_R R[d^{-1}] \rightarrow R_1[d^{-1}, t]$ , [KnOj] IV.2. Let  $a_1$  be in  $A_1$  with characteristic polynomial  $f(t)$ . Now when we localize at a height one prime  $p$  of  $R$ , there is a finite étale extension  $R'$  of  $R_p$  so that  $A_1 \otimes R'$  is of the form in corollary 5.2. Since these orders are subrings of matrix rings, their reduced characteristic polynomials certainly have coefficients in

$R'$ . So we see that the coefficients of  $f(t)$  are in  $R'_1 \cap R_1[d^{-1}]$ . Since  $R_1$  is a finite étale extension  $R'_1 \cap R_1[d^{-1}] = R_p[\varepsilon]/\varepsilon^2$ . Since  $R$  is normal,

$$R = \bigcap_{\text{ht } p=1} R_p.$$

Hence the coefficients of  $f(t)$  are in  $R_1$ .  $\square$

Given an ideal  $I$  of  $R$  the set of deformations of  $I$  in  $R$  correspond to elements of  $N_{I/R} = \text{Hom}_R(I, R/I)$ , the module of normal vector fields to  $\text{Spec } R/I$  in  $\text{Spec } R$ . A deformation of  $A$  in  $\text{AlgExt}_R(A, A)$  gives a deformation of the discriminant ideal  $\Delta R$  in  $R$ .

**Lemma 5.6** *There is an canonical  $R$ -module map*

$$\text{AlgExt}_R(A, A) \rightarrow \text{Hom}_R(\Delta R, R/\Delta R).$$

**Proof.** This map is given by the reduced trace. Let  $\{x_i\}$  be an  $R_1$ -basis of  $A_1$ . Then the deformation of the discriminant ideal is given by

$$\Delta \mapsto \frac{1}{\varepsilon}(\det(\text{rtr}(x_i x_j)) - \Delta).$$

$\square$

Let  $q = \Delta/d = f_1^{e_1-1} \cdots f_n^{e_n-1}$ . There is a natural injective map

$$\text{Hom}(dR, R/dR) \rightarrow \text{Hom}(\Delta R, R/\Delta R).$$

This map is the composition of injective maps

$$\text{Hom}(dR, R/dR) \xrightarrow{s} \text{Hom}(\Delta R, qR/\Delta R) \rightarrow \text{Hom}(\Delta R, R/\Delta R).$$

The map  $s$  is given by tensoring with  $qR$ .

We let  $D = V(d)$ , the discriminant locus in  $X = \text{Spec } R$ . Let  $N_{X/D}$  denote the module of normal vector fields  $\text{Hom}_R(dR, R/dR)$  where  $X = \text{Spec } R$  and  $D = V(d)$ . A deformation of  $A$  in  $\text{AlgExt}_R(A, A)$  also gives a deformation of the reduced discriminant ideal  $dR$  in  $R$ .

**Proposition 5.7** *There is a canonical  $R$ -module map*

$$\psi : \text{AlgExt}_R(A, A) \rightarrow N_{X/D}.$$

*The natural map  $\Theta_X \rightarrow N_{X/D}$  factors through this map.*

**Proof.**

It suffices to show that the map of lemma 5.6 takes  $\Delta$  into  $qR/\Delta R$ . Using 4.2 and 5.2 we can verify this statement for  $R'$  étale over  $R_p$  for any height one prime  $p$  of  $R$ . Hence it follows for  $R$ . The last statement follows from the description of the maps.  $\square$

**Lemma 5.8** *Let  $R$  be a d.v.r with parameter  $t$ . Let  $A = \text{Stan}^n(R, tR)^{m \times m}$ . Then the map  $\phi : \text{AlgExt}_R(A, A) \rightarrow N_{tR/R}$  is an isomorphism.*

**Proof.** The discriminant of  $A$  is  $\Delta = t^n$  and the reduced discriminant is  $d = t$ . The discriminant of a deformation of  $A$  as described in 5.2 is  $(t + \varepsilon t')^n = (t^n + n\varepsilon t^{n-1})$ . The map from  $\text{AlgExt}_R(A, A) \rightarrow \text{Hom}_R(\Delta R, R/\Delta R)$  sends  $t' + tR$  in  $R/t \simeq \text{AlgExt}_R(A, A)$  to the map defined by  $\Delta \mapsto nt'^{n-1} + \Delta R$ . So the image of  $t' + tR$  in  $\text{Hom}_R(dR, R/dR)$  is the map  $d \mapsto nt' + dR$ .  $\square$

Let  $\Theta(-\log D)$  be the kernel of the map  $\Theta_X \rightarrow N_{X/D}$ . So  $\Theta(-\log D)$  is the module of those vector fields on  $X$  that are tangent to  $D$ , or alternatively derivations  $\delta$  on  $R$  that take  $dR$  into  $dR$ . Let  $T_1$  be the module of deformations of  $D$  as described in [Sch]. It is known that  $T_1$  is the cokernel of the map  $\Theta_X \rightarrow N_{X/D}$ . See [Artin76] for a proof of this fact. We denote by  $S(A)$  the cokernel of the map  $\Theta_R \rightarrow \text{AlgExt}_R(A, A)$ .

**Lemma 5.9** *If we have maps  $A \xrightarrow{\psi} B \xrightarrow{\phi} C$  in an abelian category then there is a canonical exact sequence*

$$0 \rightarrow \ker\psi \rightarrow \ker\phi\psi \rightarrow \ker\phi \rightarrow \text{cok}\psi \rightarrow \text{cok}\phi\psi \rightarrow \text{cok}\phi \rightarrow 0.$$

The proof is a diagram chase.

We use the above lemma for the maps  $\Theta_X \rightarrow \text{AlgExt}_R(A, A) \xrightarrow{\psi} N_{X/D}$ . This gives the following diagram, with zeroes on the periphery. The sequence

wrapping around the triangle in the centre is exact.

**Theorem 5.11** Let  $A$  be a maximal  $R$ -order with discriminant locus  $D$ . Then

1. The module  $S(A)$  is a finite length  $R$ -module.
  2. There is an exact sequence  $0 \rightarrow S(A) \rightarrow \text{AlgExt}_k(A, A) \rightarrow I_D\Theta_X^2$ .
  3. There is a natural map  $S(A) \rightarrow T_1$ .
  4. If  $\theta = 0$  then
    - (a)  $\text{OutDer}_k(A, A) \simeq \Theta(-\log D)$ .
    - (b)  $\text{AlgExt}_k(A, A) \simeq S(A) \oplus I_D\Theta_X^2$ .

**Proof.** The map  $\psi$  is an isomorphism and so the map  $\theta$  is zero for standard orders by 5.8, so the first statement follows. The statements 3 and 4a follow

immediately from the diagram 5.10. To show 2 we will consider the map  $\phi_2 : \text{AlgExt}_k(A, A) \rightarrow \Theta_X^2$  from the exact sequence 4.10. We must show that the image of  $\phi_2$  is contained in  $I_D\Theta_X^2$ . There is a natural isomorphism  $\wedge^2\Theta(-\log D) \simeq I_D\Theta_X^2$ . Recall that the cup product of Hochschild cohomology for  $H_k^*(A, A)$  is  $R$ -linear 2.3 and that the edge map of the spectral sequence gives us a map of graded skew commutative algebras 2.10. Hence we get the diagram

$$\begin{array}{ccc} \wedge^2 \text{OutDer}_k(A, A) & \xrightarrow{\wedge^2\phi_1} & \wedge^2\Theta \\ \downarrow \sim & & \parallel \sim \\ \text{AlgExt}_k(A, A) & \xrightarrow{\phi_2} & \Theta_X^2 \end{array} \quad (5.12)$$

where the horizontal maps are given by the cup product. Since we have an isomorphism on the right, and the map  $\phi_1$  factors through  $\Theta(-\log D)$  we get the following commutative square

$$\begin{array}{ccc} \wedge^2 \text{OutDer}_k(A, A) & \longrightarrow & \wedge^2\Theta(-\log D) \\ \downarrow \sim & & \downarrow \sim \\ \text{AlgExt}_k(A, A) & \xrightarrow{\phi_2} & \Theta_X^2. \end{array} \quad (5.13)$$

The left cup product map is an isomorphism when we localize at a codimension one prime by 5.3 and 5.2, and since  $\wedge^2\Theta(-\log D) \simeq I_D\Theta_X^2$ , the module  $\wedge^2\Theta(-\log D)$  is reflexive. So we use the following lemma 5.14 to conclude statement 2. The statement 4b follows from 4.11 and the fact that  $I_D\Theta_X^2$  is torsion free.  $\square$

**Lemma 5.14** *Suppose we have the following commutative diagram of  $R$ -modules*

$$\begin{array}{ccc} P & \xrightarrow{b} & Q \\ a \downarrow & & \downarrow d \\ T & \xrightarrow{c} & S. \end{array}$$

*Suppose that  $Q$  is reflexive and that the map  $a$  becomes an isomorphism upon localizing at height one primes. Then there is a map  $e : T \rightarrow Q$  so that  $de = c$ .*

### Proof.

Let  $^{**}$  denote the reflexive hull, and let  $i_M : M \rightarrow M^{**}$  be the natural map from  $M$  to its reflexive hull. Upon taking reflexive hulls, the map  $a^{**}$  becomes an isomorphism, and so the desired map is  $i_Q^{-1}b^{**}a^{**-1}i_T$  as in the following diagram.

$$\begin{array}{ccccc}
P & \xrightarrow{b} & Q & & \\
a \downarrow & & \downarrow i_Q & & \\
P^{**} & \xrightarrow{b^{**}} & Q^{**} & & \\
\downarrow & & \downarrow & & \\
T & \xrightarrow{a^{**}} & S & & \\
i_T \downarrow & & \downarrow & & \\
T^{**} & \xrightarrow{\quad} & S^{**} & &
\end{array}$$

□

## 6 Deformations at Singularities of the Discriminant

For this section we suppose that  $A$  has global dimension two. In this case we can prove that the map  $\theta$  of the diagram 5.10 is zero and so get the stronger statement of 5.11. We will compute the diagram 5.10 for all of the possible complete local structures except for types IV' and V'. It was verified that  $\theta = 0$  for these algebras by using Macaulay 2. The Macaulay computation used the same techniques as the other types, but used the basis of  $A$  as an  $R$ -module as algebra generators. Since the number of generators and relations was much larger it was not possible to do the computation by hand.

The possible maximal orders  $A$  of global dimension two over the complete local ring  $R = k[[u, v]]$  are classified in [Artin86] 3.4, 5.31-5.33. We summarize the main result of that paper in the following table.

Type	Order $A$	Reduced discriminant $d$
$I_k$	$x^k - u$ $y^k - v$ $yx - \zeta_k xy$	$uv$
$II_k^i$	$x^2 - u^{2i+1}$ $y^2 - u^{k-2i}v$ $yx + xy - 2v$	$v(v - u^{k+1})$
$III_k$	$x^2 - v$ $y^2 - uv$ $yx + xy - 2u^{k+1}$	$u(v^2 - u^{2k+1})$
$IV$	$x^3 - v$ $y^3 - u^2 + v$ $yx - \zeta_3 xy$	$v(v - u^2)$
$IV'$	$\{1, w, w^2, xw^2,$ $wxw^2, ux, uxw,$ $uwx, uwxw\}$	
$V$	$x^2 - v$ $y^2 - v^2 + u^3$ $yx + xy$	$v(v^2 - u^3)$
$V'$	$\{1, ux, y - v,$ $u^{-1}x(y - v)\}$	

In the above table  $k$  is an arbitrary integer  $\geq 1$ ,  $\zeta_k$  is a primitive  $k^{th}$  root of unity. For type  $II_k^i$ ,  $i$  is chosen so that  $0 \leq i \leq k/2$ . The types  $IV'$  and  $V'$  are described by an  $R$ -basis of  $A$ . They satisfy the same relations as types  $IV$  and  $V$ . For type  $IV'$ ,  $w = u^{-1}(x + y)^2$ .

We will compute  $S(A)$  for all of the above algebras except  $IV'$  and  $V'$ . It was verified that  $\theta = 0$  for these algebras by using Macaulay 2. The same techniques were used, but since  $IV'$  and  $V'$  are specified by bases rather than a small number of relations, the computation is much larger. The following table describes the module  $S(A)$  and its relation to  $T_1$  through the map  $S(A) \rightarrow T_1 \rightarrow \text{cok } \psi \rightarrow 0$ . The module  $S(A)$  is given by its relations on the  $R$ -module generators  $\alpha, \beta, \gamma$  which correspond to the infinitesimal changes in each of the three relations describing  $A$  as an  $R$ -algebra. The module  $T_1$  has a single generator and so is described by the annihilator ideal. The cokernel  $\text{cok } \psi$  is described in the same manner. The map  $S(A) \rightarrow T_1$  is described in

the proof below.

Type	$\dim S(A)$	$S(A)$	$T_1$	$\text{cok } \psi$
$I_k$	1	$\alpha, \beta, u\gamma, v\gamma$	$u, v$	$u, v$
$II_k^i$	$2k + 1$	$\gamma + u^{k-2i}\beta, v\alpha,$ $u^{2k-2i+1}\beta, (2i+1)u^{2i}\alpha$ $+(k-2i)u^{k-2i-1}v\beta,$ $u^{2i+1}\alpha, (2v-u^{k+1})\beta$	$2v - u^{k+1}$ $u^{2k+1}$	$u^{2i+1}, v$
$III_k$	$2k + 2$	$\alpha + u\beta, u^{k+2}\beta, v\beta,$ $u^{k+1}\gamma, uv\gamma, v^2\gamma$ $2u2k+1\beta + v\gamma$	$uv,$ $v^2 - (2k+2)u^{2k+1}$	$u^{k+1}, v$
$IV$	3	$\alpha - \beta, u\alpha$ $v\alpha, u^2\gamma, v\gamma$	$2v - u^2, u^3$	$u^2, v$
$V$	7	$\alpha + 2v\beta, u^2\beta$ $v^2\beta, v\gamma, u^3\gamma$	$3v^2 - u, u^2v$	$u^3, v$

We use the description of the deformations of diagonal algebras in 3.13. These calculations will apply to cases  $I_k, IV$ , and  $V$ . We now suppose  $a, b$  form a regular sequence as is true in the cases we are considering. Since  $I_D = dR = abR$  is a principal ideal, we identify  $N_{X/D}$  with  $R/(ab)$ . Recall that

$$\text{AlgExt}_R(A, A) \simeq \frac{R}{(a)} \oplus \frac{R}{(b)} \oplus \frac{R}{(a, b)}.$$

Computing the change in the reduced discriminant  $d = ab$ , we write the map  $\text{AlgExt}_R(A, A) \rightarrow N_{X/D}$  as the map

$$\frac{R}{(a)} \oplus \frac{R}{(b)} \oplus \frac{R}{(a, b)} \xrightarrow{(b,a,0)} \frac{R}{(ab)}.$$

We get the exact sequence

$$0 \rightarrow \frac{R}{(a, b)} \rightarrow \text{AlgExt}_R(A, A) \rightarrow N_{X/D} \rightarrow \frac{R}{(a, b)} \rightarrow 0.$$

We now compute the diagram 5.10. If we write  $\Theta_R = R\frac{\partial}{\partial u} \oplus R\frac{\partial}{\partial v}$ , the map  $\Theta_R \rightarrow \text{AlgExt}_R(A, A)$  may be written as

$$\begin{pmatrix} a_u & a_v \\ b_u & b_v \\ 0 & 0 \end{pmatrix}.$$

This map restricted to  $\Theta(-\log D)$  is zero since its image must lie in the kernel of  $\text{AlgExt}_R(A, A) \rightarrow N_{X/D}$ .

We immediately get

**Corollary 6.1** *Let  $A$  be a diagonal algebra over  $R$  so that  $a, b$  form a regular sequence. Then  $\text{OutDer}_k(A, A) \simeq \Theta(-\log D)$  and*

$$0 \rightarrow \frac{R}{(a, b)} \rightarrow S(A) \rightarrow T_1 \rightarrow \frac{R}{(a, b)} \rightarrow 0$$

is exact.

It is now an easy matter to obtain  $S(A)$  for diagonal algebras as in the table 6.

We now compute the deformations for the cases  $II_k^i$  and  $III_k$ . We first compute  $\text{AlgExt}_R(A, A)$  for a more general algebra including these two cases. We define an  $R$ -algebra  $A$  to be quaternion if it is generated by  $x, y$  with the relations  $x^2 - a, y^2 - b, yx + xy - 2c$ , for some  $a, b, c$  in  $R$ . We will say  $A$  is non-degenerate if  $a$  or  $c$  is a regular element of  $R$ , and  $b$  or  $c$  is a regular element of  $R$ . We will be considering the case where  $R$  is a domain and  $ab - c^2 \neq 0$  since  $(ab - c^2)$  is the reduced discriminant of  $A$ . Let  $R_1 = R[\varepsilon]/\varepsilon^2$ .

**Proposition 6.2** *Let  $R$  be a commutative  $k$ -algebra. Let  $a, b, c \in R$  so that the quaternion algebra  $A$  above is non-degenerate.*

1. *Any first order deformation over  $R_1$  of the quaternion algebra  $A$  is isomorphic to an  $R_1$ -algebra  $A_1$  generated by  $x, y$  satisfying the relations  $x^2 - a - \alpha\varepsilon, y^2 - b - \beta\varepsilon, yx + xy - 2c - 2\gamma\varepsilon$ , where  $\alpha, \beta, \gamma \in R$ .*
2. *If  $A_1$  is given by  $\alpha, \beta, \gamma$  and  $A'_1$  is given by  $\alpha', \beta', \gamma'$  then  $A_1 \simeq A'_1$  iff  $(\alpha, \beta, \gamma) \equiv (\alpha', \beta', \gamma')$  modulo the relations  $(2a, 0, c), (2c, 0, b), (0, 2c, a), (0, 2b, c)$ . We let*

$$M = \begin{pmatrix} 2a & 2c & 0 & 0 \\ 0 & 0 & 2c & 2b \\ c & b & a & c \end{pmatrix}.$$

So

$$\text{AlgExt}_R(A, A) \simeq \text{cok } M.$$

**Proof.** We proceed by using the complex in proposition 3.12 as in the diagonal case. The four overlaps  $x^3, y^3, y^2x, x^2y$  yield the conditions

$$[x, \alpha] = 0, \quad [y, \beta] = 0,$$

$$[\beta, x] + 2[\gamma, y] = 0,$$

$$[y, \alpha] + 2[x, \gamma] = 0.$$

Writing

$$\alpha = \alpha_{00} + \alpha_{01}x + \alpha_{10}y + \alpha_{11}xy,$$

and similarly for  $\beta, \gamma$ , we get

$$0 = [x, \alpha] = 2(\alpha_{10}xy - c\alpha_{10} + a\alpha_{11}y + c\alpha_{11}).$$

So by the non-degeneracy hypothesis,  $\alpha_{10} = \alpha_{11} = 0$ . So we write  $\alpha = \alpha_0 + \alpha_1x$  and similarly  $\beta = \beta_0 + \beta_1y$ . Now the third overlap gives us

$$0 = [y, \alpha] + 2[x, \gamma] = 2((c\alpha_1 - 2c\gamma_{10} + 2c\gamma_{11}) + 2a\gamma_{11}y + (2\gamma_{10} - \alpha_1)xy).$$

Using the non-degeneracy hypothesis again, we get that  $\gamma_{11} = 0$  and  $2\gamma_{10} = \alpha_1$ . So we may write

$$\gamma = \gamma_0 + \frac{1}{2}\beta_1x + \frac{1}{2}\alpha_1y,$$

completing the description of the kernel.

We now determine the image resulting from infinitesimal changes in  $x$  and  $y$ . We again denote these changes by  $d(x)$  and  $d(y)$ . We compute that  $\alpha$  is taken modulo

$$xd(x) + d(x)x = 2((d_{01}(x)a + d_{10}(x)c) + (d_{00}(x) + d_{11}(x)c)x).$$

So  $\gamma$  is taken modulo

$$d(y)x + yd(x) + xd(y) + d(x)y =$$

$$2(d_{00}(y)x + d_{01}(y)a + d_{10}(y)c + d_{12}(y)cx + d_{00}(x)y + d_{01}(x)c + d_{10}(x)b + d_{11}(x)cy).$$

We can kill  $\alpha_1, \beta_1$  by using  $d_{00}(x), d_{00}(y)$ . We see that  $\alpha_0, \beta_0, \gamma_0$  may be chosen in  $R$  modulo the presentation matrix in the statement of the proposition where the relations are generated by  $d_{01}(x), d_{10}(x), d_{01}(y), d_{10}(y)$ .  $\square$

We now compute  $S(A)$  and  $\theta$  for the quaternion algebras. Since  $I_D = dR = (ab - c^2)R$  is a principal ideal we may identify  $N_{X/D}$  with  $R/dR$ . The

map  $\text{AlgExt}_R(A, A) \rightarrow N_{X/D}$  may be written as

$$\begin{array}{ccccccc} R^4 & \xrightarrow{M} & R^3 & \longrightarrow & \text{AlgExt}_R(A, A) & \longrightarrow & 0 \\ \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right) \Big| & & \left( \begin{array}{c} b \\ a \\ -2c \end{array} \right) \Big| & & \Big| & & \\ R & \xrightarrow{\delta} & R & \longrightarrow & N_{X/D} & \longrightarrow & 0. \end{array}$$

By the right exactness of cokernel, the sequence

$$\text{AlgExt}_R(A, A) \rightarrow N_{X/D} \rightarrow R/(a, b, c) \rightarrow 0.$$

is exact.

Let  $d_u = \partial d / \partial u$ ,  $d_v = \partial d / \partial v$ . Suppose  $d_u, d_v$  form a regular sequence and  $d \in (d_u, d_v)$ . This holds in the cases  $II_k^i$  and  $III_k$  we are considering. If  $d = \alpha d_u + \beta d_v$  then the sequence

$$0 \rightarrow \Theta(-\log D) \rightarrow \Theta_R \rightarrow N_{X/D} \rightarrow T_1 \rightarrow 0$$

may be written as

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} \alpha & -d_v \\ \beta & d_u \end{pmatrix}} R^2 \xrightarrow{(d_u, d_v)} R/dR \longrightarrow R/(d_u, d_v) \longrightarrow 0.$$

Let

$$\frac{\partial(a, b, c)}{\partial(u, v)} = \begin{pmatrix} a_u & a_v \\ b_u & b_v \\ c_u & c_v \end{pmatrix}$$

Computations show that the columns of

$$\frac{\partial(a, b, c)}{\partial(u, v)} \begin{pmatrix} \alpha & -d_v \\ \beta & d_u \end{pmatrix}$$

are in the column space of  $M$  in each of the cases  $II_k^i$  and  $III_k$ . So the induced map from  $\Theta(-\log D)$  to  $\ker(\text{AlgExt}_R(A, A) \rightarrow N_{X/D})$  is zero. This

shows that  $\text{OutDer}_k(A, A) \simeq \Theta(-\log D)$  by using lemma 5.9 as before. So using corollary 4.11 we get that the image of  $\underline{\text{AlgExt}}_k(A, A) \rightarrow \Theta_R^2$  is  $I_D \Theta_R^2$ . The module  $S(A)$  is presented by the matrix

$$\begin{pmatrix} \frac{\partial(a,b,c)}{\partial(u,v)} & M \end{pmatrix}.$$

## 7 Deformations of Orders over Surfaces

Let  $X$  be a smooth projective surface over  $k$ . Let  $A$  be a maximal  $\mathcal{O}_X$ -order. We will write  $\underline{H}^i$  for the sheafification of Hochschild cohomology. We have again the basic results in proposition 2.2,

$$\begin{aligned} \underline{H}_k^0(A, M) &= (M)^A \\ \underline{H}_k^1(A, M) &= \underline{\text{OutDer}}_k(A, M) \\ \underline{H}_k^2(A, M) &= \underline{\text{AlgExt}}_k(A, M) \\ \underline{H}_k^3(A, A) &= \underline{\text{Obs}}_k(A, A). \end{aligned}$$

Let  $D$  be the discriminant locus of  $A$ . Let  $T_1$  be sheaf of deformations of  $D$ . So  $T_1$  is the cokernel of the map  $\Theta_X \rightarrow N_{X/D}$ . We define  $S(A)$  to be the cokernel of the map  $\Theta_X \rightarrow \underline{\text{AlgExt}}_X(A, A)$ . We write  $K$  for the canonical divisor of  $X$ . We now extend theorem 5.11 to the global case.

**Corollary 7.1** *Let  $A$  be a maximal order over a smooth surface  $X$ . Then there are exact sequences*

$$0 \rightarrow S(A) \rightarrow H^0(X, \underline{\text{AlgExt}}_k(A, A)) \rightarrow H^0(X, \mathcal{O}_X(-K - D)).$$

$$0 \rightarrow A/\mathcal{O}_X \rightarrow \underline{\text{Der}}_k(A, A) \rightarrow \Theta(-\log D).$$

If  $A$  has global dimension two then there is an isomorphism

$$H^0(X, \underline{\text{AlgExt}}_k(A, A)) \simeq S(A) \oplus H^0(X, \mathcal{O}_X(-K - D)),$$

and an exact sequence

$$0 \rightarrow A/\mathcal{O}_X \rightarrow \underline{\text{Der}}_k(A, A) \rightarrow \Theta_X \rightarrow N_{X/D} \rightarrow T_1 \rightarrow 0.$$

**Proof.** Since  $\mathcal{O}_X(-K) \simeq \Theta_X^2$ , and  $I_D$  is the ideal sheaf of  $D$ , we have that  $I_D\Theta^2 \simeq \mathcal{O}_X(-K - D)$ . We now show the direct sum decomposition of  $H^0(X, \underline{\text{AlgExt}}_k(A, A))$  for orders of global dimension two. Since  $I_D \otimes \Theta_X^2$  is locally free,  $\underline{\text{Ext}}^p(I_D \otimes \Theta_X^2, S(A)) = 0$  for  $p \geq 1$ . So the local to global spectral sequence

$$\underline{\text{Ext}}_X^{p+q}(I_D \otimes \Theta_X^2, S(A)) \Leftarrow H^q(X, \underline{\text{Ext}}^p(I_D \otimes \Theta_X^2, S(A))) = E_2^{pq}$$

collapses to give

$$\underline{\text{Ext}}_X^1(I_D \otimes \Theta_X^2, S(A)) = H^1(X, \underline{\text{Hom}}(I_D \otimes \Theta_X^2, S(A))).$$

Since  $S(A)$  has support in  $\text{Sing } D$ , the above module is zero. So the extension

$$0 \rightarrow S(A) \rightarrow \underline{\text{AlgExt}}_k(A, A) \rightarrow I_D \otimes \Theta_X^2 \rightarrow 0$$

is globally split for orders of global dimension two. The rest follows from theorem 5.11.  $\square$

The following propositions follow from standard patching of deformations.

**Proposition 7.2** *Let  $A$  be a maximal order over  $X$ .*

1. *A first order deformation is determined locally up to isomorphism by a section in  $H^0(X, \underline{\text{AlgExt}}_k(A, A))$ .*
2. *There is an obstruction to globalizing a local deformation given by a map  $H^0(X, \underline{\text{AlgExt}}_k(A, A)) \rightarrow H^2(X, \underline{\text{Der}}_k(A, A))$ .*
3. *If the obstruction vanishes, the possible global deformations with the given local isomorphism class is a principal homogeneous space under  $H^1(X, \underline{\text{Der}}_k(A, A))$ .*

**Proposition 7.3** *Let  $S$  be an Artin local ring with residue field  $k$ . Let  $S'$  be a quotient of  $S$  so that the kernel of the map  $S' \rightarrow S$  is dimension one vector space over  $k$ . Let  $A_{S'}$  be a deformation of  $A$  over  $S'$ . We consider extending  $A_{S'}$  to a deformation over  $S$ .*

1. *There is an obstruction to extending the local deformation given by a map  $H^0(X, \underline{\text{AlgExt}}_k(A, A)) \rightarrow \underline{H}^0(X, \underline{\text{Obs}}_k(A, A))$ . If this obstruction vanishes the deformation  $A_{S'}$  can be extended to  $S$  locally.*

2. Suppose that this obstruction vanishes. Then the deformation can be extended locally, but the obstruction to finding compatible local deformations is in  $H^1(X, \underline{\text{AlgExt}}_k(A, A))$ .
3. If this obstruction also vanishes, then the set of possible local extensions is a principal homogeneous space under  $H^0(X, \underline{\text{AlgExt}}_k(A, A))$ .
4. The obstruction to finding a global deformation over  $S$  is in  $H^2(X, \underline{\text{Der}}_k(A, A))$ . If this obstruction vanishes then the set of possible global extensions is a principal homogeneous space under  $H^1(X, \underline{\text{Der}}_k(A, A))$ .

The bracket of a first order deformation  $A$  is defined to be the image of the class of the deformation under the natural map

$$H^0(X, \underline{\text{AlgExt}}_k(A, A)) \rightarrow H^0(X, \Theta_X^2).$$

as in 4.10. If a first order deformation of an order  $A$  gives rise to a non-zero bracket in  $H^0(X, \Theta_X^2)$ , we must have that  $H^0(X, \mathcal{O}_X(-K - D)) \neq 0$  by 5.11. More generally, if  $H^0(X, \mathcal{O}_X(-D - K)) = 0$  then  $H^0(X, \underline{\text{AlgExt}}_k(A, A)) \simeq S(A)$  by 7.1, and so all local deformations of  $A$  are deformations as orders. So all deformations of  $A$  over arbitrary Artin algebras are finite over their centres. So in order to find a deformation not finite over its centre, we certainly require that  $-K$  is effective. Denote by  $\text{kod}(X)$  the Kodaira dimension of  $X$  as in [Beau] VII.1.

**Proposition 7.4** *If the anticanonical divisor  $-K$  is effective, then  $X$  is either*

1. birational to a ruled surface,
2. a minimal K3 surface or an abelian surface.

**Proof.** In order for  $-K$  to be effective we either have  $\text{kod}(X) = 0$  with  $\omega_X \simeq \mathcal{O}_X$ , or  $\text{kod}(X) = -\infty$ . These are the above cases by the Enriques-Kodaira classification of surfaces as shown in [Beau] VII.3, VI.20, VIII.2.  $\square$

**Proposition 7.5** *If  $A$  is a maximal order over an abelian surface or a minimal K3 surface that has deformations with a non-trivial bracket then  $A$  is an Azumaya algebra.*

**Proof.** Since in both of these cases  $\mathcal{O}_X \simeq \mathcal{O}_X(-K)$ ,  $D$  is the trivial divisor. Hence  $A$  must be unramified.  $\square$

We recall some well known generalities about algebraic surfaces, and in particular ruled surfaces. We will be using [H] as our main reference for these results. The following is in [BPV] II.10.

**Proposition 7.6** *The arithmetic genus of an effective divisor  $D$  is given by the formula*

$$2p_a(D) - 2 = D.(D + K).$$

In particular,  $p_a(-K) = 1$ . The next statement follows from [H] V 3.2, V 3.3, V 3.6.

**Proposition 7.7** *Let  $\pi : X_p \rightarrow X$  be the blowup of  $X$  at  $p$ . Let  $K'$  be the canonical divisor of  $X_p$ . We denote by  $\tilde{C}$  the proper transform of a divisor  $C$  in  $X$ . Let  $E$  be the exceptional divisor. Let  $m$  denote the multiplicity at  $p$  of the divisor  $-K$ . Then*

$$-K' = \pi^*(-K) - E = -\tilde{K} + (m-1)E.$$

*The effective  $-K'$  are obtained by blowing up at  $p \in K$ . If  $m = 1$  then  $-K' \simeq -K$ .*

We now introduce the notation of [H] V 2.8.1 for ruled surfaces that will be used throughout this section.

**Notation 7.8** *Let  $X$  be a minimal ruled surface over a curve  $C$  of genus  $g$ . Then there is a rank two vector bundle  $V$  on  $C$  so that  $X \simeq \mathbb{P}(V)$ . Moreover  $V$  may be chosen so that  $H^0(C, V) \neq 0$  and if  $L$  is a line bundle on  $C$  with  $\deg L < 0$  then  $H^0(C, V \otimes L) = 0$ . The number  $e = -\deg(\wedge^2 V)$  is an invariant of  $X$ . Let  $\mathcal{O}_X(1)$  be the line bundle on  $X$  that is dual to the tautological line bundle that associates a point to the corresponding line in  $V$ . There is a section  $C_0$  of  $C$  so that  $\mathcal{O}_X(C_0) \simeq \mathcal{O}_X(1)$ . We will write the linear equivalence class of a divisor in terms of  $\text{Pic } X \simeq \mathbb{Z}C_0 \oplus \text{Pic}(C)f$  where  $f$  is a fibre.*

The following proposition, [H] V 2.18, 2.20, is an application of Nakai's criterion.

**Proposition 7.9** *Let  $X$  be a ruled surface over  $C$ . If  $aC_0 + Pf$  is linearly equivalent to a irreducible curve  $\neq C_0, f$  then*

1.  *$a > 0$  and  $\deg(P) \geq ae$ .*
2. *If in addition  $X$  is a rational ruled surface then  $a > 0, \deg P > ae$  or  $e > 0, a > 0, \deg P = ae$ .*

The possible discriminant curves of an order with deformations not finite over its centre are restricted by the Brauer group. We first analyze the case of a rational surface. A *polygon* of  $n$  smooth rational curves is a sum  $C = C_0 + \dots + C_{n-1}$  of smooth rational curves so that  $C_i$  meets each of the curves  $C_{i-1}$  and  $C_{i+1}$  transversely in a single point and does not meet the other curves at all, where the indices are taken modulo  $n$ .

**Proposition 7.10** *Suppose  $A$  is a maximal order over a rational surface  $X$ . If  $A$  has deformations with a non-trivial bracket then either  $D = 0$  and  $A = \underline{\text{End}}V$  for some vector bundle  $V$  on  $X$ , or  $D = -K$  and  $-K$  is one of the following*

- $E$  smooth elliptic curve,*
- $A_1$  rational curve with one node,*
- $A_n$  a polygon of  $n$  smooth rational curves.*

We will prove this proposition by classifying all possible curves in the linear system  $| -K |$ , and then using the Brauer group to shorten the list to the above one.

**Lemma 7.11** *If  $X$  is rational surface with  $-K$  effective then  $-K$  is one of the following*

- $E$  a smooth elliptic curve,*
- $A_1$  a rational curve with one node,*
- $A'_1$  a rational curve with one cusp,*
- $A'_2$  a sum of two smooth rational curves meeting at a single point with order two,*

$A'_3$  a sum of three smooth rational curves meeting transversely at a single point,

$A_n$  a polygon of  $n$  smooth rational curves,

$F$  a non-reduced divisor supported on a tree of smooth rational curves.

**Proof.** We will show that this list of curves is stable under blowing up a point by using 7.7. Let  $m$  be the multiplicity of  $-K$  at  $p$ , the point to be blown up. If  $m = 0$  then  $-K'$  is not effective. If  $m = 1$  then  $-K' \simeq -K$ . If we blow up at  $p$  with  $m \geq 2$  then  $-K \simeq A_i$  gives  $-K' \simeq A_{i+1}$ , and  $-K \simeq A'_i$  gives  $-K' \simeq A'_{i+1}$  for  $i = 1, 2$ . Also  $-K \simeq A'_3$  and  $-K \simeq F$  both give  $-K' \simeq F$ .

So we only need to verify the statement for minimal rational surfaces. The sequence

$$0 \rightarrow \mathcal{O}_X(K) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{-K} \rightarrow 0$$

shows that  $h^0(X, \mathcal{O}_{-K}) = 1$  and so  $-K$  is connected. Using the notation of 7.8 we write

$$X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$$

for some  $e \geq 0$ . Now  $-K = 2C_0 + (2+e)f$ . Using the genus formula we see that  $p_a(-K) = 1$ . For any subcurve  $C = aC_0 + bf$  with  $0 \leq a \leq 2, 0 \leq b \leq e+2$  and  $C \neq 0, C \neq -K$ , we have  $p_a(C) \leq 0$ . So if  $-K$  is integral it must be  $E, A_1$  or  $A'_1$ . If  $-K$  is reducible it must be a sum of smooth rational curves. If  $-K$  is not reduced then  $p_a(\text{supp}(-K)) = 0$  since  $-K$  is connected. So  $-K$  must be supported on a tree of smooth rational curves [BPV] II 11(c).

Suppose  $e > 2$ , then  $C_0$  is a fixed component of  $-K$  by the comparison sequence

$$0 \rightarrow \mathcal{O}_X(-K - C_0) \rightarrow \mathcal{O}_X(-K) \rightarrow \mathcal{O}_{C_0}(-K) \rightarrow 0.$$

since  $-K.C_0 = 2 - e < 0$ . The only smooth rational subcurves of  $-K$  are linearly equivalent to one of  $f, C_0, C_0 + ef, C_0 + (e+1)f$ , or  $C_0 + (e+2)f$  by 7.9. Computing the intersection numbers of these possible subcurves, we see that the only possible reduced decompositions of  $-K$  by using these curves are  $A_2, A'_2, A_3, A'_3$  or  $A_4$ .

If  $e = 2$  then  $f, C_0, C_0 + 2f, C_0 + 2f, C_0 + 3f, C_0 + 4f$  and  $-K$  are all possible integral subcurves. We obtain the same possible cases and the possibility that  $-K$  itself is integral.

If  $e = 0$  then  $X = \mathbb{P}^1 \times \mathbb{P}^1$  then the possible integral subcurves are linearly equivalent to one of  $f, C_0, C_0 + f, 2C_0 + f, C_0 + 2f$ , and  $-K$ . We again obtain the same possibilities as when  $e = 2$ . The last minimal surface to consider is  $\mathbb{P}^2$ . This case follows from looking at all cubic curves in  $\mathbb{P}^2$ . The possible cases are  $E, A_1, A'_1, A_2, A'_2, A_3, A'_3$ , and  $F$ .  $\square$

**Lemma 7.12** *Let  $D$  be the discriminant locus of a maximal order  $A$  over a surface  $X$ . Suppose  $f : \mathbb{P}^1 \rightarrow D$  is a nonconstant map. Then  $f^{-1}(\text{Sing } D)$  contains at least two distinct points.*

**Proof.** There is a canonical exact sequence

$$0 \rightarrow \text{Br } X \rightarrow \text{Br } K(X) \rightarrow \bigoplus_C H^1(K(C), \mathbb{Q}/\mathbb{Z}) \rightarrow \bigoplus_p \mu[-1] \rightarrow \mu[-1] \rightarrow 0 \quad (7.13)$$

for a simply connected surface  $X$  that is proved in [AM]. The argument of [AM] using the sequence 3.1 and 3.2 of that paper show that this sequence is a complex for any projective surface. This complex is exact except at  $\bigoplus_C H^1(K(C), \mathbb{Q}/\mathbb{Z})$ . The cohomology at that term is  $H^3(S, \mu)$ . Since a maximal order is contained in the central simple algebra  $A \otimes K(X)$  we get a class in  $\text{Br } K(X)$ . This sequence gives constraints on the possible ramification curves  $D$ . The order  $A$  determines a cyclic extension of  $K(C)$  for each component  $C$  of the ramification curve  $D$ . This cyclic extension corresponds to a ramified cover of the normalization of the curve  $C$ . The exact sequence requires that the ramification of the covers of the various curves at a point  $p$  must cancel to yield a element of  $\text{Br } K(X)$ . Any nontrivial cover of a smooth rational curve must ramify at two or more points. So if  $f : \mathbb{P}^1 \rightarrow D$  then  $f^{-1}(\text{Sing } D)$  contains at least two distinct points.  $\square$

**Proof.** We now prove proposition 7.10. The only possible discriminant curves are the subcurves of 7.11. Using 7.12 we eliminate the cases  $A'_1, A'_2, A'_3, F$ , and any of their subcurves and the proper subcurves of  $A_n$  leaving the desired list. The statement for the unramified case follows from the fact that the Brauer group of a rational surface is trivial [DeMF] 1.1. So if  $A$  is unramified then  $A \simeq \underline{\text{End}}V$  for some locally free sheaf  $V$ .  $\square$

We now consider the case of a surface ruled over a higher genus curve. First we describe the possible curves  $-K$ .

**Lemma 7.14** *Let  $\pi : X \rightarrow C$  be a birationally ruled surface over a curve  $C$  of genus  $g \geq 2$  with  $-K$  effective. Then*

$$-K = 2C_0 + \pi^*T$$

where  $C_0$  is a section of  $C$  and  $T$  is a divisor on  $C$ . Also,  $\pi^*T = T_1 + \cdots + T_n$  where the  $T_i$  are fibres of the map  $\pi$ . Each  $T_i$  is a tree of smooth rational curves with  $T_i \cdot C_0 = 1$ .

**Proof.** Suppose  $X$  is a minimal model and write  $X = \mathbb{P}(V)$  for  $V$  a rank two vector bundle as above. Now  $-K$  is linearly equivalent to  $2C_0 + Pf$  where  $P \in \text{Pic } C$  is of degree  $2 - 2g + e$ . Since  $-K$  is effective,  $e \geq 2$ . Now by the genus formula and 7.9, the only irreducible subcurves of  $-K$  are  $C_0$  and  $f$ . Using the comparison sequence twice,

$$|2C_0 + Pf| \simeq |C_0 + Pf| \simeq |Pf|$$

since  $C_0 \cdot (2C_0 + Pf) = 2 - 2g - e < 0$ ,  $C_0 \cdot (C_0 + Pf) = 2 - 2g < 0$ . So  $2C_0$  is a fixed component of  $-K$ . The decomposition of  $-K$  into irreducible subcurves depends only on  $P$ . So  $-K$  is as described in the statement of the lemma. The statement is preserved when we blow up at a point  $p$  in  $-K$  by applying 7.7.  $\square$

**Lemma 7.15** *Let  $\pi : X \rightarrow E$  be a birationally ruled surface over an elliptic curve with  $-K$  effective. Then either*

$$-K = 2E_0 + \pi^*T$$

where  $E_0$  is section of  $E$  and  $T$  is a divisor on  $E$ . Also,  $\pi^*T = T_1 + \cdots + T_n$  where the  $T_i$  are fibres of the map  $\pi$ . Each  $T_i$  is a tree of smooth rational curves with  $T_i \cdot E_0 = 1$ . or

$$-K = E_0 + E'_0$$

where  $E_0$  and  $E'_0$  are disjoint sections of  $E$ .

**Proof.** Suppose  $X$  is a minimal model and write  $X = \mathbb{P}(V)$  for a rank two vector bundle as above. Now  $-K$  is linearly equivalent to  $2E_0 + Pf$  where  $P \in \text{Pic } E$ . Since  $-K$  is effective,  $\deg(P) = e \geq 0$ , and if  $\deg(P) = 0$  then  $P$  is the trivial divisor. Now by the genus formula and 7.9, the only possible

irreducible subcurves of  $-K$  are  $E_0$ ,  $f$ , and  $E_0 + Pf$ . Using the comparison sequence, if  $e < 0$ ,

$$|2E_0 + Pf| \simeq |E_0 + Pf|,$$

since  $E_0.(2E_0 + Pf) = -e$ . So  $E_0$  is a fixed component of  $-K$  if  $e < 0$ . The possible decompositions of  $-K$  are as two disjoint sections  $E_0$  and  $E'_0 \in |E_0 + Pf|$  or as  $2E_0$  and a sum of fibres. If  $e = 0$  then  $X \simeq \mathbb{P}^1 \times E$ . In this case  $-K$  is linearly equivalent to  $2E_0$  and the linear system of  $-K$  is isomorphic to the pullback of the linear system of quadratics on  $\mathbb{P}^1$ . So  $-K$  is as described in the statement of the lemma. The statement is preserved when we blow up at a point  $p$  in  $-K$  by 7.7.  $\square$

Using 7.12 and the classification of possible anti-canonical curves in 7.14 and 7.15, we see that the only possible ramification loci are

1.  $X$  is birational to a surface ruled over an elliptic curve and the discriminant locus is either
  - (a) an elliptic curve that is a section of the base curve
  - (b) two disjoint sections of the base curve.
2.  $X$  is birational to a surface ruled over a curve  $B$  of genus  $\geq 2$  and the discriminant locus is a section of  $B$ .

**Theorem 7.16** *Let  $A$  be a maximal order over a surface  $X$ . Suppose that  $X$  is birational to a surface ruled over a curve of genus  $\geq 1$ . Suppose also that  $X$  has an effective anticanonical divisor  $-K$ . If  $A$  has deformations with a non-trivial bracket then one of the two following cases holds:*

1.  $A$  is unramified and is isomorphic to  $\underline{\text{End}}V$  for some vector bundle  $V$  on  $X$ .
2.  $X$  is birational to an elliptic ruled surface and the discriminant curve is the sum of two disjoint elliptic curves  $E + E' \in |-K|$ .

We first need the following proposition on the Brauer group.

**Proposition 7.17** *Let  $X$  be a smooth surface. Let  $D$  be a smooth curve in  $X$ . Let  $U = X - D$ . There is a canonical exact sequence*

$$0 \rightarrow Br(X) \rightarrow Br(U) \rightarrow H^1(D, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\gamma} H^3(X, \mu) \rightarrow$$

$$H^3(U, \mu) \rightarrow H^2(D, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\gamma} H^4(X, \mu) \rightarrow H^4(U, \mu) \rightarrow 0$$

where  $\mu$  is the group of roots of unity in  $k$  and  $\gamma$  is Poincaré dual to the restriction maps  $H^p(X, \mu) \rightarrow H^p(D, \mu)$  for  $p=0,1$ .

**Proof.** Let  $j : U \rightarrow X$  and  $i : D \rightarrow X$  be the inclusions. Let  $\mathbb{G}_m$  be the sheaf of units of  $\mathcal{O}_X$  and  $\mu_n$  be the group of  $n^{th}$  roots of unity. Local calculations in [Artin62] IV (3.4), show that  $R^k j_* \mathbb{G}_m = 0$  for  $k > 0$  and  $R^k j_* \mu_n = 0$  for  $k > 1$ . There is a canonical exact sequence of sheaves on  $X$

$$0 \rightarrow \mathbb{G}_m \rightarrow j_* \mathbb{G}_m \rightarrow i_* \mathbb{Z} \rightarrow 0,$$

as shown in [SGAIV] p.542, p.609. So by the Kummer exact sequence and the snake lemma we get the following diagram of sheaves on  $X$ ,

$$\begin{array}{ccccccc} \mu_n & \longrightarrow & j_* \mu_n & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{G}_m & \longrightarrow & j_* \mathbb{G}_m & \longrightarrow & i_* \mathbb{Z} \\ \downarrow n & & \downarrow n & & \downarrow n \\ \mathbb{G}_m & \longrightarrow & j_* \mathbb{G}_m & \longrightarrow & i_* \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & R^1 j_* \mu_n & \longrightarrow & i_* \mathbb{Z}/n\mathbb{Z}. \end{array} \quad (7.18)$$

All rows and columns of this diagram are exact with zeroes on the periphery. Given a bounded complex  $A^*$ ,

$$0 \rightarrow A^0 \rightarrow A^1 \rightarrow \cdots \rightarrow A^n \rightarrow 0$$

we will write the image in the derived category as

$$(A^0 \rightarrow A^1 \rightarrow \cdots \rightarrow A^n)$$

Shifting is denoted by  $A[k]^i = A^{i+k}$ . We will define three complexes  $R, S, T$  by taking the middle two rows of 7.18. The diagram 7.18 gives the following quasi-isomorphisms in the derived category of étale  $\mathcal{O}_X$ -sheaves

$$(\mu_n) \simeq (\mathbb{G}_m \xrightarrow{n} \mathbb{G}_m) = R, \quad (7.19)$$

$$(i_* \mathbb{Z}/n\mathbb{Z})[-1] \simeq (i_* \mathbb{Z} \xrightarrow{n} i_* \mathbb{Z}) = T. \quad (7.20)$$

Since  $\mathbb{G}_m$  is  $j_*$  acyclic,

$$(\mathbb{R}j_*)(\mathbb{G}_m \xrightarrow{n} \mathbb{G}_m) \simeq (j_*\mathbb{G}_m \xrightarrow{n} j_*\mathbb{G}_m) = S.$$

By the functoriality of  $\mathbb{R}j_*$  applied to the quasi-isomorphism 7.19 on  $U$

$$\mathbb{R}j_*(\mu_n) \simeq (\mathbb{R}j_*)(\mathbb{G}_m \xrightarrow{n} \mathbb{G}_m).$$

Unless otherwise noted all cohomology groups are taken over  $X$ . We have a short exact sequence of complexes  $0 \rightarrow R \rightarrow S \rightarrow T \rightarrow 0$  by the diagram 7.18. Let  $\Gamma$  be the global section functor. We denote by  $\mathbb{H}$  the hypercohomology functor  $\mathbb{R}\Gamma$ . The Leray spectral sequence in the derived category provides the quasi-isomorphism

$$\mathbb{R}\Gamma(X, \mathbb{R}j_*\mu_n) \simeq \mathbb{R}\Gamma(U, \mu_n),$$

[Wb] 10.8.3. The cohomology of the left hand side is the hypercohomology of the complex  $\mathbb{R}j_*\mu_n$ . The cohomology of the right hand side is  $H^p(U, \mu_n)$ . Hence

$$\mathbb{H}^p(X, S) \simeq H^p(U, \mu_n).$$

The hypercohomology of  $R, T$  may be computed by the quasi-isomorphisms 7.19, and 7.20. So the long exact sequence in hypercohomology coming from the short exact sequence of complexes is

$$\rightarrow H^p(\mu_n) \rightarrow H^p(U, \mu_n) \rightarrow H^{p-2}(i_*\mathbb{Z}/n\mathbb{Z}) \rightarrow H^{p+1}(\mu_n) \rightarrow .$$

Each complex  $A^*$  has a spectral sequence abutting to its hypercohomology, as in [Wb] 5.7.10, that is formed by taking sheaf cohomology first and then the cohomology of the resulting complex

$$\mathbb{H}^{p+q}(X, A^*) \Leftarrow H^p(H^q(X, A)^*) = E_2^{pq}. \quad (7.21)$$

Since each of the complexes  $R, S, T$  have only two terms this spectral sequence will give a long exact sequence. By the functoriality of this spectral sequence we get the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccc} H^2(\mu_n) & \longrightarrow & H^2(U, \mu_n) & \longrightarrow & H^1(i_*\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & H^3(\mu_n) \\ \downarrow & & \downarrow & & \downarrow \alpha_n & & \downarrow \beta_n \\ H^2(\mathbb{G}_m) & \longrightarrow & H^2(j_*\mathbb{G}_m) & \longrightarrow & H^2(i_*\mathbb{Z}) & \longrightarrow & H^3(\mathbb{G}_m) \\ \downarrow n & & \downarrow n & & \downarrow n & & \downarrow n \\ H^2(\mathbb{G}_m) & \longrightarrow & H^2(j_*\mathbb{G}_m) & \longrightarrow & H^2(i_*\mathbb{Z}) & \longrightarrow & H^3(\mathbb{G}_m). \end{array} \quad (7.22)$$

The long exact sequences given by the spectral sequence 7.21 are written vertically. It is shown in [Groth] p.71 that  $H^q(\mathbb{G}_m)$  is torsion for  $q \geq 2$ . So in the direct limit over  $n$  the maps  $\alpha_n, \beta_n$  give isomorphisms since  $H^p(i_*\mathbb{Q}/\mathbb{Z}) \simeq H^{p+1}(i_*\mathbb{Z})$  and  $H^q(\mu) \simeq H^q(\mathbb{G}_m)$  via these maps for  $p \geq 1$  and  $q \geq 3$  [Milne] 2.22. Since  $\mathbb{G}_m$  is  $\mathbb{R}j_*$  acyclic  $H^p(X, j_*\mathbb{G}_m) \simeq H^p(U, \mathbb{G}_m)$ . So the middle exact sequence of 7.22 gives us the exact sequence to be shown. The top sequence of the diagram 7.22 is the isomorphic to the long exact sequence induced by the Leray spectral sequence

$$E_2^{pq} = H^p(R^q j_* \mu_n) \Rightarrow H^{p+q}(U, \mu_n).$$

So the maps  $H^{2-q}(i_*\mathbb{Z}/n\mathbb{Z}) \rightarrow H^{4-q}(\mu_n)$  are Poincaré dual to the restriction map  $H^q(\mu_n) \rightarrow H^q(i_*\mu_n)$  for  $q = 0, 1, 2$  by [Milne] 11.5,5.4.  $\square$

We now prove the theorem.

**Proof.** Let  $X_0$  be birational to a surface ruled over  $B$ , a curve of genus  $\geq 1$ . There is a sequence of maps

$$X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n \xrightarrow{\pi} B$$

where the maps  $X_i \rightarrow X_{i+1}$  are blow-ups at points and  $X_n$  is a  $\mathbb{P}^1$ -bundle over  $B$  with projection  $\pi$ . Using the Leray spectral sequence for each of these maps we get that the natural restriction map  $H^1(X, \mu) \rightarrow H^1(B, \mu)$  is an isomorphism. So if  $D$  is a section of  $\pi$  then the map  $H^1(D, \mathbb{Q}/\mathbb{Z}) \rightarrow H^3(X, \mu)$  is Poincaré dual to an isomorphism. Hence if  $D$  is a single section of the  $\mathbb{P}^1$ -bundle  $S_n \rightarrow B$  then the order  $A$  must be unramified. If  $D$  is the sum of two disjoint sections then the map  $H^1(D, \mu) \rightarrow H^3(X, \mu)$  is a sum of two isomorphisms and so cancellation is possible.  $\square$

## References

- [Anick] D. Anick, *On the Homology of Associative algebras*, Trans. A.M.S. 296, 2 (1986), 641-659.
- [Artin62] M. Artin, *Grothendieck Topologies*, Harvard University, Notes (1962).
- [SGAIV] M. Artin, A. Grothendieck, J. L. Verdier, *Théorie des Topos et Cohomologie Étale des Schémas*, Springer-Verlag, L.N.M. 305, (1973).

- [Artin76] M. Artin, *Deformations of Singularities*, Tata Institute of Fundamental Research, Bombay, (1976).
- [Artin82a] M. Artin, *Local Structure of Maximal Orders on Surfaces*, Springer-Verlag, L.N.M. 917 (1982), 146-181.
- [Artin82b] M. Artin, *Left Ideals in Maximal Orders*, Springer-Verlag, L.N.M. 917 (1982), 182-193.
- [Artin86] M. Artin, *Maximal orders of global dimension and Krull dimension two*, Invent. math. 195-222 (1986).
- [AM] M. Artin and D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London Math. Soc. 3 (1972), 75-79.
- [AG] M. Auslander and O. Goldman, *The Brauer Group of a Commutative Ring*, Trans. A.M.S. 97 (1960), 367-409.
- [BPV] W. Barth, C. Peters, A. Van de Ven, *Compact Complex Surfaces*. Berlin, Heidelberg, New York, Tokyo: Springer-Verlag, 1984.
- [Beau] Beauville, *Complex Algebraic Surfaces*, London Math. Soc. Stud. Texts 34, University Press, Cambridge, (1996).
- [Brg] Bergman, *The Diamond Lemma in Ring Theory* Adv. in Math. 29 (1978), 178-218.
- [DeMF] F.R. DeMeyer, T.J. Ford *On the Brauer Group of Surfaces and Subrings of  $k[x,y]$* , Springer-Verlag, L.N.M. 917 (1982), 211-221.
- [E] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*. Berlin, Heidelberg, New York: Springer-Verlag, 1995.
- [EG] E.G. Evans, P. Griffith, *Syzygies*. London: London Math Soc. Lect. Notes, 106, 1985.
- [FD] B.Farb and R.K.Dennis, *Noncommutative Algebra*, Springer-Verlag, Berlin, Heidelberg, New York, 1993.
- [Grst] M. Gerstenhaber, *Deformations of Rings and Algebras*. Ann. of Math. (2) 79 (1964), 59-103.

- [Groth] A. Grothendieck, *Dix exposés sur la Cohomologie des Schémas*. IHES.
- [H] R. Hartshorne, *Algebraic Geometry*. Berlin, Heidelberg, New York: Springer-Verlag, 1977.
- [Janusz] G. Janusz, *Tensor Products of Orders* J.London Math. Soc., (2), 20 (1979) 186-192.
- [KnOj] M. Knus, M. Ojanguren, *Théorie de la Descent et Algèbres d'Azumaya*, Springer-Verlag, L.N.M. 389 (1974).
- [MacLane] S. MacLane, *Homology*, Springer-Verlag, Berlin, 1963.
- [Milne] Milne, *Étale Cohomology*, Princeton, New Jersey, Princeton University Press, 1980.
- [Reiner] I. Reiner, *Maximal Orders*, London, New York, San Francisco: Academic Press, 1975.
- [Rowen] L. Rowen, *Ring Theory*, Student Edition, Academic Press, London, 1991.
- [Sanada] K. Sanada, *Hochschild Cohomology of Minimal Hereditary Orders*, J. Algebra 176, 786-805 (1995).
- [Sch] M. Schlessinger, *Functors on Artin Rings*, Trans. A.M.S. 130, 208-222.
- [Wb] C. Weibel, *Homological Algebra*. Cambridge: Cambridge University Press, 1994.