Logarithmic Centres of Twisted Noncommutative Crepant Resolutions are Kawamata Log Terminal

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Theorem (I,Yasuda)

The logarithmic centres of twisted NCCRs are Kawamata log terminal.

Corollary and Generalization of

Theorem (Stafford and Van den Bergh)

The centres of homologically homogeneous algebras (TNCCRs) have rational singularities.

Geometric Definitions

- k - field of characteristic zero.
- $X = \operatorname{Spec} Z$ - normal irreducible variety,
- D - $\mathbb{Q}$ divisor
- (X, D) - log variety
- D is the *boundary* of (X, D)
- $K_X + D$ is $\mathbb{Q}$ Cartier, there exists $m \in \mathbb{Z}$ such that $m(K_X + D)$ is Cartier.

Definition (Discrepency)

Let $f : Y \to X$ be proper, birational with Y smooth. Define $\{E_i\}$ be the set of irreducible exceptional divisors ($\dim f(E_i) < \dim E_i$) for f. Then $K_Y = f^*K_X + \sum_i a_i E_i$. Discrepency a is the minimum a_i over all maps f.

Definition

(X, D) is	terminal	$a > 0$
	canonical	$a \geq 0$
	log terminal	$a > -1$
	log canonical	$a \geq -1$

Definition

If $a = 0$ then $f : Y \to X$ is crepant. (0 discrepency).

Algebraic Definitions

- Λ is a finitely generated k-algebra
- $Z = Z(\Lambda)$ is a normal integral domain with fraction field K.
- Λ is an order $\Lambda \subseteq \Lambda \otimes_Z K = A$ central simple K-algebra.
- Λ is a finitely generated Z module.
- Λ is homologically homogeneous
    - $\mathrm{pd}_\Lambda S$ constant for all simple Λ modules S.
- homologically homogeneous $\Rightarrow \Lambda$ is a *tame* order:
    - reflexive over Z
    - hereditary in codimension one.

Geometric data from Λ

- $\alpha \in \operatorname{Br} K$ where α is Brauer class of A.
- Discriminant of Λ:
- Let p be a height one prime in Z. There is an étale extension R of Z_p such that
- $$\Lambda \otimes_Z R \overset{\text{Morita}}{\simeq} \begin{pmatrix} R & \dots & \dots & R \\ p & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ p & \dots & p & R \end{pmatrix} \subseteq R^{e_p \times e_p}$$

  e_p is the ramification index of Λ.

Discriminant:

$$D = \sum_{\substack{p \in \operatorname{Spec} Z \\ \mathrm{ht}\, p = 1}} \left(\frac{e_p - 1}{e_p} \right) V(p)$$

Usual discriminant is $\lceil D \rceil = V(\sqrt{\det(x_i x_j)})$.

Definition (log centre)

$(\operatorname{Spec} Z, D) = \log Z(\Lambda)$ is the log centre of Λ.

Definition (TNCCR)

If Λ is homologically homogeneous we say Λ is an α-twisted NCCR of $\log Z(\Lambda) = (X, D)$.

- There is a n such that

$$\omega_{\Lambda}^{(n)} = \omega_{Z}^{(n)}(nD)$$

where $(-)^{(n)} = (-)^{\otimes n**}$.

- Follows from adjunction formula:

$$\omega_{\Delta} = \mathrm{Hom}_Z(\Lambda, \omega_Z) = \begin{pmatrix} R & p^{-1} & \dots & p^{-1} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & p^{-1} \\ R & \dots & \dots & R \end{pmatrix}$$

in codimension one.

- Write

$$K_{\Lambda} := \pi^{*}(K_X + D).$$

where $\pi : \mathrm{Spec}\,\Lambda \to X$.

Examples

$\alpha = 0$ Λ is an NCCR of (X, D), $\Lambda \otimes K \simeq K^{n \times n}$ and $\Lambda = \mathrm{End}_{\mathrm{Flag}}(M)$

$D = 0$ Λ is Azumaya on $X \setminus \operatorname{Sing} X$ and $\Lambda \in \mathrm{Br}(X \setminus \operatorname{Sing} X)$ Λ is an α-twisted NCCR of X.

$\alpha = D = 0$ $\Lambda = \mathrm{End}_Z(M)$ usual NCCR

Theorem (Brown, Goodearl)

Let H be a Hopf Algebra that is a finitely generated algebra over a field and is a finite modules over its centre and has finite global dimension.

Example

- $\mathcal{O}_\zeta(G)$ where G is semisimple and ζ is a root of unity.
- $U_\zeta(\mathfrak{g})$ where $\mathfrak{g}$ is semisimple and ζ is a root of unity.

Theorem (Stafford, Zhang)

Let R be a connected graded affine noetherian PI ring. If gldim R *is finite, then R is an twisted NCCR.*

Example (Stephenson,Van den Bergh)

$$\frac{k\langle a, b, c\rangle}{(ac + ca, bc + cb, ab - 2c^3 - ba)}$$

is a TNCCR of $x^2 + y^2 + z^2 + t^3$ with a boundary.

- There are examples with twisted NCCRs which do not have NCCRs.

Example

Koszul duals of complete intersections of four quadrics in $\mathbb{P}^3$ localised at one of 10 ordinary double points of their centres.

Theorem (I,Yasuda)

Let Λ be a finitely generated k-algebra which is an order and is homologically homogeneous (twisted NCCR). Then $\log Z(\Lambda)$ *is Kawamata log terminal.*

Corollary and Generalization of:

Theorem (Stafford, Van den Bergh)

Let Λ be a finitely generated k-algebra which is an order and is homologically homogeneous (twisted NCCR). Then $Z(\Lambda)$ has rational singularities.

Corollary (I,Yasuda)

If Λ as above, then

- *$K_Z + D$ is $\mathbb{Q}$-Gorenstein.*
- *X has rational singularities.*
- *If K_X is $\mathbb{Q}$-Cartier, then X has log terminal singularities.*

Proof.

- Let m be the index of (X, D), the smallest positive integer such that $m(K_X + D)$ is Cartier. Define the canonical, (index one, Artin) cover:

$$\tilde{Z} = \bigoplus_{i=0}^{m-1} \omega_X^{(i)}(\lfloor iD \rfloor).$$

$$\tilde{\Lambda} = \bigoplus_{i=0}^{m-1} \omega_\Lambda^{(i)}$$

$$\tilde{X} = \operatorname{Spec} \tilde{Z}.$$

- $$\begin{array}{ccccccc} \tilde{Z} & \longrightarrow & \tilde{\Lambda} & \quad & \tilde{X} & \longleftarrow & \operatorname{Spec} \tilde{\Lambda} \\ \uparrow & & \uparrow & & \downarrow \phi & & \downarrow \\ Z & \longrightarrow & \Lambda & & X & \longleftarrow & \operatorname{Spec} \Lambda \end{array} .$$

Properties:

- $\tilde{X}$ irreducible, normal, $K_{\tilde{X}}$ is $\mathbb{Q}$-Cartier.

$$K_{\tilde{X}} = \phi^*(K_X + D)$$

- $\tilde{\Lambda}$ is prime, homologically homogeneous and

$$\omega_{\tilde{\Lambda}} \simeq \tilde{\Lambda}$$

locally as bimodules (Stafford, Van den Bergh).
- Calculation:

$$Z(\tilde{\Lambda}) = \tilde{Z}$$

by computing

$$Z\left(\begin{pmatrix} R & \dots & \dots & R \\ p & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ p & \dots & p & R \end{pmatrix}^i\right)$$

- By Stafford, Van den Bergh, $\tilde{\Lambda}$ is homologically homogeneous and so $\tilde{X} = \operatorname{Spec} Z(\tilde{\Lambda})$ has rational singularities.

Proposition (Kollar, Mori)

If $K_{\tilde{X}}$ is Cartier, then $\tilde{X}$ has rational singularities if and only if $\tilde{X}$ has canonical singularities.

Proposition (Kollar,Mori)

Let $\phi : \tilde{X} \to X$ be a finite map. If $K_{\tilde{X}} = \phi^*(K_X + D)$, then $\tilde{X}$ has canonical singularities if and only if (X, D) has Kawamata log terminal singularities.

□

- Question: When does there exist twisted NCCRs?