Rationality of Brauer-Severi Varieties of Sklyanin Algebras

C. Ingalls

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Azumaya on $\mathbb{P}^2 - E/\langle\sigma\rangle$.

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If $k = \overline{k}$ *and* $\operatorname{char} k = 0$ *then* $\mathrm{BSV}(\mathcal{A})$ *is rational over* k*.*

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k is a C_1 field if any form f of degree $d < n$ variables has k rational point in $\mathbb{P}^{n-1}$.

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If k is C_1 then $\operatorname{Br} k = 0$

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Proof.

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- Replace X with $\mathrm{BSV}(\mathcal{A})$

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$k(X) \simeq k(\mathrm{BSV}(\mathcal{A}))(t_1, \ldots, t_d)$.

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- $p \in \mathrm{Sing}\,\pi$ then $R = \mathcal{O}^{sh}_{\mathbb{P}^1,p} \simeq \overline{k}[\![t]\!]$

$$\mathcal{A} \otimes R \simeq \overline{k}(n\text{-cycle})^h \simeq \overline{k}[\![x]\!] \rtimes \mathbb{Z}/\mathbb{Z}n$$

Interlude on Toric Varieties

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Varieties	Lattice
$X, \mathcal{O}(H)$	convex lattice polytope
$H^0(\mathcal{O}(H))$	lattice points in polytope
$\mathbb{P}^2, \mathcal{O}(1)$	triangle
$\mathbb{P}^2, \mathcal{O}(2)$	bigger triangle
toric subvarieties	faces
toric divisors	facets
changing $\mathcal{O}(H)$	sliding facets in and out
Blowing up	Cutting off subvariety
$\mathrm{Bl}_p\, \mathbb{P}^2$	trapezoid
$\mathbb{P}^1 \times \mathbb{P}^1$	square

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- F, G are complete flags in fibres over $0, \infty$

Theorem (Artin,Hille,Bocklandt,I,...)

- $\Delta = \operatorname{Spec} R$ *strictly hensel*
- $F = (0 \subset F_1 \subset \cdots F_n = \mathbb{P}^n)$ *is a complete flag*
- $X_\Delta = \mathrm{Bl}_F\, P^n_\Delta = \mathrm{BSV}(n\text{-}cycle) =$ *toric quiver variety of wagon wheel*
- *Picture*
- *Puzzle*

- $\overline{X} = \mathrm{Bl}_{F,G}\, \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ where
- $\mathcal{E}$ is a vector bundle over $\mathbb{P}^1$
- F, G are complete flags in fibres over $0, \infty$

Proposition

$\overline{X} \to \mathbb{P}^1$ *is toric and F, G are toric invaraiant flags.*

Lemma

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Proposition

X has a Galois invariant section or $\overline{X} \simeq \mathrm{Bl}_{F,G}(\mathbb{P}^n \times P^1)$ *with* F, G *complementary flags.*

- $\overline{X}$ has Galois invariant divisors $K_{\overline{X}}$ and $F = \pi^{-1}(p)$ for $p \in \mathbb{P}^1$.

$\square$

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- Effective cone spanned by $-K_\pi = -K - 2F$ and F.

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- $\mathrm{Bl}_p\, Y \overset{|H-nE_p|}{\dashrightarrow} \mathbb{P}^{n+1}$

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Polytopes over the effective cone