17. $\dfrac{\sqrt{2}-1}{2}$.

$$\begin{aligned} I &\equiv \int_0^{\sqrt{\pi}/2} x\sec(x^2)\tan(x^2)\ dx \\ &= \int_0^{\sqrt{\pi}/2} \frac{1}{2}\frac{d}{dx}\sec(x^2)\ dx \\ &= \frac{1}{2}\sec(x^2)\Big|_0^{\sqrt{\pi}/2} = \frac{1}{2}\left(\sec\frac{\pi}{4}-\sec 0\right) = \frac{1}{2}(\sqrt{2}-1). \end{aligned}$$

18. 0. Let $\square = x^2$ So $D\square = 2x$ and the antiderivative looks like

$$\frac{1}{2}\int \frac{1}{1+\square^2}\frac{d\square}{dx}\ dx,$$

which reminds one of the derivative of the Arctangent function. In fact,

$$\int_{-1}^{1}\frac{x}{1+x^4}\ dx = \frac{1}{2}\tan^{-1}x^2\Big|_{-1}^{1} = \frac{1}{2}(\tan^{-1}1-\tan^{-1}1) = 0.$$

(Notice that 0 is the expected answer because the integrand is an odd function.)

19. Following the hint, we have $\dfrac{d}{dx}\displaystyle\int_0^{x^2} e^t\ dt = e^{x^2}\dfrac{d}{dx}x^2 = 2xe^{x^2}$.

20. These identities can be seen from the respective symmetry in the graph of f. Here is an analytic argument. Assume that f is even: $f(-x) = f(x)$. Let $\mathcal{F}(x) = \int_0^x f(t)dt$, $(-\infty < x < \infty)$. Then $\frac{d}{dx}\mathcal{F}(x) = f(x)$ and

$$\begin{aligned} \int_{-x}^{x} f(t)dt &= \int_{-x}^{0} f(t)dt + \int_0^x f(t)dt \\ &= -\int_0^{-x} f(t)dt + \int_0^x f(t)dt = -\mathcal{F}(-x)+\mathcal{F}(x). \end{aligned}$$

Thus we will have $\int_{-x}^{x} f(t)dt = 2\int_0^x f(t)dt$ if we can show $-\mathcal{F}(-x) = \mathcal{F}(x)$. Let $\mathcal{G}(x) = -\mathcal{F}(-x)$. We are going to show $\mathcal{G} = \mathcal{F}$. Now

$$\begin{aligned} \frac{d}{dx}\mathcal{G}(x) &= \frac{d}{dx}(-\mathcal{F}(-x)) = -\left(\frac{d}{dx}\mathcal{F}(-x)\right) \\ &= -\big(\mathcal{F}'(-x)\cdot(-1)\big) = \mathcal{F}'(-x) = f(-x) = f(x). \end{aligned}$$

Thus, by the Fundamental Theorem of Calculus, $\mathcal{G}(x) = \int_0^x f(t)\ dt + C$ for some constant C, or $\mathcal{G}(x) = \mathcal{F}(x)+C$. Now $\mathcal{G}(0) = -\mathcal{F}(-0) = -\mathcal{F}(0) = -0 = 0$, which is the same as $\mathcal{F}(0)$ $(=0)$. So C must be zero. Thus $\mathcal{G} = \mathcal{F}$. Done! (The second part of the exercise which involves an even function f can be dealt with in the same manner.)

7.3 Exercise Set 36

1. $\displaystyle\sum_{i=1}^{10} i$.

2. $\displaystyle\sum_{i=1}^{9}(-1)^{i-1}$, or $\displaystyle\sum_{i=1}^{9}(-1)^{i+1}$, or $\displaystyle\sum_{i=0}^{8}(-1)^{i}$.

3. $\displaystyle\sum_{i=1}^{5}\sin i\pi$.

4. $\displaystyle\sum_{i=1}^{n} \frac{i}{n}$

5. -0.83861

6. 0.19029.

7. 0. Note that $\sin n\pi = 0$ for any integer n.

8. $1 + 2 + 3 + \cdots + 50 = \dfrac{50 \times 51}{2} = 1275.$

9. $1^2 + 2^2 + \cdots + 100^2 = \dfrac{100 \times 101 \times 201}{6} = 338350.$

10. $\displaystyle\sum_{i=1}^{n} \frac{i}{n} = \frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2}.$

11. $\displaystyle\sum_{i=1}^{n} 6\left(\frac{i}{n}\right)^2 = \frac{6}{n^2}\sum_{i=1}^{n} i^2 = \frac{6}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{n}.$

12. This is a telescoping sum:

$$\sum_{i=1}^{6}(a_i - a_{i-1}) = (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_6 - a_5) = a_6 - a_0.$$

The final expression stands for what is left after many cancellations.

13. We prove this identity by induction. For $n = 1$, we have

$$\text{LHS} = \sum_{i=1}^{1}(a_i - a_{i-1}) = a_1 - a_0 = \text{RHS}.$$

Now we assume $\sum_{i=1}^{k}(a_i - a_{i-1}) = a_k - a_0$, that is, the identity holds for $n = k$. Then, for $n = k + 1$, we have

$$\begin{aligned}\sum_{i=1}^{k+1}(a_i - a_{i-1}) &= \sum_{i=1}^{k}(a_i - a_{i-1}) + (a_{k+1} - a_k) \\ &= (a_k - a_0) - (a_{k+1} - a_k) = a_{k+1} - a_0.\end{aligned}$$

So the identity is also valid for $n = k + 1$. Done.

14. Indeed,

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2 &= \left(\frac{1}{n}\right)^3 \sum_{i=1}^{n} i^2 = \left(\frac{1}{n}\right)^3 \frac{n(n+1)(2n+1)}{6} \\ &= \frac{n}{n}\ \frac{(n+1)}{n}\ \frac{(2n+1)}{n}\ \frac{1}{6} = \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\ \frac{1}{6}.\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2 = 1 \cdot 2 \cdot \frac{1}{6} = \frac{1}{3}.$$

15. For convenience, we write

$$A_n = \sum_{i=1}^{n-1} \frac{n^3}{n^4 + in^3 + p_n}.$$

We have to show that $\lim_{n\to\infty} A_n = \ln 2$. We know that $\int_1^2 \frac{1}{x}\, dx = \ln 2$. Divide the interval $[1, 2]$ into n subintervals of the same length $1/n$ by means of subdivision points $x_i = 1 + \frac{i}{n}$ $(i = 0, 1, 2, \ldots, n-1)$ and form the corresponding Riemann sum S_n for the function $f(x) = 1/x$:

$$S_n = \sum_{i=0}^{n-1} f(x_i) \cdot \Delta x_i = \sum_{i=0}^{n-1} \frac{1}{x_i} \cdot \Delta x_i = \sum_{i=0}^{n-1} \frac{n}{n+i} \cdot \frac{1}{n} = \sum_{i=0}^{n-1} \frac{1}{n+i}.$$

Since f is continuous on $[1,2]$, from the theory of Riemann integration we know that $\lim_{n\to\infty} S_n = \ln 2$. It suffices to show that $\lim_{n\to\infty}(S_n - A_n) = 0$. Now

$$S_n - A_n = \sum_{i=0}^{n-1} \left(\frac{1}{n+i} - \frac{n^3}{n^4 + in^3 + p_n} \right)$$

$$= \sum_{i=0}^{n-1} \frac{(n^4 + in^3 + p_n) - n^3(n+i)}{(n^4 + in^3 + p_n)(n+i)} = \sum_{i=0}^{n-1} \frac{p_n}{(n^4 + in^3 + p_n)(n+i)}.$$

Thus

$$0 \le S_n - A_n \le \sum_{i=0}^{n-1} \frac{p_n}{n^4 \cdot n} = p_n/n^4;$$

(dropping something positive from the denominator of a positive expression would diminish the denominator and hence would increase the size of this expression.) By the *Hint*, we have $p_n < 36n \ln n$. It is well-known that $\ln x \le x$ for all $x > 0$. So $p_n < 36n^2$ for all $n \ge 2$. Thus $0 < S_n - A_n \le p_n/n^4 < 36n^2/n^4 = 36/n^2$ for $n \ge 2$. Now it is clear that $S_n - A_n$ tends to 0 as $n \to \infty$, by the Sandwich Theorem of Chapter 2.

7.4

7.5 Chapter Exercises

1. $\frac{1}{27}(x+1)^{27} + C$. Use Table 7.5, $\square = x+1, r = 26$.
2. $\frac{1}{2}\sin 2x + C$.
3. $\frac{1}{3}(2x+1)^{3/2} + C$.
4. $-\frac{1}{12}(1-4x^2)^{3/2} + C$. Use Table 7.5, $\square = 1-4x^2, r = 1/2$.
5. $-\cos x + \sin x + C$.
6. $-\frac{1}{3}(5-2x)^{3/2} + C$.
7. $-\frac{1}{2}\cos(2x) + C$. Use Table 7.6, $\square = 2x$.
8. $0.4\, x^{2.5} + 0.625\, \cos(1.6x) + C$.
9. $3\tan x + C$.
10. $\frac{1}{200}(x^2+1)^{100} + C$. Use Table 7.5, $\square = x^2+1, r = 99$.
11. $-\frac{1}{3}\csc 3x + C$.
12. $-\frac{1}{6}e^{-3x^2} + C$. Use Table 7.5, $\square = -3x^2$.
13. $-\frac{1}{k}e^{-kx} + C$.
14. $\frac{\sin kx}{k} + C$. Use Table 7.6, $\square = kx$.
15. $-\frac{\cos kx}{k} + C$.
16. 2. $\int_0^1 (2x+1)\, dx = x^2 + x \Big|_0^1 = 2$.
17. 0. Note that $f(x) = x^3$ is an *odd function*.
18. 10. $I = \int_0^2 (3x^2 + 2x - 1)\, dx = x^3 + x^2 - x \Big|_0^2 = 10$.

19. $\frac{1}{5}$. $\int_0^{\pi/2} \sin^4 x \cos x \, dx = \frac{\sin^5 x}{5}\Big|_0^{\pi/2} = \frac{1}{5}$.

20. $\frac{1}{\ln 3}$. $\int_0^1 x \cdot 3^{x^2} \, dx = \frac{1}{2}\frac{3^{x^2}}{\ln 3}\Big|_0^1 = \frac{1}{\ln 3}$. Use Table 7.5, $\square = x^2, a = 3$.

21. $\frac{1}{\ln 4}$. $\int_0^1 2^{-x} dx = -\frac{1}{\ln 2}\, 2^{-x}\Big|_0^1 = -\frac{1}{\ln 2}\left(\frac{1}{2} - 1\right) = \frac{1}{2\ln 2} = \frac{1}{\ln 4}$.

22. $\frac{2}{3}$. $\int_0^{\pi} \cos^2 x \cdot \sin x \, dx = -\frac{\cos^3 x}{3}\Big|_0^{\pi} = \left(-\frac{(-1)^3}{3}\right) - \left(-\frac{1}{3}\right) = \frac{2}{3}$.

23. $\frac{28}{3}$. Note that $f(x) = x^2 + 1$ is an *even function.*

24. $\frac{\pi}{6}$. $\int_0^{0.5} \frac{1}{\sqrt{1-x^2}} \, dx = \text{Arcsin } x \Big|_0^{0.5} = \frac{\pi}{6}$.

25. $\frac{1}{2}e^4 - \frac{1}{2}$. Use Table 7.5, $\square = x^2$.

26. $\sum_{i=1}^{n} 12\left(\frac{i}{n}\right)^2 = \frac{12}{n^2}\sum_{i=1}^{n} i^2 = \frac{12}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{2(n+1)(2n+1)}{n}$.

27. Consider the partition

$$0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$$

with $x_i = \frac{i}{n}$, which divides $[0,1]$ into n subintervals $[\frac{i}{n}, \frac{i+1}{n}]$ of the same length $1/n$, $(i = 0, 1, 2, \ldots, n-1)$. In each subinterval $[\frac{i}{n}, \frac{i+1}{n}]$ we take c_i to be the left end point $\frac{i}{n}$. Then the corresponding Riemann sum for the function $f(x) = e^x$ is

$$\sum_{i=0}^{n-1} f(c_i)(x_{i+1} - x_i) = \sum_{i=0}^{n-1} e^{i/n}\left(\frac{i+1}{n} - \frac{i}{n}\right) = \sum_{i=0}^{n-1} e^{i/n} \cdot \frac{1}{n}.$$

But we know from the definition of Riemann integration that

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}) = \int_0^1 f(x)dx = \int_0^1 e^x dx = e - 1.$$

Now the assertion is clear.

28. Indeed,

$$\left(\frac{n}{n^2+0^2} + \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots + \frac{n}{n^2+(n-1)^2}\right) =$$

$$\left(\frac{n}{n^2(1+\left(\frac{0}{n}\right)^2)} + \frac{n}{n^2(1+\left(\frac{1}{n}\right)^2)} + \frac{n}{n^2(1+\left(\frac{2}{n}\right)^2)} + \ldots + \frac{n}{n^2(1+\left(\frac{n-1}{n}\right)^2)}\right) =$$

$$\left(\frac{1}{n\,(1+\left(\frac{0}{n}\right)^2)} + \frac{1}{n\,(1+\left(\frac{1}{n}\right)^2)} + \frac{1}{n\,(1+\left(\frac{2}{n}\right)^2)} + \ldots + \frac{1}{n\,(1+\left(\frac{n-1}{n}\right)^2)}\right) =$$

$$\frac{1}{n}\left(\frac{1}{(1+\left(\frac{0}{n}\right)^2)} + \frac{1}{(1+\left(\frac{1}{n}\right)^2)} + \frac{1}{(1+\left(\frac{2}{n}\right)^2)} + \ldots + \frac{1}{(1+\left(\frac{n-1}{n}\right)^2)}\right) =$$

$$\sum_{i=0}^{n-1} \frac{1}{(1+\left(\frac{i}{n}\right)^2)}\left(\frac{1}{n}\right) \quad = \quad \sum_{i=0}^{n-1} f(c_i)\,(\Delta x_i),$$

once we choose the c_i as $c_i = x_i = i/n$ and f as in the Hint. Next, we let $n \to \infty$ so that the norm of this subdivision approaches 0 and, by the results of this Chapter, the Riemann Sum approaches the definite integral

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=0}^{n-1} f(c_i)\Delta x_i &= \int_0^1 \frac{1}{1+x^2}\,dx,\\ &= \text{Arctan}\,1 - \text{Arctan}\,0 = \frac{\pi}{4}.\end{aligned}$$

29. **Method 1** First we interpret the integral $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$ (whose value is $\frac{\pi}{2}$) as the limit of a sequence of Riemann sums S_n defined as follows. For fixed n, we divide $[0,1]$ into n subintervals of length $1/n$ by $x_i \equiv i/n$ $(0 \le i \le n)$ and we take c_i to be x_i. Then the corresponding Riemann sum for $f(x) \equiv \frac{1}{\sqrt{1-x^2}}$ is

$$S_n = \sum_{i=0}^{n-1} f(c_i)(x_{i+1}-x_i) = \sum_{i=0}^{n-1} \frac{1}{\sqrt{1-(i/n)^2}}\cdot\frac{1}{n} = \sum_{i=0}^{n-1}\frac{1}{\sqrt{n^2-i^2}}.$$

For convenience, let us put $A_{n,i} = n^8 - i^2n^6 + 2ip_n - p_n^2$. It is enough to show that $S_n - \sum_{i=0}^{n-1}\frac{n^3}{\sqrt{A_{n,i}}} \to 0$ as $n \to \infty$. Now, for each n and each i,

$$\frac{1}{\sqrt{n^2-i^2}} - \frac{n^3}{\sqrt{A_{n,i}}} = \frac{\sqrt{A_{n,i}} - n^3\sqrt{n^2-i^2}}{\sqrt{n^2-i^2}\sqrt{A_{n,i}}} = \frac{A_{n,i} - n^6(n^2-i^2)}{(n^2-i^2)\sqrt{A_{n,i}} + A_{n,i}\sqrt{n^2-i^2}}.$$

The denominator is too bulky here and we have to sacrifice some terms to tidy it up. But we have to wait until the numerator is simplified:

$$A_{n,i} - n^6(n^2-i^2) = (n^8 - i^2n^6 + 2ip_n - p_n^2) - (n^8 - i^2n^6) = 2ip_n - p_n^2.$$

Now we drop every thing save $A_{n,i}$ in the denominator. Then within

$$A_{n,i} \equiv n^8 - in^6 + 2ip_n - p_n^2 = n^6(n^2-i^2) + 2ip_n - p_n^2$$

we drop the positive term $2ip_n$ and the factor $n^2 - i^2$ which is ≥ 1. (We still have to keep the burdensome $-p_n^2$ because it is negative.) Ultimately, the denominator is replaced by a smaller expression, namely $n^6 - p_n^2$. Recall that $p_n < 36n^2$ for $n \ge 2$; (see Exercise 15 in the previous **Exercise Set.**) Using this we see that

$$n^6 - p_n^2 \ge n^6 - 36n^2 = n^2(n^4-36).$$

Thus, for $n \ge 2$, $n^2(n^4-36)$ is a lower bound of the denominator. Next we get an upper bound for the numerator:

$$|2ip_n - p_n^2| \le 2ip_n + p_n^2 \le 2np_n + p_n^2 \le 2n(36n^2) + (36n^2)^2 = 72n^3 + 1296n^4.$$

Now we can put all things together:

$$\left|S_n - \sum_{i=0}^{n-1}\frac{n^3}{\sqrt{A_{n,i}}}\right| \le \sum_{i=1}^{n-1}\left|\frac{1}{\sqrt{n^2-i^2}} - \frac{n^3}{\sqrt{A_{n,i}}}\right| \le n\cdot\frac{72n^3+1296n^4}{n^2(n^4-36)}.$$

The last expression approaches to 0 as n tends to infinity. Done!

Method 2 Let $f(x) = \dfrac{1}{\sqrt{1-x^2}}$, on $[0,1)$. Let $\mathcal{P}$ denote the partition with $x_0 = 0$, and $x_i = \frac{i}{n}$, $i = 1, 2, \ldots, n$. It is clear that, as $n \to \infty$, the norm of this partition approaches 0. Next, by Sierpinski's estimate we know that

$$p_n < 36n\ \ln n.$$

But by L'Hospital's Rule, $\lim_{n\to\infty}\dfrac{\ln n}{n^3} = 0$. This means that

$$\lim_{n\to\infty}\frac{p_n}{n^4} \le \lim_{n\to\infty}\frac{36\ln n}{n^3} = 0.$$

So
$$\lim_{n\to\infty} \frac{p_n}{n^4} = 0$$
by the Sandwich Theorem of Chapter 2. Okay, now choose our interior points t_i in the interval $(x_i,\ x_{i+1})$, as follows: Let
$$t_i = \frac{i}{n} + \frac{p_n}{n^4}.$$
By what has been said, note that if n is sufficiently large, then t_i lies *indeed in this interval*. By definition of the Riemann integral it follows that this specific Riemann sum given by
$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\ \Delta x_i$$
tends to, as $n \to \infty$, the Riemann integral of f over $[0, 1)$. But
$$\begin{aligned}\lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\ \Delta x_i &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{\sqrt{1-{t_i}^2}} \cdot \frac{1}{n} \\ &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{\sqrt{1-\left(\frac{i}{n}+\frac{p_n}{n^4}\right)^2}} \cdot \frac{1}{n} \\ &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{n^3}{\sqrt{n^8 - i^2n^6 - 2ip_n - {p_n}^2}}.\end{aligned}$$
The conclusion follows since the Riemann integral of this function f exists on $[0, 1)$ and
$$\int_0^1 \frac{1}{\sqrt{1-x^2}} = \text{Arcsin}\,1 - \text{Arcsin}\,0 = \text{Arcsin}\,1 = \frac{\pi}{2}.$$

30. 2. $\displaystyle\int_0^\pi \sin x\ dx = -\cos x\Big|_0^\pi = 2.$

31. $2\sqrt{2} - 2$. Notice that when x runs from 0 to $\pi/2$, the cosine curve drops from 1 to 0 and the sine curve elevates from 0 to 1. Between 0 and $\pi/2$, the sine curve and the cosine curve meet at $x = \frac{\pi}{4}$. Hence
$$|\cos x - \sin x| = \begin{cases} \cos x - \sin x & \text{if } 0 \le x \le \pi/4, \\ \sin x - \cos x & \text{if } \pi/4 \le x \le \pi/2. \end{cases}$$
Thus the required integral is equal to
$$\begin{aligned} & \int_0^{\pi/4} (\cos x - \sin x)\ dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\ dx \\ = \ & (\sin x + \cos x)|_0^{\pi/4} + (-\cos x - \sin x)|_{\pi/4}^{\pi/2} = 2\sqrt{2} - 2.\end{aligned}$$

32. $1 - \dfrac{\sqrt{2}}{2}$. $\displaystyle\int_{\pi/12}^{\pi/8} \frac{\cos 2x}{\sin^2 2x}\ dx = \int_{\pi/6}^{\pi/4} \csc 2x \cot 2x\ dx$
$\displaystyle = -\frac{1}{2}\csc 2x\Big|_{\pi/12}^{\pi/8} = 1 - \frac{\sqrt{2}}{2}.$

33. $\dfrac{4}{9}\sqrt{2} - \dfrac{2}{9}$. $\displaystyle\int_0^1 t^2\sqrt{1+t^3}\ dt = \frac{1}{3}\cdot\frac{(1+t^3)^{3/2}}{3/2}\Big|_0^1 = \frac{4}{9}\sqrt{2} - \frac{2}{9}.$ Use Table 7.5, $\square\ = 1 + t^3, r = 1/2$.

34. $\displaystyle\int_0^1 \frac{x}{1+x^4}\ dx = \frac{1}{2}\text{Arctan}\ x^2\Big|_0^1 = \frac{1}{2}\left(\text{Arctan}\ 1 - \text{Arctan}\ 0\right) = \frac{\pi}{8}.$

35. $\displaystyle\frac{d}{dx}\int_1^{x^2} \frac{\sin t}{t^{3/2}} dt = \frac{\sin(x^2)}{x^3}\cdot 2x = \frac{2\sin(x^2)}{x^2} \longrightarrow 2 \quad \text{as } x \to 0+.$

36. As $x \to \infty$, we have

$$\frac{d}{dx}\int_{\sqrt{3}}^{\sqrt{x}} \frac{r}{(r+1)(r-1)}dr = \frac{x^{3/2}}{(x^{1/2}+1)(x^{1/2}-1)} \cdot \frac{1}{2\sqrt{x}} = \frac{x}{2(x-1)} \longrightarrow \frac{1}{2}.$$

37. $\frac{d}{dx}\int_0^{x^2} e^{-t^2}\, dt = 2xe^{-x^2} = 2\frac{x}{e^{x^2}}$ which is of indefinite form $\frac{\infty}{\infty}$ when $x \to \infty$. By L'Hospital's rule and the fact that $e^x \to +\infty$ as $x \to +\infty$ we see that $2\frac{x}{e^{x^2}} \to 0$ as $x \to \infty$.

38. $\frac{d}{dx}\int_1^{\sqrt{x}} \frac{\sin(y^2)}{2y}\, dy = \frac{\sin(\sqrt{x}^2)}{2\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{\sin x}{4x} \longrightarrow \frac{1}{4} \quad$ as $x \to 0$.

39.

$$\begin{aligned}
& \lim_{x\to 0^+} \frac{d}{dx}\int_1^{\sin x} \frac{\ln t}{\ln(\text{Arcsin } t)}\, dt \\
= & \lim_{x\to 0^+} \left(\frac{\ln(\sin x)}{\ln(\text{Arcsin } (\sin x))} \cdot \cos x - 0\right), \\
= & \lim_{x\to 0^+} \left(\frac{\ln(\sin x)}{\ln x} \cdot \cos x\right) \\
= & \lim_{x\to 0^+} \left(\frac{\ln(\sin x)}{\ln x}\right) \cdot \cos 0 \\
= & \lim_{x\to 0^+} \left(\frac{\cot x}{1/x}\right) \\
= & \lim_{x\to 0^+} x \ \cot x = \lim_{x\to 0^+} \frac{x}{\sin x} \cos x = (1)(1) = 1.
\end{aligned}$$

40. Indeed, as $t \to 0$,

$$\frac{d}{dt}\int_{2\pi-ct}^{2\pi+ct} \frac{\sin x}{cx}\, dx = \frac{\sin(2\pi+ct)}{c(2\pi+ct)} \cdot c - \frac{\sin(2\pi-ct)}{c(2\pi-ct)}(-c) \longrightarrow \frac{\sin 2\pi}{\pi} = 0.$$

41.

$$\begin{aligned}
& \lim_{h\to 0+} \frac{d}{dx}\left(\frac{1}{h}\int_{x-h}^{x+h} \sqrt{t}\, dt\right) = \lim_{h\to 0+} \frac{\sqrt{x+h}\ (1) - \sqrt{x-h}\ (1)}{h}, \\
= & \lim_{h\to 0+} \frac{\sqrt{x+h} - \sqrt{x-h}}{h} = \lim_{h\to 0+} \frac{2h}{h(\sqrt{x+h} + \sqrt{x-h})}, \\
= & \lim_{h\to 0+} \frac{2}{\sqrt{x+h} + \sqrt{x-h}} = \frac{1}{\sqrt{x}}.
\end{aligned}$$

42. $\lim_{x\to 0} \frac{1}{2x}\int_{-x}^{x} \cos t\ dt = \lim_{x\to 0} \frac{1}{2x}(\sin x - \sin(-x)) = \lim_{x\to 0} \frac{2\sin x}{2x} = 1$. [Remark: Actually, for every continuous function f defined on the real line, we have

$$\lim_{x\to 0+} \frac{1}{2x}\int_{-x}^{x} f(t)\ dt = f(0).$$

Do you know why?]

43. $\frac{y^5}{5} = \frac{x^4}{4} + \frac{1}{5}$.

44. $\sin(y(x)) + \cos x = C$ is the most general antiderivative. But $y = \pi/2$ when $x = 0$. This means that $\sin(\pi/2) + \cos 0 = C$, or $C = 2$. So, the solution in implicit form is given by $\sin(y(x)) + \cos x = 2$.

45. $y = \tan\left[\frac{1}{2}(e^{2x} - 1) + \frac{\pi}{4}\right]$.

46. $y = 2x^4 + \frac{4}{3}x^3 + x$.

47. $y(x) = C_1 + C_2\,x + C_3\,x^2 - x^4$ is the most general antiderivative. Now, the initial conditions $y(0) = 0, y'(0) = 0, y''(0) = -1$ imply that $C_1 = 0, C_2 = 0, C_3 = -1/2$. The required solution is given by

$$y(x) = -\frac{1}{2}\,x^2 - x^4.$$

48. $y = e^x - x - 1$.

49. $y = \dfrac{x^4}{12} + \dfrac{x^3}{3}$.

50. Since marginal cost $= \frac{dC}{dx} = 60 + \frac{40}{x+10}$,

(a) total increase in cost as x goes from 20 to 40 is

$$\int_{20}^{40} \left[60 + \frac{40}{x+10}\right]\; dx = [60x + 40ln|x+40|\;]_{20}^{40}$$

$= 60 \times 40 + 40\; ln(50) - [60 \times 20 + 40ln(30)] = 1200 + 40(ln\; 50 - ln\; 30) = 1200 + 40ln(5/3) = \$1220.43$

(b) Let $I(t)$ be value of investment at time t, t in years. $\frac{dI}{dt} = (500e^{\sqrt{t}})/\sqrt{t}$, thus

$$I(t) = \int \frac{500e^{\sqrt{t}}}{\sqrt{t}} dt = 500e^{\sqrt{t}} + C.$$

When $t = 0$, $I = 1000$, so $1000 = 500 + C$, and $C = 500$. Therefore, at $t = 4, \quad I = 500e^2 + 500 = \$4194.53$.

Chapter 8

Solutions

8.1

8.2 Exercise Set 37

1. $\frac{1}{200}(2x-1)^{100}+C.$
2. $3\cdot\dfrac{(x+1)^{6.1}}{6.1}+C.$
3. $I=\displaystyle\int_0^1(3x+1)^{-5}dx=\frac{1}{3}\cdot\frac{(3x+1)^{-4}}{-4}\bigg|_0^1\approx 0.0830.$
4. $I=\displaystyle\int(x-1)^{-2}dx=-(x-1)^{-1}+C=\frac{1}{1-x}+C.$
5. $-\dfrac{1}{202}(1-x^2)^{101}+C=\dfrac{1}{202}(x^2-1)^{101}+C.$
6. $\dfrac{1}{\ln 2}2^{x^2-1}+C.$ Let $u=x^2,\ du=2x\ dx$, etc.
7. $\displaystyle\int_0^{\pi/4}\tan x\ dx=-\ln|\cos x|=\ln|\sec x|\bigg|_0^{\pi/4}=\ln\sqrt{2}-\ln 1=\frac{\ln 2}{2}.$
8. $\dfrac{1}{3}e^{z^3}+C.$
9. $-\dfrac{3}{4}(2-x)^{4/3}+C.$
10. $\frac{1}{2}\sin 8\approx 0.49468.$
11. $I=\displaystyle\int\frac{1}{1+\sin t}\cdot\frac{d(1+\sin t)}{dt}\ dt=\ln|1+\sin t|+C.$
12. $-\sqrt{1-x^2}+C.$
13. $\frac{1}{2}\ln|y^2+2y|+C.$ Let $u=y^2+2y,\ du=2(y+1)\ dy$, etc.
14. $I=\displaystyle\int\frac{\sec^2 x\ dx}{\sqrt{1+\tan x}}=\int\frac{\left(\frac{d}{dx}\tan x\right)\ dx}{\sqrt{1+\tan x}}=2\sqrt{1+\tan x}+C.$
15. $I=-\displaystyle\int_0^{\pi/4}\frac{1}{\cos^2 x}\cdot\frac{d\cos x}{dx}\ dx=\frac{1}{\cos x}\bigg|_0^{\pi/4}=\sqrt{2}-1.$ Alternatively,

$$I=\int_0^{\pi/4}\tan x\sec x\ dx=\sec x\bigg|_0^{\pi/4}=\sqrt{2}-1.$$

16. **Hard! Very hard!** The function $\sec x+\tan x$ in the hint seems to be extremely tricky and unthinkable; see Example 364 in §8.5.2 for manipulating this integral according to the hint. Here is a slightly more natural way (although just as unthinkable): Try to put everything in terms of sines or cosines. Let's begin. Don't feel bad if you find this still too slick for you.

$$\begin{aligned}\int \sec x\, dx &= \int \frac{1}{\cos x}\, dx = \int \frac{\cos x}{\cos^2 x}\, dx = \int \frac{1}{\cos^2 x}\cdot\frac{d\sin x}{dx}\, dx \\ &= \int \frac{1}{1-\sin^2 x}\cdot\frac{d\sin x}{dx}\, dx = \int \frac{1}{1-u^2}du \quad (u=\sin x) \\ &= \int \frac{1}{(1-u)(1+u)}\, du = \int \frac{1}{2}\left[\frac{1}{1-u}+\frac{1}{1+u}\right] du \\ &= \frac{1}{2}\left[-\ln|1-u|+\ln|1+u|\right]+C = \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|+C \\ &= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|+C \\ &= \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)}\right|+C \\ &= \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{\cos^2 x}\right|+C \\ &= \ln\left|\frac{1+\sin x}{\cos x}\right|+C = \ln|\sec x+\tan x|+C.\end{aligned}$$

17. One way to do this is to multiply out everything and then integrate term by term. But this way is very messy! Observe that $4z^3+1$ is nothing but the derivative of z^4+z. So we have an easy way out:

$$I = \int (z^4+z)^4\cdot\frac{d}{dz}(z^4+z)\, dz = \frac{1}{5}(z^4+z)^5+C.$$

18. $-\text{Arctan}(\cos x)+C$. Let $u=\cos x$, $du=-\sin x\, dx$, etc.

19. $I=\frac{1}{2}\,\text{Arctan}(t^2)\Big|_0^1 = \frac{\pi}{8}$.

20. $\frac{1}{8}\sin^4(x^2+1)+C$. Let $u=x^2+1$ first, then $v=\sin u$ as the next substitution.

21. $\frac{3}{2}\ln(x^2+1)-\text{Arctan } x+C$. (Since x^2+1 is always positive, there is no need to put an absolute value sign around it.)

22. $I=\int_e^{e^2}\frac{1}{\ln x}\cdot\frac{d\ln x}{dx}\, dx = \ln(\ln x)\Big|_e^{e^2} = \ln 2-\ln 1=\ln 2.$

23. $\frac{1}{3}(\text{Arctan } x)^3+C$.

24. $I=\int\cosh(e^t)\cdot e^t\, dt=\sinh(e^t)+C$. (Recall that $D\sinh\Box=\cosh\Box\ D\Box$ and $D\cosh\Box=\sinh\Box\ D\Box$.)

25. $\frac{1}{5}\text{Arcsin } 5s+C$.

26. $I=\int_{\pi^2}^{4\pi^2}\cos\sqrt{x}\cdot 2\frac{d\sqrt{x}}{dx}\, dx = 2\sin\sqrt{x}\Big|_{\pi^2}^{4\pi^2} = 2(\sin 2\pi-\sin\pi)=0.$

27. $\frac{1}{2}e^{x^2}+C$.

28. $-\sqrt{1-y^2}+\text{Arcsin } y+C$. Split this integral up into two pieces and let $u=1-y^2$, etc.

29. $\sec(\ln x)+C$. Let $u=\ln x$, etc.

30. $I=\int \sin^{-2/3}x\cdot\frac{d}{dx}\sin x\, dx = 3\sin^{1/3}x+C.$

31. $I = \int_0^1 e^{e^t} \cdot \frac{de^t}{dt}\ dt = e^{e^t}\Big|_0^1 = e^e - e.$

32. $\frac{1}{2\ln(1.5)} 1.5^{x^2+1} + C = 1.23316\, 1.5^{x^2+1} + C.$

8.3 Exercise Set 38

1. Using the normal method, we have:

$$I = \int x\frac{d}{dx}\sin x\ dx = x\sin x - \int \sin x\ dx = x\sin x + \cos x + C.$$

2. $-x\cos x + \sin x + C.$

3. $-1/2.$

4. Using the normal method, we have:

$$\begin{aligned}\int x^2\sin x\ dx &= \int x^2\frac{d}{dx}(-\cos x)dx \\ &= -x^2\cos x + \int 2x\cdot\cos x\ dx \\ &= -x^2\cos x + \int 2x\frac{d}{dx}\sin x\ dx \\ &= -x^2\cos x + 2x\sin x - \int 2\sin x\ dx \\ &= -x^2\cos x + 2x\sin x + 2\cos x + C.\end{aligned}$$

Now you can see the advantage of the Table method over the above normal method: you don't have to copy down some expressions several times and the minus signs are no longer a worry!

5. $x\tan x + \ln|\cos x| + C.$

6. $x\sec x - \ln|\sec x + \tan x| + C.$ (Here you have to recall the answer to a very tricky integral: $\int \sec x\ dx = \ln|\sec x + \tan x|$. See Exercise Set 37, Number 16.)

7. $(x^2 - 2x + 2)e^x + C.$

8. $I = -\frac{1}{3}x^2e^{-3x} + \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x}\Big|_0^\infty = \frac{2}{27}.$ Notice that here we have used the fact $p(x)e^{-3x} \longrightarrow 0$ as $x \to +\infty$, where $p(x)$ is any polynomial, that is, the exponential growth is faster than the polynomial growth. Alternately, use L'Hospital's Rule for each limit except for the last one.

9. $\frac{1}{5}x^5\ln x - \frac{1}{25}x^5 + C.$

10. $-\frac{1}{3}\left(x^3 + x^2 + \frac{2}{3}x + \frac{2}{9}\right)e^{-3x} + C.$

11. $x\sin^{-1}x + \sqrt{1-x^2} + C.$

12. $x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C.$

13. Let $u = \ln x$. Then $x = e^u$ and $dx = e^u du$. Thus the integral can be converted to $\int u^5e^{2u}e^u\ du = \int u^5e^{3u}\ du$. Using the Table method to evaluate the last integral, we have

$$\int u^5e^{3u}\ du = e^{3u}\left(\frac{1}{3}u^5 - \frac{5}{9}u^4 + \frac{20}{27}u^3 - \frac{20}{27}u^2 + \frac{40}{81}u - \frac{40}{243}\right) + C.$$

Substituting $u = \ln x$ back, we get the answer to the original integral $\int x^2(\ln x)^5\ dx$:

$$x^3\left(\frac{1}{3}(\ln x)^5 - \frac{5}{9}(\ln x)^4 + \frac{20}{27}(\ln x)^3 - \frac{20}{27}(\ln x)^2 + \frac{40}{81}\ln x - \frac{40}{243}\right) + C.$$

14. $\dfrac{x^2}{2}\sec^{-1}x - \dfrac{1}{2}\sqrt{x^2-1} + C$, if $x > 0$.
15. Use the Table method for this problem.

$$\int (x-1)^2 \sin x \, dx = -(x-1)^2 \cos x + 2(x-1)\sin x + 2\cos x + C.$$

16. $-\dfrac{1}{13}(2\sin 3x + 3\cos 3x)e^{-2x} + C$.
17. $\dfrac{1}{17}(\cos 4x + 4\sin 4x)e^{x} + C$.
18. $-\dfrac{1}{10}\cos 5x - \dfrac{1}{2}\cos x + C$, or $-\dfrac{1}{5}(2\sin 3x \sin 2x + 3\cos 3x \cos 2x) + C$.
Use the identity $\sin A \cos B = \dfrac{1}{2}(\sin(A+B) - \sin(A-B))$ with $A = 3x$ and $B = 2x$ and integrate. Alternately, this is also a **three-row problem**: This gives the second equivalent answer.
19. $-\dfrac{1}{12}\cos 6x + \dfrac{1}{4}\cos 2x + C$, or $-\dfrac{1}{3}\cos^3 2x + \dfrac{1}{2}\cos 2x + C$, or $\dfrac{1}{12}(4\sin 2x \sin 4x + 2\cos 2x \cos 4x) + C$. This is a **three-row problem** as well. See the preceding exercise.
20. $\dfrac{1}{14}\sin 7x + \dfrac{1}{2}\sin x + C$, or $\dfrac{1}{7}(4\cos 3x \sin 4x - 3\sin 3x \cos 4x) + C$. Use the identity
$\cos A \cos B = \dfrac{1}{2}(\cos(A+B) + \cos(A-B))$ with $A = 4x$ and $B = 3x$ and integrate. Alternately, this is also a **three-row problem**: This gives the second equivalent answer.
21. $e^{2x}\left(\dfrac{1}{2}x^5 - \dfrac{5}{4}x^4 + \dfrac{5}{2}x^3 - \dfrac{15}{4}x^2 + \dfrac{15}{4}x - \dfrac{15}{8}\right) + C$. For this exercise you really should use the Table method, otherwise you will find the amount of work overwhelming!
22. $\dfrac{x}{2}(\cos \ln x + \sin \ln x) + C$. See Example 351.

8.4

8.4.1 Exercise Set 39

1. $x - 3 + \dfrac{4}{x+1}$
2. $2 - \dfrac{3x^2+x+3}{x^3+2x+1}$
3. $\dfrac{1}{3}\left(x^2 - \dfrac{2}{3} + \dfrac{7/3}{3x^2-1}\right)$
4. $x^2 - 1 + \dfrac{2}{x^2+1}$
5. $x^4 + x^3 + 2x^2 + 2x + 2 + \dfrac{3}{x-1}$
6. $\dfrac{3}{2}\left(x - 1 + \dfrac{13x+15}{6x^2+6x+3}\right)$

8.4.2 Exercise Set 40

1. $\displaystyle\int \frac{x}{x-1}\,dx = \int \left(1 + \frac{1}{x-1}\right)\,dx = x + \ln|x-1| + C.$

2. $\displaystyle\int \frac{x+1}{x}\,dx = \int\left(1+\frac{1}{x}\right)\,dx = x + \ln|x| + C.$
3. $\displaystyle\int \frac{x^2\,dx}{x+2} = \int\left(x-2+\frac{4}{x+2}\right)\,dx = \frac{x^2}{2} - 2x + 4\ln|x+2| + C.$
4. $\displaystyle\int \frac{x^2\,dx}{x^2+1} = \int\left(1-\frac{1}{x^2+1}\right)\,dx = x - \text{Arctan}\,x + C.$
5. Since the denominator and the numerator have the same degree, we have to perform the long division first

$$\begin{aligned} I &\equiv \int \frac{x^2}{(x-1)(x+1)}\,dx = \int \frac{x^2}{x^2-1}\,dx = \int\left(1+\frac{1}{x^2-1}\right)\,dx \\ &= \int\left(1+\frac{1}{(x+1)(x-1)}\right)\,dx = \int\left(1+\frac{1}{2}\cdot\frac{1}{x-1}-\frac{1}{2}\cdot\frac{1}{x+1}\right)\,dx \\ &= x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C. \end{aligned}$$

6. Put $\dfrac{2x}{(x-1)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-3}$. Then $2x = A(x-3) + B(x-1)$. Setting $x = 1$ we have $A = -1$ and setting $x = 3$ we have $B = 3$. Thus the required integral is

$$\int\left(\frac{-1}{x-1} + \frac{3}{x-3}\right)\,dx = 3\ln|x-3| - \ln|x-1| + C.$$

7. Put $\dfrac{3x^2}{(x-1)(x-2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$. Then

$$3x^2 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

Setting $x = 1,\ 2,\ 3$ respectively, we have $A = 3/2$, $B = -12$ and $C = 27/2$. Thus

$$\int \frac{3x^2\,dx}{(x-1)(x-2)(x-3)} = \frac{3}{2}\ln|x-1| - 12\ln|x-2| + \frac{27}{2}\ln|x-3| + C.$$

8. We start with long division:

$$\begin{aligned} I &\equiv \int_0^1 \frac{x^3-1}{x+1}dx = \int_0^1 \left(x^2 - x + 1 - \frac{2}{x+1}\right)\,dx \\ &= \frac{x^3}{3} - \frac{x^2}{2} + x - 2\ln|x+1|\Big|_0^1 = \frac{1}{3} - \frac{1}{2} + 1 - 2\ln 2 - 0 = \frac{5}{6} - \ln 4. \end{aligned}$$

9. Here we perform a small trick on the numerator of the integrand:

$$\begin{aligned} \int \frac{3x}{(x-1)^2}\,dx &= \int \frac{3(x-1)+3}{(x-1)^2}\,dx \\ &= \int \frac{3}{(x-1)^2}\,dx + \int \frac{3}{x-1}\,dx \\ &= 3(1-x)^{-1} + 3\ln|x-1| + C. \end{aligned}$$

10. Put

$$\frac{2x-1}{(x-2)^2(x+1)} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

Then $2x-1 = A(x-2)^2 + B(x-2)(x+1) + C(x+1)$. Setting $x = -1$, we have $-3 = A(-3)^2$ and hence $A = -1/3$. Setting $x = 2$, we have $3 = 3C$; so $C = 1$. Comparing the coefficients of x^2 on both sides, we get $0 = A + B$, which gives $B = -A = 1/3$. Thus

$$\begin{aligned} \int \frac{2x\,dx}{(x-2)^2(x+1)} &= \int\left(-\frac{1}{3}\cdot\frac{1}{x+1} + \frac{1}{3}\cdot\frac{1}{x-2} + \frac{1}{(x-2)^2}\right)\,dx \\ &= \frac{1}{2-x} + \frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C. \end{aligned}$$

11. By long division, we get $\dfrac{x^4+1}{x^2+1} = x^2 - 1 + \dfrac{2}{x^2+1}$. So

$$\int \frac{x^4+1}{x^2+1}\,dx = \frac{x^3}{3} - x + 2\,\text{Arctan}\,x + C.$$

12. Putting $u = x^2$, the integrand becomes $\dfrac{1}{(u+1)(u+4)} = \dfrac{1}{3}\cdot\dfrac{1}{u+1} - \dfrac{1}{3}\dfrac{1}{u+4}$. So

$$\begin{aligned}\int \frac{dx}{(x^2+1)(x^2+4)} &= \frac{1}{3}\int\frac{dx}{x^2+1} - \frac{1}{3}\int\frac{dx}{x^2+4}\\ &= \frac{1}{3}\,\text{Arctan}\,x - \frac{1}{6}\,\text{Arctan}\,\frac{x}{2} + C.\end{aligned}$$

13. Put

$$\frac{1}{x^2(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+2}.$$

The we have

$$1 = Ax(x-1)(x+2) + B(x-1)(x+2) + Cx^2(x+2) + Dx^2(x-1).$$

Setting $x = 1,\ 0,\ -2$ respectively, we have $C = 1/3$, $B = -1/2$ and $D = -1/12$. Comparing coefficients of x^3 on both sides, we have $0 = A + C + D$, or $A + \frac{1}{3} - \frac{1}{12} = 0$ and hence $A = -\frac{1}{4}$. Thus

$$\begin{aligned}\int\frac{dx}{x^2(x-1)(x+2)} &= -\frac{1}{4}\int\frac{dx}{x} - \frac{1}{2}\int\frac{dx}{x^2} + \frac{1}{3}\int\frac{dx}{x-1} - \frac{1}{12}\int\frac{dx}{x+2}\\ = \quad & -\frac{1}{4}\ln|x| + \frac{1}{2x} + \frac{1}{3}\ln|x-1| - \frac{1}{12}\ln|x+2| + C.\end{aligned}$$

14. Put

$$\frac{x^5+1}{x(x-2)(x-1)(x+1)(x^2+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-1} + \frac{D}{x+1} + \frac{Ex+F}{x^2+1}.$$

Using the method of "covering" described in this section, we get $A = 1/2$, $B = 11/10$, $C = -1/2$ and $D = 0$. By using the "plug-in method" described in the present section we have $E = -\frac{1}{10}$ and $F = \frac{3}{10}$. Thus the partial fraction decomposition for the integrand is

$$\frac{1}{2}\cdot\frac{1}{x} + \frac{11}{10}\cdot\frac{1}{x-2} - \frac{1}{2}\cdot\frac{1}{x-1} - \frac{1}{10}\cdot\frac{x}{x^2+1} + \frac{3}{10}\cdot\frac{1}{x^2+1}.$$

Thus the required integral is

$$\frac{1}{2}\ln|x| + \frac{11}{10}\ln|x-2| - \frac{1}{2}\ln|x-1| - \frac{1}{20}\ln(x^2+1) + \frac{3}{10}\text{Arctan}\,x + C.$$

15. Putting

$$\frac{2}{x(x-1)^2(x^2+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{Dx+E}{x^2+1},$$

we have

$$2 = A(x-1)^2(x^2+1) + Bx(x-1)(x^2+1) + Cx(x^2+1) + (Dx+E)x(x-1)^2.$$

Setting $x = 0$, we obtain $A = 2$. Setting $x = 1$, we get $C = 1$. Next we set $x = 2$. This gives us an identity relating the unknowns from A to E. Substituting $A = 2$ and $C = 1$ in this identity and then simplifying, we get a relation

$$5B + 2D + E = -9$$

between B, D and E. Setting $x = 3$ will give us another such a relation:

$$5B + 3D + E = -9.$$

From these two relations we can deduce that $D = 0$ and $5B + E = -9$. Finally, setting $x = -1$ will give us yet another relation among B, D and E:

$$B + D - E = -3.$$

Now it is not hard to solve for B and E: $B = -2$, $E = 1$. (Remark: if you are familiar with complex numbers, you can find D and E efficiently by setting $x = i$ to arrive at $2 = (Di + E)i(i-1)^2$, which gives $Di + E = 1$ and hence $D = 0$ and $E = 1$, in view of the fact that D and E are real numbers.) We conclude

$$\frac{2}{x(x-1)^2(x^2+1)} = \frac{2}{x} - \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x^2+1}.$$

So the required integral is equal to

$$\int \frac{2\;dx}{x(x-1)^2(x^2+1)} = 2\ln|x| - 2\ln|x-1| - \frac{1}{x-1} + \text{Arctan } x + C.$$

8.5

8.5.1 Exercise Set 41

1. Let $u = \cos 3x$ so that $du = -3 \sin 3x\ dx$ and $\sin^2 3x = 1 - u^2$.

$$\begin{aligned} \int \sin^3 3x\ dx &= \int \sin^2 3x \cdot \sin 3x\ dx \\ &= \int (1-u^2)\cdot(-1/3)du = -\frac{u}{3} + \frac{u^3}{9} + C \\ &= -\frac{\cos 3x}{3} + \frac{\cos^3 3x}{9} + C. \end{aligned}$$

2. Let $u = \sin(2x-1)$ so that $du = 2\cos(2x-1)\ dx$ and $\cos^2(2x-1) = 1 - u^2$.

$$\begin{aligned} \int \cos^3(2x-1)\ dx &= \int \cos^2(2x-1)\cdot\cos(2x-1)\ dx \\ &= \int (1-u^2)\cdot\frac{1}{2}du = \frac{u}{2} - \frac{u^3}{6} + C \\ &= \frac{\sin(2x-1)}{2} - \frac{\sin^3(2x-1)}{6} + C. \end{aligned}$$

3. Let $u = \sin x$ so that $du = \cos x\ dx$. Notice that $x = 0 \Rightarrow u = 0$ and $x = \frac{\pi}{2} \Rightarrow u = 1$. Thus

$$\int_0^{\pi/2} \sin^2 x \cos^3 x\ dx = \int_0^1 u^2(1-u^2)du = \left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Bigg|_0^1 = \frac{2}{15}.$$

4. Let $u = \cos(x-2)$. Then $du = -\sin(x-2)\ dx$ and

$$\begin{aligned} \int \cos^2(x-2)\sin^3(x-2)\ dx &= \int u^2(1-u^2)(-du) = -\frac{u^3}{3} + \frac{u^5}{5} + C \\ &= -\frac{1}{3}\cos^3(x-2) + \frac{1}{5}\cos^5(x-2) + C. \end{aligned}$$

5. Let $u = \sin x$. Then $du = \cos x\ dx$. Also, $x = \pi/2 \Rightarrow u = 1$ and $x = \pi \Rightarrow u = 0$. So

$$\int_{\pi/2}^{\pi} \sin^3 x \cos x\ dx = \int_1^0 u^3\ du = \frac{u^4}{4}\Bigg|_1^0 = -\frac{1}{4}.$$

The negative value in the answer is acceptable because $\cos x$ is negative when $\pi/2 < x < \pi$.

6. Set $u = x^2$. Then $du = 2x\ dx$. So

$$\begin{aligned} \int x \sin^2(x^2)\cos^2(x^2)\ dx &= \frac{1}{2}\int \sin^2 u \cos^2 u\ du = \frac{1}{8}\int \sin^2 2u\ du \\ &= \frac{1}{8}\int \left(\frac{1-\cos 4u}{2}\right)\ du = \frac{u}{16} - \frac{\sin 4u}{64} + C \\ &= \frac{x^2}{16} - \frac{\sin(4x^2)}{64} + C. \end{aligned}$$

7. We use the “double angle” formulae several times:

$$\begin{aligned} \int \sin^4 x \cos^4 x\ dx &= \frac{1}{16}\int \sin^4 2x\ dx = \frac{1}{16}\int \sin^2 2x\ \sin^2 2x\ dx \\ &= \frac{1}{16}\int \left(\frac{1-\cos 4x}{2}\right)^2\ dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{64}\int (1 - 2\cos 4x + \cos^2 4x)\ dx \\
&= \frac{1}{64}x - \frac{1}{128}\sin 4x + \frac{1}{64}\int \frac{1+\cos 8x}{2}\ dx \\
&= \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C.
\end{aligned}
$$

8. Let $u = \sin x$. Then $du = \cos x\ dx$ and $\cos^2 x = 1 - u^2$. So

$$
\begin{aligned}
\int \sin^4 x \cos^5 x\ dx &= \int u^4(1-u^2)^2\ du = \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C \\
&= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C.
\end{aligned}
$$

9. Use the "double angle formula" twice:

$$
\begin{aligned}
\int \cos^4 2x\ dx &= \int \left(\frac{1+\cos 4x}{2}\right)^2 dx \\
&= \frac{1}{4}\int (1 + 2\cos 4x + \cos^2 4x)\ dx \\
&= \frac{x}{4} + \frac{\sin 4x}{8} + \frac{1}{4}\int \frac{1+\cos 8x}{2}\ dx \\
&= \frac{3x}{8} + \frac{\sin 4x}{8} + \frac{\sin 8x}{64} + C.
\end{aligned}
$$

10. Let $u = \sin x$. Then $du = \cos x\ dx$ and $\cos^2 x = 1 - u^2$. So

$$
\int \sin^5 x \cos^3 x\ dx = \int u^5(1-u^2)du = \frac{u^6}{6} - \frac{u^8}{8} + C = \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C.
$$

11. Set $u = \cos x$. Then $du = -\sin x\ dx$ and $\sin^2 x = 1 - u^2$. So

$$
\begin{aligned}
\int \sin^5 x \cos^4 x\ dx &= \int \sin^4 x \cos^4 x \cdot \sin x\ dx = \int (1-u^2)^2 u^4\ (-du) \\
&= \int (-u^4 + 2u^6 - u^8)\ du \\
&= -\frac{1}{5}u^5 + \frac{2}{7}u^7 - \frac{1}{9}u^9 + C \\
&= -\frac{1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + C.
\end{aligned}
$$

12. We use the "double angle formula" several times.

$$
\begin{aligned}
\int \sin^6 x\ dx &= \int \left(\frac{1-\cos 2x}{2}\right)^3 dx \\
&= \frac{1}{8}\int (1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x)\ dx \\
&= \frac{x}{8} - \frac{3}{16}\sin 2x + \frac{3}{8}\int \frac{1+\cos 4x}{2}\ dx - \\
&\quad \frac{1}{8}\int (1 - \sin^2 2x)\cdot \frac{d}{dx}\left(\frac{\sin 2x}{2}\right)\ dx \\
&= \frac{5}{16}x - \frac{1}{4}\sin 2x + \frac{1}{48}\sin^3 2x + \frac{3}{64}\sin 4x + C.
\end{aligned}
$$

13. Let $u = \sin x$. Then $du = \cos x\ dx$ and $\cos^6 x = (1 - \sin^2 x)^3 = (1-u^2)^3$. So

$$
\begin{aligned}
\int \cos^7 x\ dx &= \int (1-u^2)^3\ du = \int (1 - 3u^2 + 3u^4 - u^6)\ du \\
&= u - u^3 + \frac{3}{5}u^5 - \frac{1}{7}u^7 + C \\
&= \sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C.
\end{aligned}
$$

8.5.2 Exercise Set 42

1. $-\ln|\cos x| + C = \ln|\sec x| + C$. Let $u = \cos x,\ du = -\sin x\ dx$.
2. $\dfrac{1}{3}\tan(3x+1) + C$. Let $u = 3x + 1$.
3. $\sec x + C$, since this function's derivative is $\sec x\ \tan x$.
4. $\dfrac{\tan^2 x}{2} + C$. Let $u = \tan x,\ du = \sec^2 x\ dx$.
5. $\dfrac{\tan^3 x}{3} + C$. Let $u = \tan x,\ du = \sec^2 x\ dx$.
6. $\dfrac{\tan^6 x}{6} + C$. Let $u = \tan x,\ du = \sec^2 x\ dx$.
7. $\dfrac{\sec^3 x}{3} - \sec x + C$. Case m, n both ODD. Use (8.59) then let $u = \sec x,\ du = \sec x\ \tan x\ dx$.
8. $\dfrac{\tan^5 x}{5} + \dfrac{\tan^7 x}{7} + C$. Case m, n both EVEN. Solve for one copy of $\sec^2 x$ then let $u = \tan x,\ du = \sec^2 x\ dx$, in the remaining.
9. $\dfrac{\sec^5 x}{5} - \dfrac{\sec^3 x}{3} + C$. Case m, n both ODD. Factor out one copy of $\sec x\ \tan x$, use (8.59), then let $u = \sec x,\ du = \sec x\ \tan x\ dx$ in the remaining.
10. $\dfrac{\sec^7 2x}{14} - \dfrac{\sec^5 2x}{5} + \dfrac{\sec^3 2x}{6} + C$. Let $u = 2x$ and use Example 394.
11. $\dfrac{\tan^6 2x}{12} + C$. Let $u = 2x,\ du = 2\ dx$, and use Exercise 6, above or, more directly, let $v = \tan 2x,\ dv = 2\sec^2 2x\ dx$.
12. $\dfrac{\tan^2 x}{2} + \ln|\cos x|$. Solve for $\tan^2 x$ in (8.59), break up the integral into two parts, use the result in Exercise 1 for the first integral, and let $u = \tan x$ in the second integral.
13. $\dfrac{1}{6}\sec^5 x\ \tan x + \dfrac{5}{24}\sec^3 x\ \tan x + \dfrac{5}{16}(\sec x\ \tan x + \ln|\sec x + \tan x|)$. Use Example 392 with $k = 7$, and then apply Example 395.
14. See Example 388.
15. $\dfrac{1}{4}\sec^3 x\ \tan x - \dfrac{1}{8}(\sec x\ \tan x + \ln|\sec x + \tan x|)$. The case where m is ODD and n is EVEN. Solve for $\tan^2 x$ and use Example 392 with $k = 5$ along with Example 387.

8.6

8.6.1 Exercise Set 43

1. $\displaystyle\int_0^1 \frac{1}{1+x^2}\ dx = \text{Arctan } x\Big|_0^1 = \frac{\pi}{4}$.
2. $\displaystyle\int \frac{2\ dx}{x^2 - 2x + 2} = \int \frac{2\ dx}{(x-1)^2 + 1} = 2\,\text{Arctan }(x-1) + C$.
3. $\displaystyle I = \int \frac{dx}{(x-1)^2 + 4} = \frac{1}{2}\text{Arctan } \frac{x-1}{2} + C$.
4. There is no need to complete a square:

$$\int \frac{dx}{x^2 - 4x + 3} \quad = \quad \int \frac{dx}{(x-1)(x-3)}$$

$$\begin{aligned} &= \int \frac{1}{2}\left(\frac{1}{x-3} - \frac{1}{x-1}\right)\, dx \\ &= \frac{1}{2}\ln|x-3| - \frac{1}{2}\ln|x-1| + C. \end{aligned}$$

5. We need to complete the square in the denominator of the integrand:

$$\begin{aligned} \int \frac{4}{4x^2+4x+5}\, dx &= \int \frac{4\ dx}{(2x+1)^2+4} \\ &= \int \frac{dx}{\left(x-\frac{1}{2}\right)^2+1} = \text{Arctan}\left(x+\frac{1}{2}\right) + C. \end{aligned}$$

6. The minus sign in front of x^2 should be taken out first.

$$\begin{aligned} \int \frac{dx}{4x-x^2-3} &= -\int \frac{dx}{(x-1)(x-3)} \\ &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x-3| + C. \end{aligned}$$

(For the last step, see the answer to Exercise 4 above.)

7. We have

$$\begin{aligned} \int \frac{1}{\sqrt{4x-x^2}}\, dx &= \int \frac{dx}{\sqrt{-(x^2-4x+4-4)}} = \int \frac{dx}{\sqrt{4-(x-2)^2}} \\ &= \frac{1}{2}\int \frac{dx}{\sqrt{1-\left(\frac{x-2}{2}\right)^2}} = \text{Arcsin}\ \frac{x-2}{2} + C. \end{aligned}$$

8. We have

$$\begin{aligned} \int_{-1}^{0} \frac{1}{4x^2+4x+2}\, dx &= \int_{-1}^{0} \frac{1}{(2x+1)^2+1}\, dx \\ &= \frac{1}{2}\text{Arctan}\ (2x+1)\Big|_{-1}^{0} = \pi/4. \end{aligned}$$

9. $\displaystyle\int \frac{dx}{\sqrt{2x-x^2+1}} = \int \frac{dx}{\sqrt{2-(x-1)^2}} = \text{Arcsin}\ \frac{x-1}{\sqrt{2}} + C.$

10. $\displaystyle\int \frac{dx}{x^2+x+1} = \int \frac{dx}{(x+1/2)^2+3/4} = \frac{2}{\sqrt{3}}\text{Arctan}\ \left(\frac{2x+1}{\sqrt{3}}\right) + C.$

11. The roots of x^2+x-1 are $(-1\pm\sqrt{5})/2$; (these interesting numbers are related to the so-called Golden Ratio and the Fibonacci sequence.) We have the following partial fraction decomposition:

$$\frac{1}{x^2+x-1} = \frac{1}{\left(x-\frac{(-1+\sqrt{5})}{2}\right)\left(x-\frac{(-1-\sqrt{5})}{2}\right)} = \frac{1}{\sqrt{5}}\left(\frac{1}{x-\frac{(-1+\sqrt{5})}{2}} - \frac{1}{x-\frac{(-1-\sqrt{5})}{2}}\right).$$

So $\displaystyle\int \frac{dx}{x^2+x-1} = \frac{1}{\sqrt{5}}\left\{\ln\left|x-\frac{(-1+\sqrt{5})}{2}\right| - \ln\left|x-\frac{(-1-\sqrt{5})}{2}\right|\right\} + C.$

12. $\displaystyle I = \int \frac{dx}{(2x+1)\sqrt{(2x+1)^2-1}} = \frac{1}{2}\text{Arcsec}\ (2x+1) + C,$

since $|2x+1| = 2x+1$ for $x > -1/2$ (see Table 7.7).

8.6.2 Exercise Set 44

1. Set $x = 2\sin\theta$. Then $dx = 2\cos\theta\ d\theta$ and $\sqrt{4-x^2} = 2\cos\theta$. So

$$\begin{aligned}\int \sqrt{4-x^2}\ dx &= \int 2\cos\theta \cdot 2\cos\theta\ d\theta = 4\int \cos^2\theta\ d\theta \\ &= 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C \\ &= 2\,\text{Arcsin}\ (x/2) + \frac{1}{2}x\sqrt{4-x^2} + C.\end{aligned}$$

2. Let $x = 3\tan\theta$. Then $\sqrt{x^2+9} = 3\sec\theta$ and $dx = 3\sec^2\theta\ d\theta$. Hence

$$\begin{aligned}\int \sqrt{x^2+9}\ dx &= \int 3\sec\theta \cdot 3\sec^2\theta\ d\theta = 9\int \sec^3\theta\ d\theta \\ &= \frac{9}{2}\{(\tan\theta\sec\theta + \ln|\sec\theta + \tan\theta|\} + C \\ &= \frac{x}{2}\sqrt{x^2+9} + \frac{9}{2}\ln\left[\sqrt{x^2+9} + x\right] + C.\end{aligned}$$

(A constant from the ln term is absorbed by C.)

3. Let $x = \sec\theta$. Then $\sqrt{x^2-1} = \tan\theta$ and $dx = \sec\theta \cdot \tan\theta\ d\theta$. Hence

$$\begin{aligned}\int \sqrt{x^2-1}\ dx &= \int \sec\theta\ \tan^2\theta\ d\theta \\ &= \frac{1}{2}\tan\theta\ \sec\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C \\ &= \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln\left|x + \sqrt{x^2-1}\right| + C.\end{aligned}$$

4. Let $x - 2 = 2\sin\theta$. Then $dx = 2\cos\theta\ d\theta$ and

$$\sqrt{4x-x^2} = \sqrt{-(x^2-4x+4-4)} = \sqrt{4-(x-2)^2} = 2\cos\theta.$$

So we have

$$\begin{aligned}\int \sqrt{4x-x^2}\ dx &= \int 2\cos\theta \cdot 2\cos\theta \cdot d\theta \\ &= 2\theta + \sin 2\theta + C \\ &= 2\theta + 2\sin\theta\cos\theta + C \\ &= 2\sin^{-1}\frac{x-2}{2} + \frac{x-2}{2}\sqrt{4x-x^2} + C.\end{aligned}$$

5. Let $x = 2\sin u$. Then $dx = 2\cos u\ du$ and $(4-x^2)^{1/2} = 2\cos u$. Thus

$$\begin{aligned}\int \frac{dx}{(4-x^2)^{3/2}} &= \int \frac{2\cos u\ du}{2^3\cos^3 u} = \frac{1}{4}\int \sec^2 u\ du \\ &= \frac{1}{4}\tan u + C = \frac{1}{4}\cdot\frac{\sin u}{\cos u} + C = \frac{1}{4}\cdot\frac{x}{\sqrt{4-x^2}} + C.\end{aligned}$$

6. Let $x = 3\sin u$. Then $dx = 3\cos u\ du$ and $(9-x^2)^{1/2} = 3\cos u$. Thus

$$\begin{aligned}\int \frac{x^2\ dx}{(9-x^2)^{3/2}} &= \int \frac{3^2\sin^2 u \cdot 3\cos u\ du}{3^3\cos^3 u} = \int \tan^2 u\ du \\ &= \int (\sec^2 u - 1)\ du = \tan u - u + C = \frac{\sin u}{\cos u} - u + C \\ &= \frac{x}{\sqrt{9-x^2}} - \text{Arcsin}\ \frac{x}{3} + C.\end{aligned}$$

7. Let $x = 2\sec\theta$. Then $\sqrt{x^2-4} = 2\tan\theta$ and $dx = 2\sec\theta\,\tan\theta\;d\theta$. Also notice that $\cos\theta = 1/\sec\theta = 2/x$.

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2-4}} &= \int \frac{2\sec\theta\cdot\tan\theta\;d\theta}{4\sec^2\theta\cdot 2\tan\theta}\\ &= \frac{1}{4}\int \cos\theta\;d\theta = \frac{1}{4}\sin\theta + C = \frac{1}{4}\sqrt{1-\cos^2\theta}+C\\ &= \frac{1}{4}\sqrt{1-(2/x)^2}+C = \frac{\sqrt{x^2-4}}{4x}+C.\end{aligned}$$

8. Let $2x-1=\tan\theta$. Then $2\;dx = \sec^2\theta\;d\theta$ and

$$\sqrt{4x^2-4x+2} = \sqrt{(2x-1)^2+1} = \sqrt{\tan^2\theta+1} = \sec\theta.$$

Therefore we have

$$\begin{aligned}&\int \sqrt{4x^2-4x+2}\;dx = \frac{1}{2}\int \sec^3\theta\;d\theta\\ =\;& \frac{1}{4}\left(\tan\theta\,\sec\theta + \ln|\sec\theta+\tan\theta|\right)+C\\ =\;& \frac{1}{4}(2x-1)\sqrt{4x^2-4x+2}+\frac{1}{4}\ln\left|2x-1+\sqrt{4x^2-4x+2}\right|+C.\end{aligned}$$

9. Let $x=3\tan\theta$. Then $9+x^2 = 9\sec^2\theta$ and $dx = 3\sec^2\theta\;d\theta$. So

$$\begin{aligned}\int \frac{dx}{(9+x^2)^2} &= \int \frac{3\sec^2\theta\;d\theta}{81\sec^4\theta} = \frac{1}{27}\int \cos^2\theta\;d\theta\\ &= \frac{1}{27}\left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)+C\\ &= \frac{1}{27}\left\{\frac{1}{2}\text{Arctan}\,\frac{x}{3}+\frac{1}{4}\sin\left(2\text{Arctan}\,\frac{x}{3}\right)\right\}+C.\end{aligned}$$

NOTE: It is possible to simplify the expression for $\sin(2\,\text{Arctan}\,x/3) \equiv \sin 2\theta$:

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta = 2\frac{\sin\theta}{\cos\theta}\cdot\cos^2\theta = 2\tan\theta\cdot\frac{1}{\sec^2\theta}\\ &= \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\cdot x/3}{1+(x/3)^2} = \frac{6x}{9+x^2}.\end{aligned}$$

10. The easiest way to solve this exercise is to use the substitution $u=\sqrt{4-x^2}$. (This is highly nontrivial! At first sight one would try the trigonometric substitution $x = 2\sin\theta$. This method works, but the computation involved is rather tedious and lengthy.) Then $u^2 = 4-x^2$ and hence $2u\;du = -2x\;dx$, which gives $u\;du = -x\;dx$. Now

$$\frac{dx}{x} = \frac{x\;dx}{x^2} = \frac{-u\;du}{4-u^2} = \frac{u\;du}{u^2-4}$$

and hence

$$\begin{aligned}\int \frac{\sqrt{4-x^2}}{x}\;dx &= \int u\cdot\frac{u\;du}{u^2-4} = \int\left(1+\frac{4}{u^2-4}\right)\;du\\ &= u+\int\left(\frac{1}{u-2}-\frac{1}{u+2}\right)\;du\\ &= u+\ln|u-2|-\ln|u+2|+C\\ &= u+(\ln|2-u|+\ln|2+u|)-2\ln|2+u|+C\\ &= u+\ln|4-u^2|-2\ln|u+2|+C\\ &= \sqrt{4-x^2}+2\ln|x|-2\ln|2+\sqrt{4-x^2}\,|+C.\end{aligned}$$

11. Let $x = 5\tan\theta$. Then we have $dx = 5\sec^2\theta\ d\theta$, $(x^2+25)^{1/2} = 5\sec\theta$ and $(x^2+25)^{3/2} = 5^3\sec^3\theta$. Hence

$$\begin{aligned}\int \frac{dx}{(x^2+25)^{3/2}} &= \int \frac{5\sec^2\theta\ d\theta}{5^3\sec^3\theta} \\ &= \frac{1}{25}\int \cos\theta\ d\theta \\ &= \frac{1}{25}\sin\theta + C = \frac{1}{25}\frac{x}{\sqrt{x^2+25}} + C.\end{aligned}$$

12. Let $x = 2\sin\theta$. Then $\sqrt{4-x^2} = 2\cos\theta$ and $dx = 2\cos\theta\ d\theta$. So

$$\begin{aligned}\int \frac{\sqrt{4-x^2}}{x^2}\ dx &= \int \frac{2\cos\theta}{4\sin^2\theta}\cdot 2\cos\theta\ d\theta \\ &= \int \cot^2\theta\ d\theta \\ &= \int (\csc^2\theta - 1)\ d\theta = -\cot\theta - \theta + C \\ &= -\frac{\sqrt{4-x^2}}{x} - \operatorname{Arcsin}\frac{x}{2} + C.\end{aligned}$$

13. Let $x = a\sin\theta$. Then $dx = a\cos\theta\ d\theta$ and hence

$$\begin{aligned}\int \frac{dx}{x^4\sqrt{a^2-x^2}} &= \int \frac{a\cos\theta\ d\theta}{a^4\sin^4\theta\cdot a\cos\theta} \\ &= a^{-4}\int \frac{d\theta}{\sin^4\theta} = a^{-4}\int \csc^4\theta\ d\theta \\ &= a^{-4}\int (\csc^2\theta + \csc^2\theta\cot^2\theta)\ d\theta \\ &= a^{-4}\left(-\cot\theta - \frac{\cot^3\theta}{3}\right) + C, \\ &= -\frac{1}{a^4}\cdot\frac{(a^2-x^2)^{1/2}}{x} - \frac{1}{3a^4}\cdot\frac{(a^2-x^2)^{3/2}}{x^3} + C.\end{aligned}$$

14. Let $x = a\sec\theta$. Then $dx = a\sec\theta\ \tan\theta$ and hence

$$\begin{aligned}\int \frac{dx}{x^4\sqrt{x^2-a^2}} &= \int \frac{a\sec\theta\ \tan\theta\ d\theta}{a^4\sec^4\theta\cdot a\tan\theta} \\ &= \frac{1}{a^4}\int \cos^3\theta\ d\theta = \frac{1}{a^4}\int (1-\sin^2\theta)\cos\theta\ d\theta \\ &= \frac{1}{a^4}\left(\sin\theta - \frac{\sin^3\theta}{3}\right) + C \\ &= \frac{1}{a^4}\left(\frac{(x^2-a^2)^{1/2}}{x} - \frac{1}{3}\frac{(x^2-a^2)^{3/2}}{x^3}\right) + C.\end{aligned}$$

Notice that

$$\sin\theta = (1-\cos^2\theta)^{1/2} = (1-\sec^{-2}\theta)^{-1/2} = \left(1-\frac{a^2}{x^2}\right)^{1/2} = \frac{(x^2-a^2)^{1/2}}{x},$$

for $x > 0$.

15. We have

$$\begin{aligned} I &\equiv \int \frac{\sqrt{x^2+2x-3}}{x+1}\ dx \\ &= \int \frac{\sqrt{(x+1)^2-4}}{x+1}\ dx = \int \frac{\sqrt{u^2-4}}{u}\ du,\end{aligned}$$

where $u = x+1$. Use the tricky substitution similar to the one in Exercise 8 above: $v = \sqrt{u^2-4} \equiv \sqrt{x^2+2x-3}$. Then $v^2 = u^2 - 4$ and hence $2vdv = 2udu$, or $vdv = udu$. Thus

$$\frac{du}{u} = \frac{udu}{u^2} = \frac{vdv}{v^2+4}.$$

Now we can complete our evaluation as follows:

$$\begin{aligned} I &= \int \frac{v^2dv}{v^2+4} = \int \left(1 - \frac{4}{v^2+4}\right) dv \\ &= v - 2\text{Arctan}\,\frac{v}{2} + C \\ &= \sqrt{x^2+2x-3} - 2\text{Arctan}\,\frac{\sqrt{x^2+2x-3}}{2} + C. \end{aligned}$$

16. Let $u = x^2+2x+5$. Then $du = (2x+2)dx$ and hence

$$\begin{aligned} \int \frac{(2x+1)\,dx}{\sqrt{x^2+2x+5}} &= \int \frac{(2x+2-1)\,dx}{\sqrt{x^2+2x+5}} \\ &= \int \frac{du}{\sqrt{u}} - I = 2\sqrt{u} - I = 2\sqrt{x^2+2x+5} - I, \end{aligned}$$

where

$$I = \int \frac{dx}{\sqrt{x^2+2x+5}} \equiv \int \frac{dx}{\sqrt{(x+1)^2+4}}.$$

Let $x+1 = 2\tan\theta$. Then $dx = 2\sec^2\theta\,d\theta$ and $\sqrt{x^2+2x+5} = 2\sec\theta$. So

$$\begin{aligned} I &= \int \frac{2\sec^2\theta\,d\theta}{2\sec\theta} = \int \sec\theta\,d\theta = \ln|\tan\theta + \sec\theta| + C \\ &= \ln\left|x+1+\sqrt{x^2+2x+5}\right| + C, \end{aligned}$$

where a factor of $\frac{1}{2}$ inside the logarithm symbol is absorbed by the integral constant C. Thus our final answer is

$$\int \frac{(2x+1)\,dx}{\sqrt{x^2+2x+5}} = 2\sqrt{x^2+2x+5} - \ln\left|x+1+\sqrt{x^2+2x+5}\right| + C.$$

8.7

8.7.1

8.7.2 Exercise Set 45

1. $\boxed{\mathcal{T}_4 = 2.629 \text{ and } \mathcal{S}_4 = 2.408}$. Here $n = 4$, $a = 1, b = 10$, $f(x) = 1/x$. So, $h = (b-a)/n = 9/4$, $x_i = a + i(b-a)/n = 1 + 9i/n$, and

$$\begin{aligned} \mathcal{T}_4 &= \frac{h}{2} \cdot (y_0 + 2y_1 + 2y_2 + 2y_3 + y_4), \\ &= \frac{9}{8} \cdot \left(1 + 2 \cdot \sum_{i=1}^{3} \frac{1}{1+\frac{9i}{4}} + \frac{1}{1+10}\right), \\ &= \frac{99}{80} + \frac{9}{4} \cdot \sum_{i=1}^{3} \frac{1}{1+\frac{9i}{4}}, \\ \mathcal{T}_4 &\approx \boxed{2.629} \end{aligned}$$

On the other hand, $\mathcal{S}_4$ is given by,

$$\begin{aligned}
\mathcal{S}_4 &= \frac{h}{3} \cdot (y_0 + 4y_1 + 2y_2 + 4y_3 + y_4), \\
&= \frac{1}{12} \cdot \left(1 + 4 \cdot \frac{1}{1+\frac{9}{4}} + 2 \cdot \frac{1}{1+\frac{9\cdot 2}{4}} + 4 \cdot \frac{1}{1+\frac{9\cdot 3}{4}} + \frac{1}{1+\frac{9\cdot 4}{4}}\right), \\
&= \frac{9}{12} \cdot \left(1 + \frac{16}{13} + \frac{8}{22} + \frac{16}{31} + \frac{1}{10}\right), \\
\mathcal{S}_4 &\approx \boxed{2.408}
\end{aligned}$$

The Actual value is given by $\ln 10 \approx 2.302$, so Simpson's Rule is closer to the Actual value.

2. $\boxed{\mathcal{T}_5 = 1.161522}$. You can't use Simpson's Rule because n is ODD. The Actual value is approximately given by $\boxed{1.19814}$ here is given in terms of so-called **elliptic functions** and so it cannot be written down nicely.

3. $\boxed{\mathcal{T}_4 = 1.49067 \text{ and } \mathcal{S}_4 = 1.46371}$. The Actual value is found to be around 1.46265.

4. $\boxed{\mathcal{T}_6 = 3.062 \text{ and } \mathcal{S}_6 = 3.110}$. Here $n = 6$, $a = 0, b = 1$, $f(x) = 4 \cdot \sqrt{1 - x^2}$. So, $h = (b-a)/n = 1/6$, $x_i = a + i(b-a)/n = 0 + i/6 = i/6$, and

$$\begin{aligned}
\mathcal{T}_6 &= \frac{h}{2} \cdot (y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + y_6), \\
&= \frac{1}{12} \cdot \left(4 + 2 \cdot \sum_{i=1}^{5} 4 \cdot \sqrt{1 - (i/6)^2} + 0\right), \\
&= \frac{1}{3} + \frac{1}{6} \cdot \sum_{i=1}^{5} \left(4 \cdot \sqrt{1 - \frac{i^2}{36}}\right), \\
\mathcal{T}_6 &\approx \boxed{3.062}
\end{aligned}$$

On the other hand, $\mathcal{S}_6$ is given by,

$$\begin{aligned}
\mathcal{S}_6 &= \frac{h}{3} \cdot (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6), \\
&\approx \boxed{3.110}.
\end{aligned}$$

Now, remember to use a trigonometric substitution like $x = \sin\theta$ to convert the integral into an easily solvable trigonometric integral like

$$\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta.$$

The Actual value is then found to be $\pi \approx 3.1416$, so Simpson's Rule is closer to the Actual value.

5. $\boxed{\mathcal{T}_{10} = 1.090607 \text{ and } \mathcal{S}_{10} = 1.089429}$. The Actual value is about 1.089429, and so Simpson's Rule gives an extremely good estimate!

6. $\boxed{1.18728}$, by using Simpson's Rule with $n = 6$. The area is given by a definite integral, namely, $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx$, since the values $\sin x \geq 0$ for every x in this interval. The Actual value is around 1.19814.

7. First, sketch the graphs and find the points of intersection (equate the $y-$values) of these two curves. You will see that they intersect when $x = 0$ and $x = 1$. See the graph in the margin below.

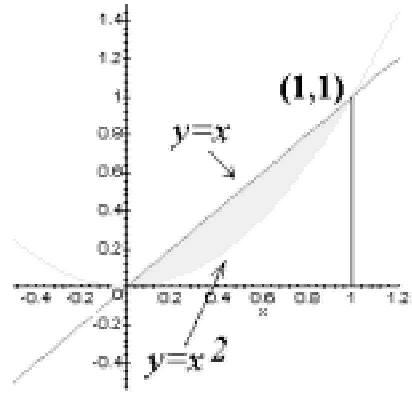
   Then we find the area under the curve $y = x$ and subtract from it the area under the curve $y = x^2$, (because the line lies *above* the parabola). So, the area of the closed lop is given by

$$\int_0^1 (x - x^2)\, dx.$$

Let $n = 6$. Applying both Rules with $a = 0, b = 1$, $f(x) = x - x^2$, $h = 1/6$, we get the values,

$$\mathcal{T}_6 = 0.1621, \quad \mathcal{S}_6 = 0.1667,$$

and the estimate using Simpson's Rule is EXACT (*i.e.,* equal to the Actual value) because the curve $y = x - x^2$ is a parabola (or a quadratic, see the margin).



8. $\boxed{\mathcal{T}_6 = 0.6695 \text{ and } \mathcal{S}_6 = 0.5957}$. Here $n = 6$, $a = 1.05, b = 1.35$, and the values of $f(x)$ are given in the Table. So, $h = (b - a)/n = 0.05$, $x_i = a + i(b - a)/n = 1.05 + i(0.05)$, and

$$\begin{aligned}
\mathcal{T}_6 &= \frac{h}{2} \cdot (y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + y_6), \\
&= 0.025 \cdot (2.32 + 2(1.26) + 2(1.48) + 2(1.6) + 2(3.6) + 2(2.78) + (3.02)), \\
&\approx 0.6695 \\
\mathcal{T}_6 &\approx \boxed{0.6695}
\end{aligned}$$

On the other hand, $\mathcal{S}_6$ is given by,

$$\begin{aligned}
\mathcal{S}_6 &= \frac{h}{3} \cdot (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6), \\
&= \frac{0.05}{3} \cdot (2.32 + 4(1.26) + 2(1.48) + 4(1.6) + 2(3.6) + 4(2.78) + 3.02) \\
\mathcal{S}_6 &\approx \boxed{0.5957}.
\end{aligned}$$

9. $\boxed{30.7655}$, using Simpson's Rule with $n = 10$. This means that there are roughly 30 primes less than 100.

10. $\boxed{L(6) \approx 0.1876}$. Use $a = 0, b = 1$, $n = 6$, and $h = 1/6$ in the expression for $\mathcal{S}_6$.

11. $\boxed{0.087817}$, using the Trapezoidal Rule with $n = 8$. This means that only about 8% of the total population has an IQ in this range.

8.8 Exercise Set 46

1. Yes, $x = 0$ is an infinite discontinuity.
2. No, the integrand is continuous on $[-1, 1]$.
3. Yes, $x = 0$ is an infinite discontinuity.
4. Yes, $x = 1$ is an infinite discontinuity (and ∞ is an upper limit).
5. Yes, $x = -1$ is an infinite discontinuity.
6. No, the integrand is continuous on $[-1, 1]$.
7. Yes, $x = -\pi, \pi$ are each infinite discontinuities of the cosecant function.
8. Yes, $\pm\infty$ are the limits of integration.
9. Yes, $x = 0$ gives an indeterminate form of the type $0 \cdot \infty$ in the integrand.
10. Yes, $\pm\infty$ are the limits of integration.

11. 2. This is because $\lim_{T\to\infty} \int_0^T x^{-1.5}\, dx = \lim_{T\to\infty} \left(\frac{-2}{\sqrt{T}} + 2 \right) = 2.$
12. $+\infty$. This is because $\lim_{T\to\infty} \int_2^T x^{-1/2}\, dx = \lim_{T\to\infty} \left(2T^{1/2} - 2\sqrt{2} \right) = +\infty.$
13. $+\infty$. Note that $\lim_{T\to 0^+} \frac{1}{2} \int_T^2 \frac{dx}{x} = \lim_{T\to 0^+} \left(\frac{1}{2} \ln|x| \right) \Big|_T^2 = \lim_{T\to 0^+} \left(\frac{1}{2}\ln 2 - \frac{1}{2}\ln T \right) =$ $-(-\infty) = +\infty.$
14. 2. Use Integration by Parts (with the Table Method) and L'Hospital's Rule twice.
    This gives $\lim_{T\to\infty} \int_0^T x^2\, e^{-x}\, dx = \lim_{T\to\infty} \left(2 - \frac{T^2 + 2T + 2}{e^T} \right) = 2.$
15. 0. Use the substitution $u = 1 + x^2$, $du = 2x\, dx$ to find an antiderivative and note that
    $$\int_{-\infty}^{\infty} \frac{2x}{(1+x^2)^2}\, dx = \int_{-\infty}^{0} \frac{2x}{(1+x^2)^2}\, dx + \int_0^{\infty} \frac{2x}{(1+x^2)^2}\, dx.$$
    $$= \lim_{T\to-\infty} \int_T^0 \frac{2x}{(1+x^2)^2}\, dx + \lim_{T\to\infty} \int_0^T \frac{2x}{(1+x^2)^2}\, dx,$$
    $$= \lim_{T\to-\infty} \left(-\frac{1}{1+x^2} \right) \Big|_T^0 + \lim_{T\to\infty} \left(-\frac{1}{1+x^2} \right) \Big|_0^T = -1 + 0 + 0 - (-1) = 0.$$
16. -1. Note that the infinite discontinuity is at $x = -1$ only. Now, use the substitution $u = 1 - x^2$, $-\frac{du}{2} = x\, dx$. Then
    $$\int_{-1}^{0} \frac{x}{\sqrt{1-x^2}}\, dx = \lim_{T\to -1} \left(-\sqrt{1-T^2} \right) \Big|_T^0 = -1 - 0 = -1.$$
17. Diverges (or does not exist). There is one infinite discontinuity at $x = 1$. First, use partial fractions here to find that
    $$\frac{1}{x^2-1} = \frac{1}{2} \cdot \frac{1}{x-1} - \frac{1}{2} \cdot \frac{1}{x+1}.$$
    Next, using the definitions, we see that
    $$\int_0^2 \frac{1}{x^2-1}\, dx = \int_0^1 \frac{1}{x^2-1}\, dx + \int_1^2 \frac{1}{x^2-1}\, dx =$$
    $$= \lim_{T\to 1^-} \int_0^T \frac{1}{x^2-1}\, dx + \lim_{T\to 1^+} \int_T^2 \frac{1}{x^2-1}\, dx = \lim_{T\to 1^-} \left(\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| \right) \Big|_0^T +$$
    $$+ \lim_{T\to 1^+} \left(\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| \right) \Big|_T^2 =$$
    $$= \lim_{T\to 1^-} \left(\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| \right) \Big|_0^T + \lim_{T\to 1^+} \left(\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| \right) \Big|_T^2 =,$$

$= \lim_{T\to 1^-} \left(\frac{1}{2} \ln \left| \frac{T-1}{T+1} \right| - \frac{1}{2} \ln |-1| \right) \; + \; \lim_{T\to 1^+} \left(\frac{1}{2} \ln \left| \frac{1}{3} \right| - \frac{1}{2} \ln \left| \frac{T-1}{T+1} \right| \right) = =$
$(-\infty - 0) + (-\frac{1}{2}\ln 3 - (-\infty) = \infty - \infty$, and so the limit does not exist.
So, the improper integral diverges.

18. $-\infty$. See the (previous) Exercise 17 above for more details. In this case the discontinuity, $x = 1$, is at an end-point. Thus, using partial fractions as before, we find that
$\int_1^2 \frac{1}{1-x^2}\, dx = \lim_{T\to 1^+} \int_T^2 \frac{1}{1-x^2}\, dx = -\lim_{T\to 1^+} \int_T^2 \frac{1}{x^2-1}\, dx =$
$= -\lim_{T\to 1^+} \left(\frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| \right) \Big|_T^2 = -\left(-\frac{1}{2} \ln 3 - (-\infty) \right) = -\infty.$

19. $\frac{1}{2}$. Use Integration by Parts and the Sandwich Theorem to find that
$\int_0^\infty e^{-x} \; \sin x \; dx = \lim_{T\to\infty} \int_0^T e^{-x} \; \sin x \; dx = \lim_{T\to\infty} \frac{1}{2} \left(-e^{-x} \cos x - e^{-x} \sin x \right) \Big|_0^T$
$= \lim_{T\to\infty} \left(\frac{1}{2} \left(-\frac{\cos T}{e^T} - \frac{\sin T}{e^T} \right) - \left(-\frac{1}{2} \right) \right) = \frac{1}{2}.$
Recall that the Sandwich Theorem tells us that, in this case,

$$0 \le \lim_{T\to\infty} \left| \frac{\cos T}{e^T} \right| \le \lim_{T\to\infty} \left| \frac{1}{e^T} \right| = 0,$$

and so the required limit is also 0. A similar argument applies to the other limit.

20. $+\infty$. The infinite discontinuity is a $x = 1$. Use the substitution $u = \ln x$, $du = \frac{dx}{x}$. Then
$\int_1^2 \frac{dx}{x \ln x} = \lim_{T\to 1^+} \int_T^2 \frac{dx}{x \ln x} = \lim_{T\to 1^+} \ln(\ln x)) \Big|_T^2 =$
$= \lim_{T\to 1^+} (\ln(ln 2) - \ln(\ln T)) = -(-\infty) = +\infty.$

21. $\frac{10}{7}$. The integrand is the same as $\int_{-1}^1 \left(x^{2/5} + x^{-3/5} \right) \; dx$ and so the infinite discontinuity (at $x = 0$) is in the second term only. So, $\int_{-1}^1 \left(x^{2/5} + x^{-3/5} \right) \; dx =$
$\int_{-1}^1 x^{2/5} \; dx + \int_{-1}^1 x^{-3/5} \; dx = \frac{10}{7} + \int_{-1}^0 x^{-3/5} \; dx + \int_0^1 x^{-3/5} \; dx = \frac{10}{7} + \lim_{T\to 0^-} \int_{-1}^T x^{-3/5} \; dx +$
$\lim_{T\to 0^+} \int_T^1 x^{-3/5} \; dx = \frac{10}{7} + \lim_{T\to 0^-} \left(\frac{5T^{2/5}}{2} - \frac{5}{2} \right) + \lim_{T\to 0^+} \left(\frac{5}{2} - \frac{5T^{2/5}}{2} \right) = \frac{10}{7} -$
$\frac{5}{2} + \frac{5}{2} = \frac{10}{7}.$

22. Diverges. The integrand is the same as $\int_{-1}^1 \left(x^{-2/3} + x^{-5/3} \right) \; dx$ and so the discontinuity is present in both terms. Thus, $\int_{-1}^1 \left(x^{-2/3} + x^{-5/3} \right) \; dx =$
$= \int_{-1}^0 \left(x^{-2/3} + x^{-5/3} \right) \; dx + \int_0^1 \left(x^{-2/3} + x^{-5/3} \right) \; dx$
$= \lim_{T\to 0^-} \int_{-1}^T \left(x^{-2/3} + x^{-5/3} \right) \; dx + \lim_{T\to 0^+} \int_T^1 \left(x^{-2/3} + x^{-5/3} \right) \; dx$
$= \lim_{T\to 0^-} \left(3x^{1/3} - \frac{3}{2} x^{-2/3} \right) \Big|_{-1}^T + \lim_{T\to 0^+} \left(3x^{1/3} - \frac{3}{2} x^{-2/3} \right) \Big|_T^1$
$= \lim_{T\to 0^-} \left(3T^{1/3} - \frac{3}{2} T^{-2/3} \right) - \left(-3 - \frac{3}{2} \right) + \left(3 - \frac{3}{2} \right) -$
$= \lim_{T\to 0^+} \left(3T^{1/3} - \frac{3}{2} T^{-2/3} \right) =$
$-\infty + 6 + \infty = \infty - \infty,$

and so the improper integral diverges.

23. 2. Simply rewrite this integral as $\int_{-\infty}^{\infty} e^{-|x|}\,dx = \int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{-x}\,dx$, since for $x < 0$ we have $|x| = -x$ while for $x > 0$ we have $|x| = x$. The integrals are straightforward and so are omitted.

24. Converges for $p > 1$ only. Let $u = \ln x$, $du = \frac{dx}{x}$. Then $\int_{2}^{\infty} \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^{\infty} \frac{du}{u^p} = \lim_{T\to\infty} \int_{\ln 2}^{T} \frac{du}{u^p} = \lim_{T\to\infty} \left.\frac{u^{1-p}}{1-p}\right|_{\ln 2}^{T} = \lim_{T\to\infty} \left(\frac{T^{1-p}}{1-p}\right) - \frac{(\ln 2)^{1-p}}{1-p} = 0 - \frac{(\ln 2)^{1-p}}{1-p} = \frac{1}{(p-1)\ (\ln 2)^{p-1}}$, only for $p > 1$. The case $p = 1$ is treted as in Exercise 20, above, while the case $p < 1$ leads to an integral which converges to $+\infty$.

25. No, this is impossible. There is no real number p such the stated integral converges to a finite number. Basically, this is because the integrand has a "bad" discontinuity at $x = 0$ whenever $p < 1$ and another discontinuity at $x = \infty$ whenever $p \geq 1$. The argument is based on a case-by-case analysis and runs like this:

    If $p + 1 > 0$, then $\int_{0}^{\infty} x^p\,dx = \lim_{T\to\infty} \int_{0}^{T} x^p\,dx = \lim_{T\to\infty} \left(\frac{T^{p+1}}{p+1} - \frac{1}{p+1}\right) = +\infty$. On the other hand, if $p+1 < 0$, then $\int_{0}^{\infty} x^p\,dx = \int_{0}^{1} x^p\,dx + \int_{1}^{\infty} x^p\,dx = \lim_{T\to 0^+} \left(\frac{1}{p+1} - \frac{T^{p+1}}{p+1}\right) + \lim_{T\to\infty} \left(\frac{T^{p+1}}{p+1} - \frac{1}{p+1}\right) = \left(\frac{1}{p+1} - \infty\right) + \left(0 - \frac{1}{p+1}\right) = -\infty$. Finally, if $p = 1$ the integrand reduces to x, by itself and it converges to $+\infty$. Thus, we have shown that for any value of p the improper integral cannot converge to a finite value.

26. $\sqrt{\frac{2}{\pi}} \cdot \frac{2}{4+\lambda^2}$. Use the method outlined in Exercise 19, above.

27. No, the integral must converge to $+\infty$. Follow the hints.

28. Follow the hints.

29. Follow the hints.

30. $L = \sqrt{\pi}$. Simpson's Rule with $n = 22$ gives us the value 1.7725 as an estimate for the value of this integral over the interval $[-5, 5]$. Its square is about 3.1416, which is close to $\sqrt{\pi}$.

8.9 Chapter Exercises

Please add a constant of integration, C, after every indefinite integral!

1. $\cos^2 x - \sin^2 x = \cos 2x$. Use the identity $\cos(A + B) = \cos A \cos B - \sin A \sin B$ with $A = B = x$.

2. $\cos^4 x - \sin^4 x = \cos 2x$. This is because $\cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x) = (\cos^2 x - \sin^2 x)(1) = \cos 2x$.

3. $\sec^4 x - \tan^4 x = \sec^2 x + \tan^2 x$. Use the same idea as the preceding one except that now, $\sec^2 x - \tan^2 x = 1$.

4. $\sqrt{1+\cos x} = \sqrt{2}\cdot\cos\left(\frac{x}{2}\right)$, if $-\pi \le x \le \pi$. Replace x by $x/2$ in the identity $\frac{1+\cos 2x}{2} = \cos^2 x$, and then extract the square root. Note that whenever $-\pi/2 \le \theta \le \pi/2$, we have $\cos\theta \ge 0$. Consequently, if $-\pi \le \theta \le \pi$, then $\cos\frac{\theta}{2} \ge 0$. This explains that the positive square root of $\cos^2\frac{\theta}{2}$ is $\cos\frac{\theta}{2}$.

5. $\sqrt{1-\cos x} = \sqrt{2}\cdot\sin\left(\frac{x}{2}\right)$, if $0 \le x \le 2\pi$. Replace x by $x/2$ in the identity $\frac{1-\cos 2x}{2} = \sin^2 x$.

6. $\sqrt{1+\cos 5x} = \sqrt{2}\cdot\cos\left(\frac{5x}{2}\right)$, if $-\pi \le 5x \le \pi$. Replace x by $5x/2$ in the identity $\frac{1+\cos x}{2} = \cos^2 x$.

7. $\int_0^2 (2x-1)\,dx = 2$, since the function is linear (a polynomial of degree 1). In this case, the Trapezoidal Rule always gives the Actual value.

8. $\int_0^4 (3x^2-2x+6)\,dx = 72$, using Simpson's Rule with $n=6$. Once again, since the integrand is a quadratic function, Simpson's Rule is exact and always gives the Actual value.

9. $\int_{-\pi}^{\pi} (\cos^2 x + \sin^2 x)\,dx = 2\pi$. The Trapezoidal Rule with $n=6$ and the Actual value agree exactly, since the integrand is equal to 1.

10. $\int_{-\pi}^{\pi} (\cos^2 x - \sin^2 x)\,dx = 0$, using Simpson's Rule with $n=6$. The exact answer, obtained by direct integration, is 0, since the integrand is equal to $\cos 2x$. Note that the two values agree!

11. $\int_0^1 e^{-x^2}\,dx \approx 1.4628$, using Simpson's Rule with $n=6$. The Actual value is 1.462651746

12. $\int_{-1}^2 \frac{1}{1+x^6}\,dx \approx 1.82860$, using Simpson's Rule with $n=4$. The Actual value is ≈ 1.94476. Don't try to work it out!

13. $\int_{-2}^2 \frac{x^2}{1+x^4}\,dx \approx 1.221441$, using the Trapezoidal Rule with $n=6$. The exact answer obtained by direct integration is 1.23352.

14. $\int_1^2 (\ln x)^3\,dx \approx 0.10107$, using Simpson's Rule with $n=6$. The Actual value is $2\ln^3 2 - 6\ln^2 2 + 12\ln 2 - 6 \approx 0.101097387$.

15. $\int \sqrt{3x+2}\,dx = \frac{2}{9}\left(\sqrt{3x+2}\right)^3.$

    Let $u = 3x+2$.

16. $\int \frac{1}{x^2+4x+4}\,dx = -\frac{1}{x+2}.$

    Note that $x^2+4x+4 = (x+2)^2$.
    Then let $u = x+2,\ du = dx$.

17. $\int \frac{dx}{(2x-3)^2} = -\frac{1}{2\,(2x-3)}.$

    Let $u = 2x-3,\ du = 2dx$, and so $dx = du/2$.

18. $\int \frac{dx}{\sqrt{a+bx}} = 2\frac{\sqrt{a+bx}}{b}.$

    Let $u = a+bx,\ du = bdx$, and $dx = du/b$, if $b \ne 0$.

19. $\int (\sqrt{a} - \sqrt{x})^2 \, dx = ax - \frac{4\sqrt{a}}{3} \left(\sqrt{x}\right)^3 + \frac{1}{2}x^2.$

Expand the integrand and integrate term-by-term.

20. $\int \frac{x \, dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2}.$

Let $u = a^2 - x^2$. Then $du = -2xdx$ and $x \, dx = -du/2$.

21. $\int x^2 \sqrt{x^3 + 1} \, dx = \frac{2}{9} \left(\sqrt{x^3 + 1}\right)^3.$

Let $u = x^3 + 1, \ du = 3x^2 \, dx$, so that $x^2 \, dx = du/3$.

22. $\int \frac{(x + 1)}{\sqrt[3]{x^2 + 2x + 2}} \, dx = \frac{3}{4} \left(\sqrt[3]{(x^2 + 2x + 2)} \right)^2.$

Let $u = x^2 + 2x + 2, \ du = (2x + 2) \, dx = 2(x + 1) \, dx$. So, $(x + 1) \, dx = du/2$.

23. $\int (x^4 + 4x^2 + 1)^2 (x^3 + 2x) \, dx = \frac{1}{12}(x^4 + 4x^2 + 1)^3$

Let $u = x^4 + 4x^2 + 1, \ du = (4x^3 + 8x) \, dx = 4(x^3 + 2x) \, dx$ and so, $(x^3 + 2x) \, dx = du/4$.

24. $\int x^{-1/3} \sqrt{x^{2/3} - 1} \, dx = \left(\sqrt{x^{2/3} - 1}\right)^3.$

Let $u = x^{2/3} - 1$. Then $du = (2/3)x^{-1/3} \, dx$, or $x^{-1/3} \, dx = 3 \, du/2$.

25. $\int \frac{2x \, dx}{(3x^2 - 2)^2} = -\frac{1}{3\,(3x^2 - 2)}$

Let $u = 3x^2 - 2, \ du = 6x \, dx$ and so $2x \, dx = du/3$.

26. $\int \frac{dx}{4x + 3} = \frac{1}{4} \ln |4x + 3|$

Let $u = 4x + 3, \ du = 4dx$ so that $dx = du/4$.

27. $\int \frac{x \, dx}{2x^2 - 1} = \frac{1}{4} \ln \left|2x^2 - 1\right|$

Let $u = 2x^2 - 1, \ du = 4x \, dx$ so that $x \, dx = du/4$.

28. $\int \frac{x^2 \, dx}{1 + x^3} = \frac{1}{3} \ln \left|1 + x^3\right|$

Let $u = 1 + x^3, \ du = 3x^2 \, dx$ so that $x^2 \, dx = du/3$.

29. $\int \frac{(2x + 3) \, dx}{x^2 + 3x + 2} = \ln \left|x^2 + 3x + 2\right|$

Let $u = x^2 + 3x + 2, \ du = (2x + 3) \, dx$.

30. $\int \sin(2x + 4) \, dx = -\frac{1}{2} \cos(2x + 4)$

Let $u = 2x + 4, \ du = 2 \, dx$, and $dx = du/2$.

31. $\int 2\cos(4x + 1) \, dx = \frac{1}{2} \sin(4x + 1)$

Let $u = 4x + 1, \ du = 4 \, dx$, and $dx = du/4$.

32. $\int \sqrt{1 - \cos 2x} \, dx = \sqrt{2} \cos x.$

Note that $\frac{1-\cos 2x}{2} = \sin^2 x$. The result follows upon the extraction of a square root. In actuality, we are assuming that $\sqrt{\sin^2 x} = |\sin x| = \sin x$, here (or that $\sin x \geq 0$ over the region of integration).

33. $\int \sin \frac{3x-2}{5}\ dx = -\frac{5}{3}\cos\left(\frac{3x-2}{5}\right)$

Let $u = \frac{3x-2}{5}$, $du = \frac{3dx}{5}$. Then $dx = \frac{5du}{3}$.

34. $\int x \cos ax^2\ dx = \frac{1}{2}\frac{\sin ax^2}{a}$

Assume $a \neq 0$. Let $u = ax^2$, $du = 2ax\ dx$, so that $x\ dx = du/2a$.

35. $\int x \sin(x^2+1)\ dx = -\frac{1}{2}\cos\left(x^2+1\right)$

Let $u = x^2+1$, $du = 2x\ dx$. Then $x\ dx = du/2$.

36. $\int \sec^2 \frac{\theta}{2}\ \mathrm{d}\theta = 2\tan\frac{1}{2}\theta$

Let $u = \theta/2$, $du = d\theta/2$. The result follows since $\int \sec^2 u\ du = \tan u$.

37. $\int \frac{\mathrm{d}\theta}{\cos^2 3\theta} = \frac{1}{3}\tan 3\theta$

The integrand is equal to $\sec^2 3\theta$. Now let $u = 3\theta$, $du = 3d\theta$.

38. $\int \frac{\mathrm{d}\theta}{\sin^2 2\theta} = -\frac{1}{2}\cot 2\theta$

The integrand is equal to $\csc^2 2\theta$. Now let $u = 2\theta$, $du = 2d\theta$, and note that $\int \csc^2 u\ du = -\cot u$.

39. $\int x \csc^2(x^2)\ dx = -\frac{1}{2}\cot x^2$

Let $u = x^2$, $du = 2x\ dx$, so that $x\ dx = du/2$. Note that $\int \csc^2 u\ du = -\cot u$.

40. $\int \tan \frac{3x+4}{5}\ dx = \frac{5}{3}\ln\left|\sec\frac{3x+4}{5}\right|$

Let $u = \frac{3x+4}{5}$, $du = 3dx/5$ and so $dx = 5du/3$. The result follows since $\int \tan u\ du = -\ln|\cos u| = \ln|\sec u|$.

41. $\int \frac{dx}{\tan 2x} = \frac{1}{2}\ln|\sin 2x|$

The integrand is equal to $\cot 2x$. Let $u = 2x$, $du = 2dx$. Then, $dx = du/2$, and since $\int \cot u\ du = \ln|\sin u|$, the result follows.

42. $\int \sqrt{1+\cos 5x}\ dx = \frac{2\sqrt{2}}{5}\sin\frac{5x}{2}$

Use the identity in Exercise 6, above. Since $\sqrt{1+\cos 5x} = \sqrt{2}\cdot\cos\left(\frac{5x}{2}\right)$ we let $u = \frac{5x}{2}$, $du = 5dx/2$. Then $dx = 2du/5$ and the conclusion follows.

43. $\int \csc(x+\frac{\pi}{2})\ \cot(x+\frac{\pi}{2})\ dx = -\sec x$

Trigonometry tells us that $\sin(x+\frac{\pi}{2}) = \cos x$, and $\cos(x+\frac{\pi}{2}) = -\sin x$. Thus, by definition, $\csc(x+\frac{\pi}{2})\ \cot(x+\frac{\pi}{2}) = -\sec x\tan x$. On the other hand, $\int \sec x \tan x\ dx = \sec x$.

44. $\int \cos 3x\ \cos 4x\ dx = \frac{1}{2}\sin x + \frac{1}{14}\sin 7x$

Use the identity $\cos A\cos B = \frac{1}{2}\left(\cos(A-B)+\cos(A+B)\right)$, with $A = 4x$, $B = 3x$, and integrate the terms individually. This is also a " three-row problem" using the Table method in Integration by Parts and so you can use this alternate method as well.

45. $\int \sec 5\theta \tan 5\theta \, \mathrm{d}\theta = \frac{1}{5}\sec 5\theta$

Let $u = 5\theta,\ du = 5d\theta$. Then $d\theta = du/5$ and since $\int \sec u \tan u \ du = \sec u$, we have the result.

46. $\int \frac{\cos x}{\sin^2 x}\ dx = -\frac{1}{\sin x}$

The integrand is equal to $\cot x \csc x$. The result is now clear since $\frac{1}{\sin x} = \csc x$.

47. $\int x^2 \cos(x^3+1)\ dx = \frac{1}{3}\sin\left(x^3+1\right)$

Let $u = x^3+1,\ du = 3x^2\ dx$. Then $x^2\ dx = du/3$ and the answer follows.

48. $\int \sec\theta\ (\sec\theta + \tan\theta)\ \mathrm{d}\theta = \sec\theta + \tan\theta$

Expand the integrand and integrate it term-by-term. Use the facts $\int \sec^2 u\ du = \tan u$, and $\int \sec u \tan u\ du = \sec u$

49. $\int (\csc\theta - \cot\theta)\csc\theta\ \mathrm{d}\theta = \csc\theta - \cot\theta = \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}$

Expand the integrand and integrate it term-by-term. Use the facts $\int \csc^2 u\ du = -\cot u$, and $\int \csc u \cot u\ du = -\csc u$. Rewrite your answer using the elementary functions sine and cosine.

50. $\int \cos^{-4} x \sin(2x)\ dx = \frac{1}{\cos^2 x}$

Write $\sin 2x = 2\sin x \cos x$ and simplify the integrand. Put the $\cos^3 x$-term in the denominator and then use the substitution $u = \cos x$, $du = -\sin x\ dx$. Then $-2\int u^{-3}\ du = u^{-2}$ and the result follows.

51. $\int \frac{\tan^2 \sqrt{x}}{\sqrt{x}}\ dx = 2\tan\sqrt{x} - 2\sqrt{x}$

Let $u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\ dx$, which gives $2\sqrt{x}\ du = dx$, or $dx = 2u\ du$. The integral becomes $\int \frac{2u\ \tan^2 u}{u}\ du = \int 2\tan^2 u\ du = \int 2(\sec^2 u - 1)\ du = 2\tan u - 2u$, and the result follows.

52. $\int \frac{1+\sin 2x}{\cos^2 2x}\ dx = \frac{1}{2\cos 2x} + \frac{1}{2}\frac{\sin 2x}{\cos 2x}$

Note that the integrand is equal to $\sec^2 2x + \sec 2x \tan 2x$. Let $u = 2x$, $du = 2dx$, or $dx = du/2$. Use the facts $\int \sec^2 u\ du = \tan u$, and $\int \sec u \tan u\ du = \sec u$. Now reduce your answer to elementary sine and cosine functions.

53. $\int \frac{dx}{\cos 3x} = \frac{1}{3}\ln|\sec 3x + \tan 3x|$

Let $u = 3x,\ du = 3dx,\ dx = du/3$, and use the result from Example 386, with $x = u$.

54. $\int \frac{dx}{\sin(3x+2)} = \frac{1}{3}\ln|\csc(3x+2) - \cot(3x+2)|$

Note that the integrand is equal to $\csc(3x+2)$. Now let $u = 3x+2$, $du = 3dx$, $dx = du/3$. The integral looks like $(1/3)\int \csc u\ du = (1/3)\ln|\csc u - \cot u|$ and the result follows. This last integral is obtained using the method described in Example 386, but applied to these functions. See also Table 8.9.

55. $\displaystyle\int \frac{1+\sin x}{\cos x}\,dx = \ln|\sec x+\tan x| - \ln|\cos x|$

Break up the integrand into two parts and integrate term-by-term. Note that $-\ln|\cos x| = \ln|\sec x|$ so that the final answer may be written in the form $\ln|\sec x+\tan x| + \ln|\sec x| = \ln\left|\sec^2 x + \tan x\ \sec x\right|$.

56. $\displaystyle\int (1+\sec\theta)^2\ d\theta = \theta + 2\ln|\sec\theta+\tan\theta| + \tan\theta$

Expand the integrand and integrate term-by-term.

57. $\displaystyle\int \frac{\csc^2 x\ dx}{1+2\cot x} = -\frac{1}{2}\ln|1+2\cot x|$

Let $u = 1+2\cot x,\ du = -2\csc^2 x\ dx$. So, $\csc^2\ dx = -\ \frac{du}{2}$. The integral now becomes $(-1/2)\displaystyle\int \frac{du}{u} = -(1/2)\ln|u|$.

58. $\displaystyle\int e^x\ \sec e^x\ dx = \ln|\sec(e^x)+\tan(e^x)|$

Let $u = e^x,\ du = e^x\ dx$, and use Example 386.

59. $\displaystyle\int \frac{dx}{x\ln x} = \ln|\ln x|$

Let $u = \ln x,\ du = \frac{dx}{x}$. The integral looks like $\int \frac{du}{u} = \ln|u|$ and the result follows.

60. $\displaystyle\int \frac{dt}{\sqrt{2-t^2}} = \text{Arcsin}\,\frac{1}{2}\sqrt{2}t$

The integrand contains a square root of a difference of squares of the form $\sqrt{a^2-u^2}$ where $a=\sqrt{2}$, and $u=t$. Let $t=\sqrt{2}\sin\theta,\ dt=\sqrt{2}\cos\theta\ d\theta$. Since $\sqrt{2-t^2}=\sqrt{2}\cos\theta$, the integral looks like $\int d\theta = \theta = \text{Arcsin}\,\frac{t}{\sqrt{2}}$.

61. $\displaystyle\int \frac{dx}{\sqrt{3-4x^2}} = \frac{1}{2}\text{Arcsin}\,\frac{2}{3}\sqrt{3}x$

The integrand contains a square root of a difference of squares of the form $\sqrt{a^2-u^2}$ where $a=\sqrt{3}$, and $u=2x$. Let $2x=\sqrt{3}\sin\theta,\ 2dx=\sqrt{3}\cos\theta\ d\theta$. Since $\sqrt{3-4x^2}=\sqrt{3}\cos\theta$, the integral looks like $\int(1/2)d\theta = \frac{\theta}{2} = \frac{1}{2}\text{Arcsin}\,\frac{2x}{\sqrt{3}}$, which is equivalent to the answer.

62. $\displaystyle\int \frac{(2x+3)\ dx}{\sqrt{4-x^2}} = -2\sqrt{4-x^2} + 3\text{Arcsin}\,\frac{1}{2}x$

Break up the integrand into two parts so that the integral looks like
$\displaystyle\int \frac{2x\ dx}{\sqrt{4-x^2}} + \int \frac{3}{\sqrt{4-x^2}}\ dx.$

Let $u=4-x^2,\ du=-2x\ dx$ in the first integral and $x=2\sin\theta,\ dx=2\cos\theta\ d\theta$ in the second integral. Then $\sqrt{4-x^2}=2\cos\theta$ and the second integral is an Arcsine. The first is a simple substitution.

63. $\displaystyle\int \frac{dx}{x^2+5} = \frac{1}{5}\sqrt{5}\text{Arctan}\,\frac{1}{5}x\sqrt{5}$

This integrand contains a sum of two squares. So let, $x=\sqrt{5}\tan\theta,\ dx=\sqrt{5}\sec^2\theta\ d\theta$. The integral becomes
$\displaystyle\int \frac{\sqrt{5}\sec^2\theta\ d\theta}{5\sec^2\theta} = \frac{\sqrt{5}}{5}\int d\theta$ and the result follows since $\theta = \text{Arctan}\,\frac{x}{\sqrt{5}}$, and $\frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$.

64. $\displaystyle\int \frac{dx}{4x^2+3} = \frac{1}{6}\sqrt{3}\text{Arctan}\,\frac{2}{3}\sqrt{3}x$

The integrand contains a sum of two squares, a^2+u^2 where $a=\sqrt{3}$ and $u=2x$. So let $2x=\sqrt{3}\tan\theta$, $2\ dx=\sqrt{3}\sec^2\theta\ d\theta$. The integral becomes
$\displaystyle\int \frac{(1/2)\sqrt{3}\sec^2\theta\ d\theta}{3\sec^2\theta} = \frac{\sqrt{3}}{6}\int d\theta$ and the result follows since $\theta = \text{Arctan}\,\frac{2x}{\sqrt{3}}$.

65. $\displaystyle\int \frac{dx}{x\sqrt{x^2-4}} = \frac{1}{2}\text{Arcsec}\,\frac{x}{2}$

The integrand contains a square root of a difference of two squares, $\sqrt{u^2-a^2}$ where $a=2$ and $u=x$. So let $x>2$ and $x=2\sec\theta$, $dx=2\sec\theta\ \ \tan\theta\ d\theta$. Moreover, $\sqrt{x^2-4}=2\tan\theta$. The integral becomes
$\displaystyle\int \frac{2\sec\theta\ \ \tan\theta\ d\theta}{(2\sec\theta)(2\tan\theta)} = \frac{1}{2}\int d\theta$ and the result follows since $\theta=\text{Arcsec}\,\frac{x}{2}$.

66. $\displaystyle\int \frac{dx}{x\sqrt{4x^2-9}} = \frac{1}{3}\text{Arcsec}\,\frac{2x}{3}$

The integrand contains a square root of a difference of two squares, $\sqrt{u^2-a^2}$ where $a=3$ and $u=2x$. So let $x>0$ and set $2x=3\sec\theta$, $2dx=3\sec\theta\ \ \tan\theta\ d\theta$. Moreover, $\sqrt{4x^2-9}=3\tan\theta$. The integral becomes
$\displaystyle\int \frac{(3/2)\sec\theta\ \cdot\tan\theta\ d\theta}{(3/2)\sec\theta\cdot 3\tan\theta} = \frac{1}{3}\int d\theta$ and the result follows since $\theta=\text{Arcsec}\,\frac{2x}{3}$.

67. $\displaystyle\int \frac{dx}{\sqrt{x^2+4}} = \ln\left|\frac{\sqrt{4+x^2}}{2}+\frac{x}{2}\right|$
$=\ln\left|\sqrt{4+x^2}+x\right|$, where the "missing" constants are absorbed by the constant of integration, C.

The integrand contains a square root of a sum of two squares, $\sqrt{u^2+a^2}$ where $a=2$ and $u=x$. Set $x=2\tan\theta$, $dx=2\sec^2\theta\ d\theta$. Moreover, $\sqrt{x^2+4}=2\sec\theta$. The integral becomes
$\displaystyle\int \frac{2\sec^2\theta\ d\theta}{(2\sec\theta)} = \int \sec\theta\ d\theta$ and the result follows from Example 386.

68. $\displaystyle\int \frac{dx}{\sqrt{4x^2+3}} = \frac{1}{2}\ln\left|\frac{\sqrt{4x^2+3}}{\sqrt{3}}+\frac{2x}{\sqrt{3}}\right|$
$=\ln\left|\sqrt{4x^2+3}+2x\right|$, where the "missing" constants are absorbed by the constant of integration, C.

The integrand contains a square root of a sum of two squares, $\sqrt{u^2+a^2}$ where $a=\sqrt{3}$ and $u=2x$. Set $2x=\sqrt{3}\tan\theta$, $2\ dx=\sqrt{3}\sec^2\theta\ d\theta$. Moreover, $\sqrt{4x^2+3}=\sqrt{3}\sec\theta$. The integral becomes
$\displaystyle\int \frac{(\sqrt{3}/2)\sec^2\theta\ d\theta}{\sqrt{3}\sec\theta} = (1/2)\int \sec\theta\ d\theta$ and the result follows from Example 386, once again.

69. $\displaystyle\int \frac{dx}{\sqrt{x^2-16}} = \ln\left|\frac{x}{4}+\frac{\sqrt{x^2-16}}{4}\right|$
$=\ln\left|x+\sqrt{x^2-16}\right|$, where the "missing" constants are absorbed by the constant of integration, C.

The integrand contains a square root of a difference of two squares, $\sqrt{u^2-a^2}$ where $a=4$ and $u=x$. Set $x=4\sec\theta$, $dx=4\sec\theta\ \ \tan\theta\ d\theta$. Moreover, $\sqrt{x^2-16}=4\tan\theta$. The integral becomes
$\displaystyle\int \frac{4\sec\theta\ \ \tan\theta\ d\theta}{4\tan\theta} = \int \sec\theta\ d\theta$ and the result follows from Example 386.

70. $\displaystyle\int \frac{e^x}{1+e^{2x}}\ dx = \text{Arctan}\,(e^x)$

Use a substitution here: Let $u=e^x$, $du=e^x\ dx$. The integral now looks like $\displaystyle\int \frac{1}{1+u^2}\ du = \text{Arctan}\,u$, where $u=e^x$.

71. $\displaystyle\int \frac{1}{x\sqrt{4x^2-1}}\ dx = \text{Arcsec}\,2x$

The integrand contains a square root of a difference of two squares, $\sqrt{u^2 - a^2}$ where $a = 1$ and $u = 2x$. So let $x > 0$ and set $2x = \sec\theta$, $2dx = \sec\theta \ \tan\theta \ d\theta$. Moreover, $\sqrt{4x^2 - 1} = \tan\theta$. The integral becomes
$\int \frac{(1/2)\sec\theta \ \cdot \tan\theta \ d\theta}{(1/2)\sec\theta \cdot \tan\theta} = \int d\theta$ and the result follows since $\theta = \text{Arcsec}\, 2x$.

72. $\int \frac{dx}{\sqrt{4x^2 - 9}} = \frac{1}{2}\ln\left|\frac{2x}{3} + \frac{\sqrt{4x^2 - 9}}{3}\right| = \ln\left|2x + \sqrt{4x^2 - 9}\right|$, where the "missing" constants are absorbed by the constant of integration, C.
The integrand contains a square root of a difference of two squares, $\sqrt{u^2 - a^2}$ where $a = 3$ and $u = 2x$. So let $x > 0$ and set $2x = 3\sec\theta$, $2\ dx = 3\sec\theta \ \tan\theta \ d\theta$. Moreover, $\sqrt{4x^2 - 9} = 3\tan\theta$. The integral becomes
$\int \frac{(3/2)\sec\theta \cdot \tan\theta \ d\theta}{3\tan\theta} = (1/2)\int \sec\theta \ d\theta$ and the result follows since $\int \sec\theta \ d\theta = \ln|\sec\theta + \tan\theta|$.

73. $\int e^{-3x} \ dx = -\frac{1}{3}e^{-3x}$

Let $u = -3x, \ du = -3 \ dx$. Then $dx = -du/3$.

74. $\int \frac{dx}{e^{2x}} = -\frac{1}{2}e^{-2x}$

Write the integrand as e^{-2x} and let $u = -2x$, $du = -2 \ dx$.

75. $\int (e^x - e^{-x})^2 \ dx = \frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x}$

Expand the expression and integrate term-by-term using the two preceding exercises.

76. $\int xe^{-x^2} \ dx = -\frac{1}{2}e^{-x^2}$

Let $u = -x^2, \ du = -2x \ dx$ so that $x \ dx = -du/2$.

77. $\int \frac{\sin\theta \ \mathrm{d}\theta}{\sqrt{1 - \cos\theta}} = 2\sqrt{1 - \cos\theta}$

Let $u = 1 - \cos\theta, \ du = \sin\theta \ d\theta$. We now have an easily integrable form.

78. $\int \frac{\cos\theta \ \mathrm{d}\theta}{\sqrt{2 - \sin^2\theta}} = \text{Arcsin}\left(\frac{1}{2}\sqrt{2}\sin\theta\right)$

Write $\theta = x$. Let $u = \sin x, \ du = \cos x \ dx$. The integral takes the form
$\int \frac{du}{\sqrt{2 - u^2}}$. Now set $u = \sqrt{2}\sin\theta$. (This is why we changed the name of the original variable to "x", so that we wouldn't get it confused with THIS θ). Then $du = \sqrt{2}\cos\theta \ d\theta$ and $\sqrt{2 - u^2} = \sqrt{2}\cos\theta$ and the rest of the integration is straightforward. (Note: If you want, you could set $u = \sin\theta$ immediately and proceed as above without first having to let $\theta = x$ etc.)

79. $\int \frac{e^{2x} \ dx}{1 + e^{2x}} = \frac{1}{2}\ln\left(1 + e^{2x}\right)$

Let $u = 1 + e^{2x}, \ du = 2e^{2x} \ dx$. Now, the integral gives a natural logarithm

80. $\int \frac{e^x \ dx}{1 + e^{2x}} = \text{Arctan}\ (e^x)$

Let $u = e^x, \ du = e^x \ dx$. Now, the integral is of the form
$\int \frac{du}{1 + u^2}$ and this gives an Arctangent.

81. $\int \frac{\cos\theta \ \mathrm{d}\theta}{2 + \sin^2\theta} = \frac{1}{2}\sqrt{2}\text{Arctan}\left(\frac{1}{2}\sqrt{2}\sin\theta\right)$

Write $\theta = x$. Let $u = \sin x,\ du = \cos x\ dx$. The integral takes the form $\int \frac{du}{2+u^2}$. Now set $u = \sqrt{2}\tan\theta$. (This is why we changed the name of the original variable to "x", so that we wouldn't get it confused with THIS θ). Then $du = \sqrt{2}\sec^2\theta\ d\theta$ and $2+u^2 = 2\sec^2\theta$ and the rest of the integration is straightforward. (Note: If you want, you could set $u = \sin\theta$ immediately and proceed as above without first having to let $\theta = x$ etc.)

82. $\int \sin^3 x \cos x\ dx = \frac{1}{4}\sin^4 x$

Let $u = \sin x,\ du = \cos x\ dx$.

83. $\int \cos^4 5x \sin 5x\ dx = -\frac{1}{25}\cos^5 5x$

Let $u = \cos 5x,\ du = -5\sin 5x\ dx$ or $\sin 5x = -du/5$. The rest is straightforward.

84. $\int (\cos\theta + \sin\theta)^2\ \mathrm{d}\theta = \theta - \cos^2\theta$

or, this can also be rewritten as $\theta + \sin^2\theta$

Expand and use the identities $\cos^2\theta + \sin^2\theta = 1$, along with $\sin 2\theta = 2\sin\theta\ \cos\theta$. Then use the substitution $u = 2x$, or if you prefer, let $u = \sin\theta$, etc.

85. $\int \sin^3 x\ dx = -\frac{1}{3}\sin^2 x\cos x - \frac{2}{3}\cos x$

This is the case m is even ($m = 0$) and n is odd ($n = 3$) in the text.

86. $\int \cos^3 2x\ dx = \frac{1}{6}\cos^2 2x \sin 2x + \frac{1}{3}\sin 2x$

Let $u = 2x$. The new integral is in the case where m is odd ($m = 3$) and n is even ($n = 0$) in the text.

87. $\int \sin^3 x \cos^2 x\ dx = -\frac{1}{5}\sin^2 x\cos^3 x - \frac{2}{15}\cos^3 x$

This is the case m is even ($m = 2$) and n is odd ($n = 3$) in the text. To get the polynomial in $\cos x$ simply use the identities $\sin^2 x = 1 - \cos^2 x$ whenever you see the $\sin^2 x-$term and expand and simplify.

88. $\int \cos^5 x\ dx = \frac{1}{5}\cos^4 x\sin x + \frac{4}{15}\cos^2 x\sin x + \frac{8}{15}\sin x$

This is the case m is odd ($m = 5$) and n is even ($n = 0$) in the text. To get the polynomial in $\sin x$ simply use the identities $\cos^2 x = 1 - \sin^2 x$ whenever you see a $\cos^2 x-$term and then expand and simplify.

89. $\int \sin^3 4\theta\cos^3 4\theta\ \mathrm{d}\theta = -\frac{1}{24}\sin^2 4\theta\cos^4 4\theta - \frac{1}{48}\cos^4 4\theta$

Let $u = 4\theta$. Then the new integral is in the case where m is odd ($m = 3$) and n is odd ($n = 3$) in the text.

90. $\int \frac{\cos^2 x\ dx}{\sin x} = \cos x + \ln|\csc x - \cot x|$

Write $cos^2 x = 1 - \sin^2 x$, break up the integrand into two parts, and use the fact that $\int \csc x\ dx = \ln|\csc x - \cot x|$.

91. $\int \frac{\cos^3 x\ dx}{\sin x} = \frac{1}{2}\cos^2 x + \ln|\sin x|$

Write $cos^2 x = 1 - \sin^2 x$, break up the integrand into two parts. In one, use the fact that
$\int \cot x\ dx = \ln|\sin x|$. In the other, use the substitution $u = \sin x$ in the other.

92. $\int \tan^2 x \sec^2 x\ dx = \frac{1}{3}\tan^3 x$

Let $u = \tan x,\ du = \sec^2 x\ dx$.

93. $\int \sec^2 x \tan^3 x\ dx = \frac{1}{4}\tan^4 x$

Let $u = \tan x,\ du = \sec^2 x\ dx$.

94. $\int \frac{\sin x\ dx}{\cos^3 x} = \frac{1}{2\cos^2 x}$

Let $u = \cos x,\ du = -\sin x\ dx$.

95. $\int \frac{\sin^2 x\ dx}{\cos^4 x} = \frac{1}{3}\tan^3 x$

The integrand is equal to $\tan^2 x\ \sec^2 x$. Now let $u = \tan x$.

96. $\int \sec^4 x\ dx = \frac{1}{3}\tan x\ \sec^2 x + \frac{2}{3}\tan x$

This is the case $m = 4,\ n = 0$ in the text. Note that $sec^2 x = 1 + \tan^2 x$. So, this answer is equivalent to $\tan x + \frac{\tan^3 x}{3}$ with the addition of a constant.

97. $\int \tan^2 x\ dx = \tan x\ -\ x$

The integrand is equal to $1 - \sec^2 x$. Now break up the integrand into two parts and integrate term-by-term.

98. $\int (1 + \cot\theta)^2\ \mathrm{d}\theta = -\cot\theta - \ln\left(1 + \cot^2\theta\right)$

Expand the integrand, use the identity $1 + \cot^2\theta = \csc^2\theta$ and integrate using the facts that $\int \csc^2 x\ dx = -\cot x$, and $\int \cot x\ dx = \ln|\sin x|$. Note that the second term may be simplified further using the fact that
$\ln\left(1 + \cot^2\theta\right) = \ln\csc^2\theta = -\ln\sin^2\theta = -2\ln\sin\theta$.

99. $\int \sec^4 x \tan^3 x\ dx = \frac{1}{6}\tan^4 x\ \sec^2 x + \frac{1}{12}\tan^4 x$

This is the case $m = 4,\ n = 3$ in the text.

100. $\int \csc^6 x\ dx = -\frac{1}{5}\csc^4 x\ \cot x - \frac{4}{15}\csc^2 x\ \cot x - \frac{8}{15}\cot x$

Use the same ideas as in the case $m = 6,\ n = 0$ in the secant/tangent case.

101. $\int \tan^3 x\ dx = \frac{1}{2}\tan^2 x - \frac{1}{2}\ln\left(1 + \tan^2 x\right)$

This is the case $m = 0,\ n = 3$ in the text.

102. $\int \frac{\cos^2 t\ dt}{\sin^6 t} = -\frac{1}{5}\csc^2 t\ \cot^3 t - \frac{2}{15}\cot^3 t$

The integrand is equal to $\cot^2 x\ \csc^4 x$, and this corresponds to the case $m = 4,\ n = 2$ in the secant/tangent case.

103. $\int \tan\theta\csc\theta\ \mathrm{d}\theta = \ln|\sec\theta + \tan\theta|$

The integrand is really $\sec\theta$ in disguise!

104. $\displaystyle\int \cos^2 4x\ dx = \frac{1}{8}\cos 4x \sin 4x + \frac{1}{2}x$

Use the identity $\cos^2 \square\ = \frac{1+\cos 2\square}{2}$, with $\square\ = 4x$. Then use a simple substitution $u = 8x$, and simplify your answer using the identity $\sin 8x = \sin(2 \cdot 4x) = 2\sin 4x\ \ \cos 4x$.

105. $\displaystyle\int (1+\cos\theta)^2\ \mathrm{d}\theta = \frac{3}{2}\theta + 2\sin\theta + \frac{1}{2}\cos\theta\sin\theta$

Expand the integrand, use the identity $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ and integrate term-by-term.

106. $\displaystyle\int (1-\sin x)^3\ dx = \frac{5}{2}x + \frac{11}{3}\cos x - \frac{3}{2}\cos x \sin x + \frac{1}{3}\sin^2 x \cos x$

Expand the integrand, and integrate term-by-term using the identity $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, and the case $m = 0,\ n = 3$ in the text.
Recall that $(1-\square\ )^3 = 1 - 3\square\ \ + 3\square\ ^2 - \square\ ^3$

107. $\displaystyle\int \sin^4 x\ dx = -\frac{1}{4}\sin^3 x \cos x - \frac{3}{8}\cos x \sin x + \frac{3}{8}x$

This is the case $m = 0,\ n = 4$ in the text.

108. $\displaystyle\int \sin^2 2x \cos^2 2x\ dx = -\frac{1}{8}\sin 2x \cos^3 2x + \frac{1}{16}\cos 2x \sin 2x + \frac{1}{8}x$

Let $u = 2x$ first. Then the new integral corresponds to the case $m = 2,\ n = 2$ in the text.

109. $\displaystyle\int \sin^4\theta \cos^2\theta\ \mathrm{d}\theta = -\frac{1}{6}\sin^3\theta\cos^3\theta - \frac{1}{8}\sin\theta\cos^3\theta + \frac{1}{16}\cos\theta\sin\theta + \frac{1}{16}\theta$

This is the case $m = 2,\ n = 4$ in the text.

110. $\displaystyle\int \cos^6 x\ dx = \frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\sin x \cos^3 x + \frac{5}{16}\cos x \sin x + \frac{5}{16}x$

This is the case $m = 6,\ n = 0$ in the text.

111. $\displaystyle\int \cos x \sin 2x\ dx = -\frac{1}{6}\cos 3x - \frac{1}{2}\cos x$

You can use either Table integration in a three-row problem or the identity $\cos A \sin B = \dfrac{1}{2}\sin(A+B) - \dfrac{1}{2}\sin(A-B)$ to find this integral.

112. $\displaystyle\int \sin x \cos 3x\ dx = -\frac{1}{8}\cos 4x + \frac{1}{4}\cos 2x$

You can use either Table integration in a three-row problem or the identity $\cos A \sin B = \dfrac{1}{2}\sin(A+B) - \dfrac{1}{2}\sin(A-B)$ to find this integral.

113. $\displaystyle\int \sin 2x \sin 3x\ dx = \frac{1}{2}\sin x - \frac{1}{10}\sin 5x$

You can use either Table integration in a three-row problem or the identity $\sin A \sin B = \dfrac{1}{2}\cos(A-B) - \dfrac{1}{2}\cos(A+B)$ to find this integral.

114. $\displaystyle\int \cos 2x \cos 4x\ dx = \frac{1}{4}\sin 2x + \frac{1}{12}\sin 6x$

You can use either Table integration in a three-row problem or the identity $\cos A \cos B = \dfrac{1}{2}\cos(A-B) + \dfrac{1}{2}\cos(A+B)$ to find this integral.

115. $\displaystyle\int \sin^2 2x \cos 3x\ dx = \frac{1}{6}\sin 3x - \frac{1}{4}\sin x - \frac{1}{28}\sin 7x$

Use the identity $\sin^2 \square = \frac{1-\cos 2\square}{2}$ with $\square = 2x$. Break up the integrand into two parts, and integrate using the substitution $u = 4x$ and the identity
$\cos A \cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$ to find the other integral.

116. $\displaystyle\int \sec x \csc x \; dx = \ln|\tan x|$

There are two VERY different ways of doing this one:

In the first proof we note the trigonometric identity (and this isn't obvious!),

$$\frac{\sec^2 x}{\tan x} = \frac{1}{\sin x \cos x} = \sec x \csc x,$$

so the result follows after using the substitution $u = \tan x$, $du = \sec^2 x \; dx$.

In the second proof we note that (and this isn't obvious either!)

$$\frac{1}{\sin x \cos x} = \frac{2}{\sin 2x} = 2\csc 2x.$$

Now use the substitution $u = 2x,\ du = 2dx$ and this new integral becomes
$2 \cdot \frac{1}{2}\displaystyle\int \csc u \; du = \ln|\csc u - \cot u|$. The answer is equivalent to
$\ln|\csc 2x - \cot 2x| + C$, because of the identity $1 - \cos 2x = 2\sin^2 x$.

117. $\displaystyle\int \frac{dx}{1-\cos x} = -\frac{1}{\tan \frac{1}{2}x} = -\cot\frac{x}{2}.$

Use the identity $1 - \cos 2\square = 2\sin^2 \square$, with $\square = \frac{x}{2}$. Then $\frac{1}{1-\cos x} = \frac{1}{2}\csc^2\frac{x}{2}$.
Let $u = x/2,\ du = dx/2$ and use the integral
$\int \csc^2 u \; du = -\cot u$ and simplify.

118. $\displaystyle\int \frac{dx}{\sqrt{2+2x-x^2}} = \text{Arcsin}\,\frac{1}{3}\sqrt{3}\,(x-1)$

First, complete the square to find $2 + 2x - x^2 = 3 - (x-1)^2$. Next, let $a = \sqrt{3},\ u = x - 1$. This integrand has a term of the form $\sqrt{a^2-u^2}$. So we use the trigonometric substitution
$u = x - 1 = \sqrt{3}\sin\theta$, $dx = \sqrt{3}\cos\theta \; d\theta$.
Furthermore, $\sqrt{2+2x-x^2} = \sqrt{3}\cos\theta$. So, the integral now takes the form
$\displaystyle\int \frac{\sqrt{3}\cos\theta \; d\theta}{\sqrt{3}\cos\theta} = \int d\theta = \theta$
where $\theta = \text{Arcsin}\,\frac{x-1}{\sqrt{3}}$ which is equivalent to the stated answer.

119. $\displaystyle\int \frac{dx}{\sqrt{1+4x-4x^2}} = \frac{1}{2}\text{Arcsin}\,\sqrt{2}\left(x-\frac{1}{2}\right)$

First, complete the square to find $1 + 4x - 4x^2 = 2 - (2x-1)^2$. Next, let $a = \sqrt{2},\ u = 2x - 1$. This integrand has a term of the form $\sqrt{a^2-u^2}$. So we use the trigonometric substitution

$$u = 2x - 1 = \sqrt{2}\sin\theta,\ 2dx = \sqrt{2}\cos\theta \; d\theta$$

or, $dx = \dfrac{\sqrt{2}}{2}\cos\theta \; d\theta$.

Furthermore, $\sqrt{1+4x-4x^2} = \sqrt{2}\cos\theta$. So, the integral now takes the form
$\displaystyle\frac{1}{2}\int \frac{\sqrt{2}\cos\theta \; d\theta}{\sqrt{2}\cos\theta} = \frac{1}{2}\int d\theta = \frac{\theta}{2}$
where $\theta = \text{Arcsin}\,\frac{2x-1}{\sqrt{2}}$ which is equivalent to the stated answer.

120. $\displaystyle\int \frac{dx}{\sqrt{2+6x-3x^2}} = \frac{1}{3}\sqrt{3}\text{Arcsin}\,\frac{1}{5}\sqrt{15}\,(x-1)$

This one is a little tricky: First, complete the square to find $2+6x-3x^2 = 5-3(x-1)^2$. But this is not exactly a difference of squares, yet! So we rewrite this as

$$5-3(x-1)^2 = 5-(\sqrt{3}x-\sqrt{3})^2,$$

and this is a difference of squares. Now let $a=\sqrt{5},\ u=\sqrt{3}x-\sqrt{3}$. We see that the integrand has a term of the form $\sqrt{a^2-u^2}$. So we use the trigonometric substitution
$u=\sqrt{3}x-\sqrt{3}=\sqrt{5}\sin\theta$,
$\sqrt{3}\ dx=\sqrt{5}\cos\theta\ d\theta$

or, $dx = \dfrac{\sqrt{5}}{\sqrt{3}}\cos\theta\ d\theta.$

Furthermore, $\sqrt{2+6x-3x^2}=\sqrt{5}\cos\theta$. So, the integral now takes the form
$$\int \frac{\frac{\sqrt{5}}{\sqrt{3}}\cos\theta\ d\theta}{\sqrt{5}\cos\theta} = \frac{1}{\sqrt{3}}\int d\theta = \frac{\theta}{\sqrt{3}}$$
where $\theta = \text{Arcsin}\,\frac{\sqrt{3}x-\sqrt{3}}{\sqrt{5}}$ which is equivalent to the stated answer.

121. $\displaystyle\int \frac{dx}{\sqrt{x^2+6x+13}} = \ln\left|\frac{\sqrt{x^2+6x+13}}{2}+\frac{x+3}{2}\right|$

First, complete the square to find $x^2+6x+13=(x+3)^2+4$. Next, let $a=2,\ u=x+3$. This integrand has a term of the form $\sqrt{a^2+u^2}$. So we use the trigonometric substitution

$u=x+3=2\tan\theta$,
$dx=2\sec^2\theta\ d\theta$.

Furthermore, $\sqrt{x^2+6x+13}=2\sec\theta$. So, the integral now takes the form

$$\int \frac{2\sec^2\theta\ d\theta}{2\sec\theta} = \int \sec\theta\ d\theta = \ln|\sec\theta+\tan\theta|,$$

where $\sec\theta = \text{Arcsec}\,\frac{\sqrt{x^2+6x+13}}{2}$
and $\tan\theta=\frac{x+3}{2}$ which is equivalent to the stated answer.

122. $\displaystyle\int \frac{dx}{2x^2-4x+6} = \frac{1}{4}\sqrt{2}\text{Arctan}\,\frac{1}{8}\,(4x-4)\sqrt{2}$

First, complete the square to find $2x^2-4x+6=2(x-1)^2+4$. The integral now looks like:

$$\int \frac{1}{2x^2-4x+6}\ dx = \int \frac{1}{2(x-1)^2+4}\ dx = \frac{1}{2}\int \frac{1}{(x-1)^2+2}\ dx.$$
Next, let $a=\sqrt{2},\ u=x-1$. The previous integrand has a term of the form a^2+u^2. So we use the trigonometric substitution

$u=x-1=\sqrt{2}\tan\theta$,
$dx=\sqrt{2}\sec^2\theta\ d\theta$.

Furthermore, $2x^2-4x+6=2\sec^2\theta$. So, the original integral now takes the form

$$\frac{1}{2}\int \frac{\sqrt{2}\sec^2\theta\ d\theta}{2\sec^2\theta} = \frac{\sqrt{2}}{4}\theta = \frac{\sqrt{2}}{4}\text{Arctan}\,\frac{x-1}{\sqrt{2}},$$

which is equivalent to the stated answer.

123. $\displaystyle\int \frac{dx}{(1-x)\sqrt{x^2-2x-3}} = -\frac{1}{2}\text{Arcsec}\,\frac{x-1}{2}$

First we complete the square so that $x^2-2x-3=(x-1)^2-4$. A trigonometric substitution is hard here: Let's try another approach...

Let $u = x-1, du = dx$. Then the integral becomes (note the minus sign)

$$-\int \frac{du}{u\sqrt{u^2-4}}.$$

Now we incorporate the number 4 into the square by factoring it out of the expression, thus:

$$u\sqrt{u^2-4} = 2u\sqrt{\left(\frac{u}{2}\right)^2-1}.$$

Now we use the substitution $v = \frac{u}{2},\ 2dv = du$. The integral in u now becomes

$$-\int \frac{2\ dv}{4v\sqrt{v^2-1}} = -\frac{1}{2}\int \frac{dv}{v\sqrt{v^2-1}} = -\frac{1}{2}\text{Arcsec}\, v,$$

according to Table 7.7 with $\square = v$. The answer follows after back-substitution.

124. $\displaystyle\int \frac{(2x+3)\ dx}{x^2+2x-3} = \frac{3}{4}\ln|x+3| + \frac{5}{4}\ln|x-1|$

Use partial fractions. The factors of the denominator are $(x+3)(x-1)$. You need to find two constants.

125. $\displaystyle\int \frac{(x+1)\ dx}{x^2+2x-3} = \frac{1}{2}\ln\left|x^2+2x-3\right|$

Let $u = x^2+2x-3,\ du = (2x+2)\ dx$ so that
$du = 2(x+1)\ dx$. Now the integral in u gives a natural logarithm.

Alternately, use partial fractions. The factors of the denominator are $(x+3)(x-1)$. You need to find the two constants.

126. $\displaystyle\int \frac{(x-1)\ dx}{4x^2-4x+2} = \frac{1}{8}\ln\left|4x^2-4x+2\right| - \frac{1}{4}\text{Arctan}\ (2x-1)$

The denominator is a Type II factor (it is irreducible) since $b^2-4ac = (-4)^2 - 4(4)(2) < 0$. So the expression is already in its partial fraction decomposition. So, the partial fractions method gives nothing.

So, complete the square in the denominator. This gives an integral of the form

$$\int \frac{(x-1)\ dx}{4x^2-4x+2} = \int \frac{(x-1)\ dx}{(2x-1)^2+1},$$

which can be evaluated using the trigonometric substitution,

$u = 2x-1,\ du = 2dx$ or $dx = du/2$. Solving for x we get
$x = \frac{u+1}{2}$, so $x-1 = \frac{u-1}{2}$. The $u-$integral looks like

$$\frac{1}{2}\int \frac{u-1}{1+u^2}\ du.$$

Break this integral into two parts and use the substitution

$$v = 1+u^2,\ dv = 2u\ du, \quad udu = dv/2$$

in the first, while the second one yields an Arctangent.

127. $\displaystyle\int \frac{x\ dx}{\sqrt{x^2-2x+2}} = \sqrt{x^2-2x+2} + \ln\left|\sqrt{x^2-2x+2}+x-1\right|$

Completing the square we see that $x^2-2x+2=(x-1)^2+1$. Next, we set

$x-1=\tan\theta, \quad dx=\sec^2\theta\ d\theta$
$x=1+\tan\theta,$
$\sqrt{x^2-2x+2}=\sqrt{(x-1)^2+1}=\sec\theta.$

The integral becomes

$$\int \frac{x\ dx}{\sqrt{x^2-2x+2}} = \int \frac{(1+\tan\theta)\sec^2\theta}{\sec\theta} d\theta$$

and this simplifies to

$$\int (\sec\theta+\sec\theta\tan\theta)\ d\theta = \ln|\sec\theta+\tan\theta|+\sec\theta.$$

Finally, use the back-substitutions $\sec\theta=\sqrt{x^2-2x+2}$ and $\tan\theta=x-1$.

128. $\displaystyle\int \frac{(4x+1)\ dx}{\sqrt{1+4x-4x^2}} = -\sqrt{1+4x-4x^2}+\frac{3}{2}\text{Arcsin}\sqrt{2}\left(x-\frac{1}{2}\right)$

Completing the square we see that $1+4x-4x^2=2-(2x-1)^2$. The integrand has a term of the form $\sqrt{a^2-u^2}$ where $a=\sqrt{2},\ u=2x-1$. So, we set

$2x-1=\sqrt{2}\sin\theta, \quad 2dx=\sqrt{2}\cos\theta\ d\theta$
$x=\frac{1+\sqrt{2}\sin\theta}{2},$
$4x+1=3+2\sqrt{2}\sin\theta,$
$\sqrt{1+4x-4x^2}=\sqrt{2}\cos\theta.$

The integral becomes

$$\int \frac{(4x+1)\ dx}{\sqrt{1+4x-4x^2}} = \int \left(\frac{3+2\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right)\frac{\sqrt{2}}{2}\cos\theta\ d\theta$$

which simplifies to

$$\frac{1}{2}\int (3+2\sqrt{2}\sin\theta)\ d\theta = \frac{3}{2}\theta-\sqrt{2}\cos\theta.$$

Finally, use the back-substitutions $\theta=\text{Arcsin}\,\frac{2x-1}{\sqrt{2}}$ and $\cos\theta=\frac{\sqrt{1+4x-4x^2}}{\sqrt{2}}$, to get it in a form equivalent to the stated answer.

129. $\displaystyle\int \frac{(3x-2)\ dx}{\sqrt{x^2+2x+3}} = 3\sqrt{x^2+2x+3}-5\ln\left|\frac{\sqrt{x^2+2x+3}}{\sqrt{2}}+\frac{x+1}{\sqrt{2}}\right|$

Completing the square we see that $x^2+2x+3=2+(x+1)^2$. The integrand has a term of the form $\sqrt{a^2+u^2}$ where $a=\sqrt{2},\ u=x+1$. So, we set

$x+1=\sqrt{2}\tan\theta, \quad dx=\sqrt{2}\sec^2\theta\ d\theta$
$x=\sqrt{2}\tan\theta-1,$
$3x-2=3\sqrt{2}\tan\theta-5=3\sqrt{2}\tan\theta-5,$
$\sqrt{x^2+2x+3}=\sqrt{2}\sec\theta.$

The integral becomes

$$\int \frac{(3x-2)\ dx}{\sqrt{x^2+2x+3}} = \int \left(\frac{3\sqrt{2}\tan\theta-5}{\sqrt{2}\sec\theta}\right)\sqrt{2}\sec^2\theta\ d\theta$$

which simplifies to

$$3\sqrt{2}\int \sec\theta\tan\theta\, d\theta - 5\int \sec\theta\, d\theta = 3\sqrt{2}\sec\theta - 5\ln|\sec\theta + \tan\theta|.$$

Finally, use the back-substitutions $\sec\theta = \frac{\sqrt{x^2+2x+3}}{\sqrt{2}}$, and $\tan\theta = \frac{x+1}{\sqrt{2}}$, to get it in a form equivalent to the stated answer.

130. $\displaystyle\int \frac{e^x\, dx}{e^{2x} + 2e^x + 3} = \frac{1}{2}\sqrt{2}\text{Arctan}\,\frac{1}{4}\left(2e^x + 2\right)\sqrt{2}$

Let $u = e^x$, $du = e^x\, dx$. The integral is now a rational function in u on which we can use partial fractions. The denominator is irreducible, since $b^2 - 4ac = 4 - 4(1)(3) < 0$. You need to find two constants.

131. $\displaystyle\int \frac{x^2\, dx}{x^2 + x - 6} = x - \frac{9}{5}\ln|x+3| + \frac{4}{5}\ln|x-2|$

Use long division first, then use partial fractions. The factors of the denominator are $x^2 + x - 6 = (x+3)(x-2)$. You need to find two constants.

132. $\displaystyle\int \frac{(x+2)\, dx}{x^2 + x} = 2\ln|x| - \ln|1+x|$

Use partial fractions. The factors of the denominator are $x^2 + x = x(x+1)$. You need to find two constants.

133. $\displaystyle\int \frac{(x^3+x^2)\, dx}{x^2 - 3x + 2} = \frac{1}{2}x^2 + 4x - 2\ln|x-1| + 12\ln|x-2|$

Use long division first. Then use partial fractions. The factors of the denominator are $x^2 - 3x + 2 = (x-1)(x-2)$. You need to find two constants.

134. $\displaystyle\int \frac{dx}{x^3 - x} = -\ln|x| + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|1+x|$

Use partial fractions. The factors of the denominator are $x^3 - x = x(x^2-1) = x(x-1)(x+1)$. You need to find three constants.

135. $\displaystyle\int \frac{(x-3)\, dx}{x^3 + 3x^2 + 2x} = -\frac{3}{2}\ln|x| - \frac{5}{2}\ln|x+2| + 4\ln|1+x|$

Use partial fractions. The factors of the denominator are $x^3 + 3x^2 + 2x = x(x^2+3x+2) = x(x+1)(x+2)$. You need to find three constants.

136. $\displaystyle\int \frac{(x^3+1)\, dx}{x^3 - x^2} = x + \frac{1}{x} - \ln|x| + 2\ln|x-1|$

Use partial fractions. The factors of the denominator are $x^3 - x^2 = x^2(x-1)$. You need to find three constants.

137. $\displaystyle\int \frac{x\, dx}{(x+1)^2} = \frac{1}{1+x} + \ln|1+x|$

Use partial fractions.

138. $\displaystyle\int \frac{(x+2)\, dx}{x^2 - 4x + 4} = -\frac{4}{x-2} + \ln|x-2|$

Use partial fractions. The factors of the denominator are $x^2 - 4x + 4 = (x-2)^2$. You need to find two constants.

139. $\displaystyle\int \frac{(3x+2)\, dx}{x^3 - 2x^2 + x} = 2\ln|x| - \frac{5}{x-1} - 2\ln|x-1|$

Use partial fractions. Note that $x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x-1)^2$. There are four constants to be found here!

140. $\displaystyle\int \frac{8\, dx}{x^4 - 2x^3} = \frac{2}{x^2} + \frac{2}{x} - \ln|x| + \ln|x-2|$

Use partial fractions. Note that $x^4 - 2x^3 = x^3(x-2)$. There are four constants to be found here!

141. $\displaystyle\int \frac{dx}{(x^2-1)^2} = -\frac{1}{4(x-1)} - \frac{1}{4}\ln|x-1| - \frac{1}{4(1+x)} + \frac{1}{4}\ln|1+x|$

Use partial fractions. Note that $(x^2-1)^2 = (x-1)^2(x+1)^2$.

142. $\displaystyle\int \frac{(1-x^3)\ dx}{x(x^2+1)} = -x + \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) + \operatorname{Arctan} x$

Use long division first, then use partial fractions.

143. $\displaystyle\int \frac{(x-1)\ dx}{(x+1)(x^2+1)} = -\ln|1+x| + \frac{1}{2}\ln\left(x^2+1\right)$

Use partial fractions.

144. $\displaystyle\int \frac{4x\ dx}{x^4-1} = \ln|x-1| + \ln|1+x| - \ln\left(x^2+1\right)$

Note that $x^4-1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$. Use partial fractions.

145. $\displaystyle\int \frac{3(x+1)\ dx}{x^3-1} = 2\ln|x-1| - \ln\left(x^2+x+1\right)$

Note that $x^3-1 = (x-1)(x^2+x+1)$. Use partial fractions.

146. $\displaystyle\int \frac{(x^4+x)\ dx}{x^4-4} = \frac{1}{4}\ln|x-2| - \frac{1}{12}\ln|x+2| - \frac{1}{12}\ln(x^2+2) + \frac{\sqrt{2}}{3}\operatorname{Arctan}\frac{x\sqrt{2}}{2}$

Use long division first, then use partial fractions.

147. $\displaystyle\int \frac{x^2\ dx}{(x^2+1)(x^2+2)} = -\operatorname{Arctan} x + \sqrt{2}\operatorname{Arctan}\frac{1}{2}\sqrt{2}x$

The factors are $(x^2+2)(x^2+1)$, both irreducible. Four constants need to be found. This is where the Arctangents come from!

148. $\displaystyle\int \frac{3\ dx}{x^4+5x^2+4} = -\frac{1}{2}\operatorname{Arctan}\frac{1}{2}x + \operatorname{Arctan} x$

The factors are $(x^2+4)(x^2+1)$, both irreducible. Four constants need to be found. This is where the Arctangents come from!

149. $\displaystyle\int \frac{(x-1)\ dx}{(x^2+1)(x^2-2x+3)} = -\frac{1}{2}\operatorname{Arctan} x + \frac{1}{4}\sqrt{2}\operatorname{Arctan}\frac{1}{4}(2x-2)\sqrt{2}$

Use partial fractions. Watch out, as both factors in the denominator are Type II.

150. $\displaystyle\int \frac{x^3\ dx}{(x^2+4)^2} = \frac{2}{x^2+4} + \frac{1}{2}\ln\left(x^2+4\right)$

Use partial fractions.

151. $\displaystyle\int \frac{(x^4+1)\ dx}{x(x^2+1)^2} = \ln|x| + \frac{1}{x^2+1}$

Use partial fractions.

152. $\displaystyle\int \frac{(x^2+1)\ dx}{(x^2-2x+3)^2} = -\frac{1}{x^2-2x+3} + \frac{1}{2}\sqrt{2}\operatorname{Arctan}\frac{1}{4}(2x-2)\sqrt{2}$

Use partial fractions. Note that $(x^2-2x+3)^2$ is irreducible (Type II). Now you have to find the four constants!

153. $\displaystyle\int \frac{x\ dx}{\sqrt{x+1}} = -2\sqrt{x+1} + \frac{2}{3}\left(\sqrt{x+1}\right)^3$

Let $u = x+1,\ du = dx$. Then $x = u-1$, and the integral becomes easy.

154. $\displaystyle\int x\sqrt{x-a}\ dx = \frac{2}{5}\left(\sqrt{x-a}\right)^5 + \frac{2}{3}\left(\sqrt{x-a}\right)^3 a$

Let $u = x-a,\ du = dx$. Then $x = u+a$, and the integral becomes easy.

155. $\displaystyle\int \frac{\sqrt{x+2}}{x+3}\,dx = 2\sqrt{x+2} - 2\text{Arctan}\,\sqrt{x+2}$

Let $u = \sqrt{x+2},\ u^2 = x+2$. Then $2u\ du = dx$ and $x = u^2 - 2$ which means that $x + 3 = u^2 + 1$. The integral takes the form $\int \frac{2u^2\ du}{1+u^2}$. This one can be evaluated using a long division and two simple integrations.

156. $\displaystyle\int \frac{dx}{x\sqrt{x-1}} = 2\text{Arctan}\,\sqrt{x-1}$

Let $u = \sqrt{x-1},\ u^2 = x-1$. Then $2u\ du = dx$ and so $x = 1+u^2$. The integral takes the form $\int \frac{2u\ du}{u(1+u^2)}$ which is an arctangent function...

157. $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a}\ln\left|\frac{a}{x} - \frac{\sqrt{a^2-x^2}}{x}\right|.$

Let $x = a\sin\theta,\ dx = \ a\cos\theta\ d\theta$. Then $\sqrt{a^2-x^2} = a\cos\theta$. After some simplification we find $a^{-1}\int \csc\theta\ d\theta = a^{-1}\ln|\csc\theta - \cot\theta|$. Finally, $\csc\theta = \dfrac{a}{x}$, $\cot\theta = \dfrac{\sqrt{a^2-x^2}}{x}$.

158. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{1}{a^2x}\sqrt{a^2-x^2}$

Let $x = a\sin\theta,\ dx = \ a\cos\theta\ d\theta$. Then $sqrta^2 - x^2 = a\cos\theta$. After some simplification we find $a^{-2}\int \csc^2\theta\ d\theta = -a^{-2}\cot\theta$.

159. $\displaystyle\int x^3\sqrt{x^2+a^2}\ dx = \frac{1}{5}x^2\left(\sqrt{x^2+a^2}\right)^3 - \frac{2}{15}a^2\left(\sqrt{x^2+a^2}\right)^3$

Let $x = a\tan\theta,\ dx = a\sec^2\theta\ d\theta$. Then $\sqrt{x^2+a^2} = a\sec\theta$. After some simplification you're left with an integral with an integrand equal to $\sec^2\theta\ \tan^3\theta$. Use Example 390.

160. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+a^2}} = -\frac{1}{a^2x}\sqrt{x^2+a^2}$

Let $x = a\tan\theta,\ dx = a\sec^2\theta\ d\theta$. Then $x^2 + a^2 = a^2\sec^2\theta$. After some simplification you're left with an integral with an integrand equal to $\csc\theta\ \cot\theta$. Its value is a cosecant function. Finally, use the fact that, in this case, $\csc\theta = \dfrac{\sqrt{x^2+a^2}}{x}$

161. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x + \sqrt{x^2+a^2}\right|$

Let $x = a\tan\theta,\ dx = a\sec^2\theta\ d\theta$. Then $x^2 + a^2 = a^2\sec^2\theta$. After some simplification you're left with an integral of the form in Example 386.

162. $\displaystyle\int \frac{x^2\ dx}{\sqrt{x^2+a^2}} = \frac{1}{2}x\sqrt{x^2+a^2} - \frac{1}{2}a^2\ln\left|x + \sqrt{x^2+a^2}\right|$

Let $x = a\tan\theta,\ dx = a\sec^2\theta\ d\theta$. Then $x^2 + a^2 = a^2\sec^2\theta$. After some simplification you're left with an integral of the form in Example 388.

163. $\displaystyle\int \frac{x^2\ dx}{(x^2+a^2)^2} = -\frac{1}{2}\frac{x}{x^2+a^2} + \frac{1}{2a}\text{Arctan}\,\frac{x}{a}$

Let $x = a\tan\theta,\ dx = a\sec^2\theta\ d\theta$. Then $x^2 + a^2 = a^2\sec^2\theta$. After some simplification you're left with the integral of the square of a sine function...

164. $\displaystyle\int x\cos x\ dx = \cos x + x\sin x$

Use Table integration

165. $\displaystyle\int x\sin x\ dx = \sin x - x\cos x$

Use Table integration

166. $\displaystyle\int x\sec^2 x\ dx = x\tan x + \ln|\cos x|$

Use Integration by Parts: Let $u = x,\ dv = \sec^2 x\ dx$. No need to use Table integration here.

167. $\displaystyle\int x\sec x\tan x\ dx = \ x\sec x - \ln|\sec x + \tan x|$

Use Integration by Parts: Let $u = x,\ dv = \sec x\tan x\ dx$. No need to use Table integration here.

168. $\displaystyle\int x^2 e^x\ dx = x^2e^x - 2xe^x + 2e^x$

Use Table integration

169. $\displaystyle\int x^4\ln x\ dx = \frac{1}{5}x^5\ln x - \frac{1}{25}x^5$

Use Integration by Parts: Let $u = \ln x,\ dv = x^4\ dx$. No need to use Table integration here.

170. $\displaystyle\int x^3e^{x^2}\ dx = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2}$

Write the integrand as $x^3e^{x^2} = x^2\cdot xe^{x^2}$. Then use Integration by Parts with $u = x^2,\ dv = xe^{x^2}\ dx$. Use the substitution $v = x^2$ in the remaining integral.

171. $\displaystyle\int \sin^{-1} x\ dx = x\text{Arcsin}\, x + \sqrt{(1-x^2)}$

Use Integration by Parts: Let $u = \text{Arctan}\, x,\ dv = dx$, followed by the substitution $u = 1 + x^2$, etc.

172. $\displaystyle\int \tan^{-1} x\ dx = x\text{Arctan}\, x - \frac{1}{2}\ln\left(x^2+1\right)$

Use Integration by Parts: Let $u = \text{Arctan}\, x,\ dv = dx$, followed by the substitution $u = 1 + x^2$, etc.

173. $\displaystyle\int (x-1)^2\sin x\ dx = \cos x - 2\sin x + 2x\cos x - x^2\cos x + 2x\sin x$

Use Table integration

174. $\displaystyle\int \sqrt{x^2-a^2}\ dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\ln\left|x + \sqrt{x^2-a^2}\right|$

Let $x = a\sec\theta,\ dx = a\sec\theta\ \tan\theta\ d\theta$. Then $\sqrt{x^2-a^2} = a\tan\theta$, etc.

175. $\displaystyle\int \sqrt{x^2+a^2}\ dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{1}{2}a^2\ln\left|x + \sqrt{x^2+a^2}\right|$

Let $x = a\tan\theta,\ dx = a\sec^2\theta\ d\theta$. Then $\sqrt{x^2+a^2} = a\sec\theta$, etc.

176. $\displaystyle\int \frac{x^2\ dx}{\sqrt{x^2-a^2}} = \frac{1}{2}x\sqrt{x^2-a^2} + \frac{1}{2}a^2\ln\left|x + \sqrt{x^2-a^2}\right|$

Let $x = a\sec\theta,\ dx = a\sec\theta\ \tan\theta\ d\theta$. Then $\sqrt{x^2-a^2} = a\tan\theta$, etc.

177. $\displaystyle\int e^{2x}\sin 3x\ dx = -\frac{3}{13}e^{2x}\cos 3x + \frac{2}{13}e^{2x}\sin 3x$

Use Table integration

178. $\displaystyle\int e^{-x}\cos x\ dx = -\frac{1}{2}e^{-x}\cos x + \frac{1}{2}e^{-x}\sin x$

Use Table integration

179. $\displaystyle\int \sin 3x \cos 2x \; dx = -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x$

Use a trig.. identity ... the one for $\sin A \cos B$, with $A = 3x,\; B = 2x$.

180. $\displaystyle\int_0^{\frac{\pi}{8}} \cos^3(2x)\sin(2x)\; dx = \frac{3}{32}$

Let $u = 2x$ first, $du = 2dx$, and follow this by the substitution $v = \cos u, dv = -\sin u \; du$ which allows for an easy calculation of an antiderivative.

181. $\displaystyle\int_1^4 \frac{2^{\sqrt{x}}}{2\sqrt{x}}\; dx = \frac{2}{\ln 2}$

Let $u = \sqrt{x}$. The result follows easily.

182. $\displaystyle\int_0^{\infty} x^3 e^{-2x}\; dx = \frac{3}{8}$

Use Table integration to find an antiderivative and then use L'Hospital's Rule (three times!).

183. $\displaystyle\int_{-\infty}^{+\infty} e^{-|x|}\; dx = 2$

Divide this integral into two parts, one where $x \geq 0$ (so that $|x| = x$), and one where $x < 0$ (so that $|x| = -x$). Then $\displaystyle\int_{-\infty}^{+\infty} e^{-|x|}\; dx = \int_{-\infty}^{0} e^{x}\; dx + \int_0^{\infty} e^{-x}\; dx$ and the integrals are defined by a limit.

184. $\displaystyle\int_0^{\infty} \frac{4x}{1+x^4}\; dx = \pi$

Let $u = x^2,\; du = 2x\; dx$. The integral becomes an Arctangent.

185. $\displaystyle\int_{-1}^{1} x^2 \cos(n\pi x)\; dx = \frac{4\cos n\pi}{n^2\pi^2}$, when $n \geq 1$, is an integer. Use Table integration.

186. $\frac{1}{2}\int_{-2}^{2} x^2 \sin\left(\frac{n\pi x}{2}\right)\; dx = 0$, when $n \geq 1$, is an integer. Use Table integration.

187. $\frac{1}{L}\int_{-L}^{L} (1-x)\sin\left(\frac{n\pi x}{L}\right)\; dx = 2L\frac{\cos n\pi}{n\pi}$, when $n \geq 1,\; L \neq 0$. Use Table integration.

188. $\displaystyle\int_0^2 (x^3+1)\cos\left(\frac{n\pi x}{2}\right)\; dx = 6\frac{8n^2\pi^2\cos n\pi - 16\cos n\pi + 16}{n^4\pi^4}$, when $n \geq 1$, is an integer. Use Table integration.

189. $\displaystyle\int_{-1}^{1} (2x+1)\cos(n\pi x)\; dx = \frac{2}{n\pi}\sin n\pi = 0$, when $n \geq 1$, is an integer. Use Table integration.

190. $\displaystyle\frac{1}{L}\int_{-L}^{L} \sin x\; \cos\left(\frac{n\pi x}{L}\right)\; dx = 0$, when $n \geq 1$, is an integer and $L \neq 0$. Use Table integration.

191. Total demand over 10 years is

$$\int_0^{10} 500\left(20 + t\; e^{-0.1t}\right)\; dt = \int_0^{10} 10000\; dt + 500\int_0^{10} t\; e^{-0.1t}\; dt.$$

Now integrating by parts

$$\int t\; e^{-0.1t}\; dt = -10t\; e^{-0.1t} + 10\int e^{-0.1t}\; dt = -10t\; e^{-0.1t} + 10\left(-10e^{-0.1t}\right).$$

Thus total demand $= \left[10,000t + 500\{-10t\; e^{-0.1t} + 10(-10e^{-0.1t})\}\right]_0^{10} = \left[10,000t - 5000te^{-0.1t} - 50,000e^{-0.1t}\right]_0^{10} = 100,000 - 50,000e^{-1} - 50,000e^{-1} - (0 - 0 - 50,000) = 150,000 - 100,000e^{-1} = 113212.1 \approx 113212$ units.

192. (a) Use partial fractions.

$$\frac{1}{y(y-10)} = \frac{A}{y} + \frac{B}{10-y} = \frac{A(10-y)+By}{y(10-y)}$$

If $y = 0$, then $10A = 1$, so $A = \frac{1}{10}$. If $y = 10$, then $10B = 1$, and $B = \frac{1}{10}$. Therefore,

$$\int \frac{1}{y(y-10)} dy = \frac{1}{10} \int \frac{dy}{y} + \frac{1}{10} \int \frac{dy}{10-y} dy$$

$$= \frac{1}{10}\ ln|y| - \frac{1}{10}\ ln|10-y| + C = \frac{1}{10}\ ln\left|\frac{y}{10-y}\right| + C$$

Thus

$$t = \frac{25}{10}\ ln\left|\frac{y}{10-y}\right| + C$$

When $t = 0$, $y = 1$, so $0 = 2.5ln\frac{1}{9}+C = 2.5(ln\ 1 - ln\ 9)+C = -2.5ln\ 9+C$. Thus $C = 2.5ln\ 9$ and

$$t = 2.5ln\left|\frac{y}{10-y}\right| + 2.5ln\ 9 = 2.5ln\left|\frac{9y}{10-y}\right|$$

(b) When $y = 4$, $t = 2.5ln\ \frac{4\times 9}{6} = 4.479$ hours.

(c) From (a), $\frac{t}{2.5} = ln\ \frac{9y}{10-y}$, so $e^{\frac{t}{2.5}} = \frac{9y}{10-y}$, and $(10-y)e^{0.4t} = 9y$, so $10e^{0.4t} = 9y + ye^{0.4t} = y(9 + e^{0.4t})$. Thus

$$y = \frac{10e^{0.4t}}{9+e^{0.4t}} = \frac{10}{1+e^{-0.4t}}$$

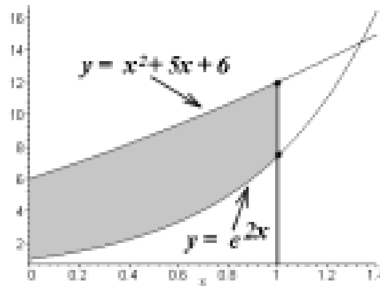
(d) At $t = 10$, $y = \frac{10}{1+9e^{-4}} = 8.58$ gm.

Chapter 9

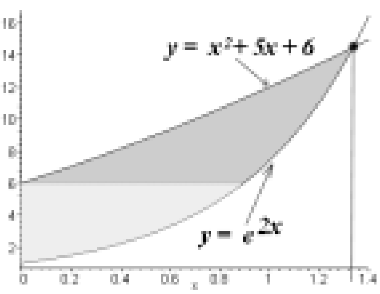
Solutions

9.1

9.2 Exercise Sets 47, 48



1. Vertical slice area $= (0 - (x^2 - 1))\ dx = (1 - x^2)\ dx$.
2. Horizontal slice area $\sqrt{y+1}\ dy$.
3. Vertical slice area $= ((x^2 + 5x + 6) - (e^{2x}))\ dx = (x^2 + 5x + 6 - e^{2x})\ dx$. Note that e^{2x} is smaller than $x^2 + 5x + 6$ on this interval. See the figure in the margin, on the left.
4. Sketch the region bounded by these curves. You should get a region like the one below:



Now, using Newton's Method with $x_0 = 1.5$ as an initial estimate, $n = 3$, and $f(x) = x^2 + 5x + 6 - e^{2x}$, we obtain the approximate value of the zero of f as 1.3358. The common value of these curves at this point is given by $e^{2(1.3358)} \approx$ 14.46. This represents the point of intersection of the curves $x^2 + 5x + 6$ and e^{2x}, in the interval $[0, 2]$. Beyond $x = 2$ we see that these curves get further apart so they cannot intersect once again. Since we are dealing with horizontal slices we need to write down the inverse function of each of these functions. For example, the inverse function of $y = x^2 + 5x + 6$ is given by solving for x in terms of y using the quadratic formula. This gives

$$x = \frac{-5 \pm \sqrt{1+4y}}{2}.$$

Since $x \geq 0$ here, we must choose the +-sign. On the other hand, the inverse function of the function whose values are $y = e^{2x}$ is simply given by $x = (\ln y)/2$.

So, the area of a typical horizontal slice in the darker region above is given by

$$\left(\frac{\ln y}{2} - \frac{-5+\sqrt{1+4y}}{2}\right)\ dy,$$

and this formula is valid provided $6 \le y \le 14.46$.
If the horizontal slice is in the lighter area above, then its area is given by

$$\left(\frac{\ln y}{2} - 0\right)\ dy = \left(\frac{\ln y}{2}\right)\ dy,$$

and this formula is valid whenever $0 \le y \le 6$.
As a check, note that both slice formulae agree when $y = 6$.

5. The horizontal line $y = 5$ intersects with the graph of $y = e^{2x}$ at the point $P \equiv (\frac{\ln 5}{2}, 5)$, approxmately $(0.8047, 5)$. Draw a vertical line through P. The area of a typical vertical slice on the left of this line is

$$((x^2+5x+6)-5)\ dx \equiv (x^2+5x+1)\ dx.$$

On the right of this line we have

$$(x^2+5x+6-e^{2x})\ dx$$

instead.

hugeExercise Set 48

1. Area $= \displaystyle\int_{-1}^{1} (1-x^2)\ dx = \frac{4}{3}$.
2. Area $= \displaystyle\int_{-2}^{2} (4-x^2)\ dx = \left(4x - \frac{x^3}{3}\right)\Bigg|_{-2}^{2} = \frac{32}{3}$.
3. Area $= \displaystyle\int_{0}^{1} (x^2+5x+6-e^{2x})dx = \left(\frac{x^3}{3} + \frac{5}{2}x^2 + 6x - \frac{e^{2x}}{2}\right)\Bigg|_{0}^{1} = \frac{28}{3} - \frac{e^2}{2}$ ≈ 5.63881.
4. Area $= \displaystyle\int_{0}^{1.3358} (x^2+5x+6-e^{2x})\ dx \approx 6.539$.
5. Area $= \displaystyle\int_{0}^{1} ye^y dy = (ye^y - e^y)\Big|_{0}^{1} = 1$.
6. $\pi^2 - 4$. This curve lies above the $x-$axis because $\sin x \ge 0$ for $0 \le x \le \pi$. It follows that $x^2 \sin x \ge 0$ for $0 \le x \le \pi$, and so the area is given by the definite integral
$$\text{Area} = \int_0^{\pi} x^2 \sin x\ dx = \pi^2 - 4,$$
where the Table method of Integration by Parts is used to evaluate it. In particular, we note that an antiderivative is given by
$$\int^x t^2 \sin t\ dt = -x^2 \cos x + 2x \sin x + 2\cos x.$$
7. Area $= \displaystyle\int_{0}^{\pi} \cos^2 x \sin x\ dx = -\frac{\cos^3 x}{3}\Big|_{0}^{\pi} = \frac{2}{3}$.

8. Using the Table method of Integration by Parts (since this is a three-row problem), we find

$$\int \sin 3x \cdot \cos 5x \; dx = \frac{5}{16} \sin 5x \cdot \sin 3x + \frac{3}{16} \cos 3x \cdot \cos 5x + C.$$

Alternatively, this integral can be computed as follows:

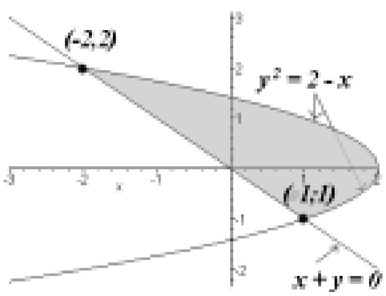
$$\int \sin 3x \cdot \cos 5x \; dx = \int \frac{1}{2}(\sin 8x - \sin 2x) \; dx = -\frac{1}{16} \cos 8x + \frac{1}{4} \cos 2x + C.$$

(Don't be fooled by its different look! This is the same answer as the above.) Notice that, for x in the interval $[\pi/10, 3\pi/10]$, $3x$ is in $[3\pi/10, 9\pi/10]$ and hence $\sin 3x$ is positive. However, for the same range of x, $5x$ is in $[\pi/2, 3\pi/2]$ and hence $\cos 3x$ is negative or zero. Hence the area of the region is the absolute value of

$$\begin{aligned} \int_{\pi/10}^{3\pi/10} \sin 3x \cdot \cos 5x \; dx &= \left(\frac{5}{16} \sin 5x \cdot \sin 3x + \frac{3}{16} \cos 3x \cdot \cos 5x \right) \Big|_{\pi/10}^{3\pi/10} \\ &= -\frac{5}{16} \left(\sin \frac{9}{10}\pi + \sin \frac{3}{10}\pi \right) = -\frac{5\sqrt{5}}{32} \approx -0.35. \end{aligned}$$

Here we use the facts that $\cos \frac{\pi}{2} = 0$, $\cos \frac{3\pi}{2} = 0$, $\sin \frac{\pi}{2} = 1$ and $\sin \frac{3\pi}{2} = -1$. It turns out that $\sin \frac{9}{10}\pi + \sin \frac{3}{10}\pi = \frac{\sqrt{5}}{2}$, which is very hard to prove!

9. $\frac{9}{2}$. Refer to the graph below:



The points of intersection of these two graphs are given by setting $y = -x$ into the expression $x + y^2 = 2$ and solving for x. This gives the two points, $x = 1$ and $x = -2$. Note that if we use vertical slices we will need two integrals. Solving for x in terms of y gives $x = -y$ and $x = 2 - y^2$ and the limits of integration are then $y = -1$ and $y = 2$. The coordinates of the endpoints of a typical horizontal slice are given by $(-y, y)$ and $(2 - y^2, y)$. So, the corresponding integral is given by

$$\text{Area} \;=\; \int_{-1}^{2} (2 - y^2 + y) \; dy = \frac{9}{2}.$$

10. The required area is

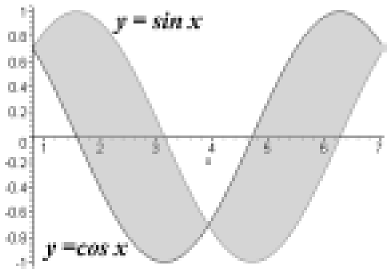
$$\int_{-2}^{2} (y^2 - (y - 5)) \; dy = \left(\frac{y^3}{3} - \frac{y^2}{2} + 5y \right) \Big|_{-2}^{2} = \frac{76}{3}.$$

11. 4 units. Note the symmetry: Since, f is an *even function*, (see Chapter 5), its graph over the interval $[-\pi, \pi]$ is symmetric with respect to the $y-$axis and so, since f is "V"-shaped and positive, the area is given by

$$\text{Area} = 2 \times (\textit{area to the right of } x = 0),$$

and this gives

$$\text{Area} \;=\; 2 \int_0^{\pi} (\sin x) \; dx = 4 \; \textit{units}.$$



12. $4\sqrt{2}$. The graph on the right represents the two curves over the interval $[\frac{\pi}{4}, \frac{9\pi}{4}]$: Using the symmetry in the graph we see that

$$\text{Area} = 2\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\ dx = 4\sqrt{2},$$

since $\cos\dfrac{5\pi}{4} = -\dfrac{\sqrt{2}}{2},\ \sin\dfrac{5\pi}{4} = -\dfrac{\sqrt{2}}{2}$.

9.3 Exercise Set 49

1. Using a vertical slice: $\int_0^1 \pi x^2\ dx$;
   using a horizontal slice $\int_0^1 (1-y)\cdot 2\pi y\ dy$.
2. Using a vertical slice: $\int_0^1 (x - x^2)\cdot 2\pi x\ dx$;
   using a horizontal slice: $\int_0^1 \pi(y - y^2)\ dy$.
3. Using a vertical slice: $\int_0^1 3\pi\ x^2\ dx$; (we do not use horizontal slices because this method is too complicated for the present problem.)
4. Using a vertical slice: $\int_0^2 2x\cdot 2\pi x\ dx$;
   using a horizontal slice: $\int_0^1 \pi(2^2 - (y/2)^2)\ dy$.
5. Using a vertical slice: $\int_0^1 (2x - x)\cdot 2\pi x\ dx = \int_0^1 2\pi x^2\ dx$.
   Using a horizontal slice:
   $$\frac{3\pi}{4}\int_0^1 y^2\ dy + \int_1^2 \pi\left(1 - \frac{y^2}{4}\right)\ dy.$$
6. $\pi/3$; $\pi/6$; 8π; $\frac{32}{3}\pi$; $\frac{2}{3}\pi$.

9.4 Exercise Set 50

1. 2. Since $y' = 0$, it follows that $L = \int_0^2 \sqrt{1}\, dx = 2$.
2. $4\sqrt{2}$. Since $y' = 1$, it follows that $L = \int_0^4 \sqrt{1+1}\, dx = 4\sqrt{2}$.
3. $2\sqrt{5}$. Here $y' = 2$, and so $L = \int_{-1}^1 \sqrt{1+4}\, dx = 2\sqrt{5}$.
4. $2\sqrt{2}$. Now $x'(y) = 1$. So, $L = \int_0^2 \sqrt{2}\, dy = 3\sqrt{2}$.
5. $3\sqrt{2}$. This is the same as $y = x+3$, so $y' = 1$, and it follows that $L = \int_{-2}^1 \sqrt{2}\, dx = 3\sqrt{2}$.
6. $\frac{52}{3}$. Now $y'(x) = \sqrt{x}$ and so (if we set $u = 1+x,\ du = dx$) we see that $L = \int_0^8 \sqrt{1+x}\, dx = \frac{52}{3}$.
7. $\frac{1}{2}\sqrt{5} + \frac{1}{4}\ln\left(\sqrt{5}+2\right)$. In this case, $L = \int_0^1 \sqrt{1+4x^2}\, dx$. Use the substitution $2x = \tan\theta,\ 2\, dx = sec^2\theta\, d\theta$, and the usual identity to obtain an

   antiderivative in the form $\frac{1}{2}\int \sec^3\theta\, d\theta$. Now, see Example 387 for this integral. We have, $\frac{1}{2}\int \sec^3\theta\, d\theta = \frac{1}{4}(\tan\theta\sec\theta + \ln|\sec\theta + \tan\theta|)$. But $\tan\theta = 2x$, and so $\sec\theta = \sqrt{1+4x^2}$. Thus, $L = \int_0^1 \sqrt{1+4x^2}\, dx = \frac{1}{4}\left(2x\sqrt{1+4x^2} + \ln|\sqrt{1+4x^2}+2x|\right)\Big|_0^1 = \frac{1}{4}\cdot\left(2\sqrt{5} + \ln(2+\sqrt{5})\right)$, and the rest follows.
8. $\sqrt{65} + \frac{1}{8}\ln\left(8+\sqrt{65}\right)$. Use the method of Example 7 above. The arc length is given by $L = \int_0^2 \sqrt{1+16x^2}\, dx$. Now use the substitution $4x = \tan\theta,\ 4\, dx = sec^2\theta\, d\theta$, and an antiderivative will look like $\frac{1}{4}\int \sec^3\theta\, d\theta$. Finally, we see that $L = \int_0^2 \sqrt{1+16x^2}\, dx = \frac{1}{8}\left(4x\sqrt{1+16x^2} + \ln|\sqrt{1+16x^2}+4x|\right)\Big|_0^2 = \frac{1}{8}\cdot\left(8\sqrt{65} + \ln(8+\sqrt{65})\right)$, and the result follows.
9. $\frac{1}{2}\sqrt{17} + \frac{1}{8}\ln(4+\sqrt{17})$. See Exercise 8, above. We know that $L = \int_0^1 \sqrt{1+16x^2}\, dx$. Use the substitution $4x = \tan\theta,\ 4\, dx = sec^2\theta\, d\theta$, and the usual identity to obtain an antiderivative in the form

   $\frac{1}{4}\int \sec^3\theta\, d\theta$. Reverting back to the original variables, we get, $L = \int_0^1 \sqrt{1+16x^2}\, dx = \frac{1}{8}\left(4x\sqrt{1+16x^2} + \ln|\sqrt{1+16x^2}+4x|\right)\Big|_0^1 = \frac{1}{8}\cdot\left(4\sqrt{17} + \ln(4+\sqrt{17})\right) = \frac{1}{2}\sqrt{17} + \frac{1}{8}\ln(4+\sqrt{17})$.
10. $\frac{181}{9}$. Note that $1 + y'(x)^2 = 1 + \left(x^6 - \frac{1}{2} + \frac{1}{16x^6}\right) = \left(x^3 + \frac{1}{4x^3}\right)^2$.

    It follows that the expression for the arc length is given by $L = \int_1^3 \left(x^3 + \frac{1}{4x^3}\right)\, dx$, giving the stated result.

11. 4π. Here, $x'(t) = -2\sin t,\ \ y'(t) = 2\cos t$ so that the length of the arc is given by $L = \int_0^{2\pi} \sqrt{4\sin^2 t + 4\cos^2 t}\ dt = \int_0^{2\pi} \sqrt{4 \cdot 1}\ dt = 2 \cdot 2\pi = 4\pi.$
12. 2π. Now, $x'(t) = -\sin t,\ \ y'(t) = -\cos t$ so that the length of the arc is given by $L = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\ dt = \int_0^{2\pi} \sqrt{1}\ dt = 1 \cdot 2\pi = 2\pi.$
13. $\sqrt{2}$. In this example, $x'(t) = 1,\ \ y'(t) = -1$ so that the length of the arc is given by $L = \int_0^1 \sqrt{1+1}\ dt = \int_0^1 \sqrt{2}\ dt = \sqrt{2}.$
14. $\frac{3}{2}$. Use the Fundamental Theorem of Calculus to show that $y'(x) = \sqrt{x^2-1}$. Then, $\sqrt{1+(y'(x))^2} = \sqrt{x^2} = x$. So, $L = \int_1^2 x\ dx = \dfrac{3}{2}.$
15. $\sqrt{2}$. Once again, use the Fundamental Theorem of Calculus to show that $y'(x) = \sqrt{\cos 2x}$. Then, $\sqrt{1+(y'(x))^2} = \sqrt{1+\cos 2x} = \sqrt{2\cos^2 x}$, by a trig. identity (which one?). So, $L = \int_0^{\pi/2} \sqrt{2\cos^2 x}\ dx = \sqrt{2}\int_0^{\pi/2} \cos x\ dx = \sqrt{2}.$
16. 10.602. See Example 483 except that we solve for x in terms of $y > 0$ (because the given interval is a $y-$interval). The length L is then given by doubling the basic integral over half the curve, that is,

$$L = 2\int_{-1}^{1} \sqrt{\frac{4-3y^2}{4(1-y^2)}}\ dy \approx \int_{-0.99}^{0.99} \sqrt{\frac{4-3y^2}{4(1-y^2)}}\ dy \approx 5.3010.$$

17. 3.3428. The length L is given by an integral of the form $L = \int_1^4 \dfrac{\sqrt{1+x^2}}{x}\ dx.$

We use a trigonometric substitution $x = \tan\theta,\ dx = sec^2\theta\ d\theta$. Then, $\sqrt{1+x^2} = \sec\theta$ and an antiderivative is given by

$$\int \frac{sec^3\theta}{\tan\theta}\ d\theta = \int \frac{sec\theta}{\tan\theta}\ \left(1+\tan^2\theta\right)\ d\theta = \int \frac{\sec\theta}{\tan\theta}\ d\theta + \int \sec\theta\ \tan\theta\ d\theta =$$
$$\int \csc\theta\ d\theta + \sec\theta = \ln|\csc\theta - \cot\theta| + \sec\theta.$$

Since $x = \tan\theta$ it follows that $\csc\theta = \dfrac{\sec\theta}{\tan\theta} = \dfrac{\sqrt{1+x^2}}{x}$, $\cot\theta = \dfrac{1}{x}$. So an antiderivative is given by $\int \dfrac{\sqrt{1+x^2}}{x}\ dx = \ln\left|\dfrac{\sqrt{1+x^2}}{x} - \dfrac{1}{x}\right| + \sqrt{1+x^2}$. Finally, we see that $L = \int_1^4 \dfrac{\sqrt{1+x^2}}{x}\ dx = \left(\ln\left|\dfrac{\sqrt{1+x^2}}{x} - \dfrac{1}{x}\right| + \sqrt{1+x^2}\right)\Bigg|_1^4$
$= \left(\ln\left|\dfrac{\sqrt{17}}{4} - \dfrac{1}{4}\right| + \sqrt{17}\right) - \left(\ln\left|\sqrt{2}-1\right| + \sqrt{2}\right) = \sqrt{17} - \sqrt{2} + \ln\left|\dfrac{\sqrt{17}-1}{4}\right| - \ln\left|\sqrt{2}-1\right| \approx 3.3428.$

18. $\ln(1+\sqrt{2}) \approx 0.8813$. Here, $y'(x) = \tan x$ and so $\sqrt{1+y'(x)^2} = \sqrt{1+\tan^2 x} = \sqrt{\sec^2 x} = \sec x$. So, the arc length is given by

$$\int_0^{\pi/4} \sec x\ dx = \ln|\sec x + \tan x|\Bigg|_0^{\pi/4} = \ln(\sqrt{2}+1) - \ln(1+0) = \ln(1+\sqrt{2}).$$

19. $\frac{1}{2}\sqrt{5} - \frac{1}{4}\ln(\sqrt{5}-2)$. See Exercise 7, above for the evaluation of the integral. Note that $-\dfrac{1}{4}\ln(\sqrt{5}-2) = \dfrac{1}{4}\ln(\sqrt{5}+2)$.
20. Follow the hints.

9.5 Exercise Set 51

1. 3.75. Set m_1 at $x_1 = 0$ and m_2 at $x_2 = 5$. Then $\overline{x} = \frac{m_1x_1+m_2x_2}{m_1+m_2} = 3.75$.
2. 1.33. Set m_1 at $x_1 = 0$, m_2 at $x_2 = 1$ and m_3 at $x_3 = 2$. Then $\overline{x} = 1.33$.
3. $(\overline{x}.\overline{y}) = \left(\frac{5}{12}, \frac{1}{3}\right)$. Note that $\overline{x} = \dfrac{\sum m_i x_i}{\sum m_i}$ and $\overline{y} = \dfrac{\sum m_i y_i}{\sum m_i}$ where (x_i, y_i) are the coordinates of m_i. In this case, $\overline{x} = \dfrac{4 \cdot 0 + 5 \cdot 1}{12} = \dfrac{5}{12}$. Similarly, $\overline{y} = \dfrac{4 \cdot 1 + 0}{12} = \dfrac{1}{3}$. Note that even though the system of masses is at the vertices of an isosceles triangle, the center of mas is not along the bisector of the right-angle (which is the line of symmetry). This doesn't contradict the Symmetry Principle since the masses are not all the same!
4. $(\overline{x}.\overline{y}) = \left(1, \frac{\sqrt{3}}{3}\right)$. As before, $\overline{x} = \dfrac{\sum m_i x_i}{\sum m_i}$ and $\overline{y} = \dfrac{\sum m_i y_i}{\sum m_i}$ where (x_i, y_i) are the coordinates of m_i. Here, $\overline{x} = \dfrac{0+6+3}{9} = 1$. Similarly, $\overline{y} = \dfrac{0+0+3\sqrt{3}}{9} = \dfrac{\sqrt{3}}{3}$. In this exercise the masses are all the same and the triangle is equilateral, so (by the Symmetry Principle) the center of mass must lie along the line of symmetry (which it does), that is, it must lie on the line $x = 1$ which bisects the base of the triangle.
5. $\left(0, \frac{4R}{3\pi}\right)$. The total mass $m = \dfrac{\pi R^2 \delta}{2}$ since we are dealing with one-half the area of a circle and δ is constant. This use of geometry saves us from actually *calculating* the mass integral which looks like $\int_{-R}^{R} \sqrt{R^2 - x^2}\ \delta\ dx$. Next, the moment about the $y-$axis is given by $M_y = \int_{-R}^{R} \overline{x}_{slice}\ \delta\ dA = \delta \int_{-R}^{R} x\sqrt{R^2 - x^2}\ dx$. Now, let $x = R\cos\theta$, etc. But even simpler is the remark that the integrand, $x\sqrt{R^2 - x^2}$, is an *odd function* defined over a symmetric interval and so its integral must be zero. Either way, this gives $M_x = \delta \int_{-R}^{R} x\sqrt{R^2 - x^2}\ dx = 0$ and so $\overline{x} = 0$, *i.e.,* the center of mass lies along the axis of symmetry (which is the $y-$axis, since δ is constant).

   Similarly we find the moment about the $x-$axis, $M_x = \int_{-R}^{R} \overline{y}_{slice}\ \delta\ dA = \dfrac{\delta}{2} \int_{-R}^{R} \left(R^2 - x^2\right)\ dx = \dfrac{\delta}{2}\dfrac{4R^3}{3} = \dfrac{2R^3\delta}{3}$. It follows that the $y-$coordinate, $\overline{y}$, of the center of mass is given by $\overline{y} = \dfrac{M_x}{m} = \dfrac{2R^3\delta}{3} \cdot \dfrac{2}{\pi R^2 \delta} = \dfrac{4R}{3\pi}$.
6. $\left(\frac{b}{2}, \frac{h}{2}\right)$. Use of geometry shows us that the total mass is its area times its density, that is, $m = bh\delta$. Next, $\overline{x} = \int_0^b \overline{x}_{slice}\ \delta\ dA = \dfrac{1}{bh\delta} \int_0^b xh\delta\ dx = \dfrac{b}{2}$. Similarly, $\overline{y} = \int_0^b \overline{y}_{slice}\ \delta\ dA = \dfrac{1}{bh\delta} \int_0^b \dfrac{h}{2} h\delta\ dx = \dfrac{h}{2}$.
7. $\left(0, \frac{2}{3}\right)$. The region is an inverted triangle with a vertex at the origin and opposite side equal to 2 units. Its total mass is its area times its density, which, in this case, is δ. So, $m = \delta$. Let $f(x) = 1$ and $g(x) = 1 - |x|$, over $[-1, 1]$. Note that the region can be described by means of these two graphs. Also, $f(x) \geq g(x)$ and so we can use the formulae already derived for the center of mass. So, $\overline{x} = \dfrac{1}{\delta} \int_{-1}^{1} \overline{x}_{slice}\ \delta\ dA = \dfrac{1}{\delta} \int_{-1}^{1} x\ (1 - |x|)\ \delta\ dx =$

$\frac{1}{\delta}\int_{-1}^{0} x\ (1+x)\ \delta\ dx + \frac{1}{\delta}\int_{0}^{1} x\ (1-x)\ \delta\ dx = 0$. Next, $\overline{y} = \frac{1}{\delta}\int_{-1}^{1} \overline{y}_{slice}\ \delta\ dA =$ $\frac{1}{\delta}\int_{-1}^{1}\left(\frac{1+|x|}{2}\right)(1-|x|)\ \delta\ dx = \frac{1}{2\delta}\int_{-1}^{1}\left(1-x^2\right)\ \delta\ dx = \frac{2}{3}.$

8. $\left(-\frac{1}{12}, \frac{1}{3}\right)$. The total mass is $m = \int_{-1}^{1}\delta(x)\ dx = \int_{-1}^{1}(1-x)\ dx = 2$. Next, $\overline{x} = \frac{1}{2}\int_{-1}^{1}\overline{x}_{slice}\ \delta\ dA = \frac{1}{2}\int_{-1}^{1} x\ (1-|x|)\ (1-x)\ dx =$ $\frac{1}{2}\int_{-1}^{0} x\ (1+x)\ (1-x)\ dx + \frac{1}{2}\int_{0}^{1} x\ (1-x)\ (1-x)\ dx = \left(\frac{1}{2}\right)\cdot\left(\frac{-1}{6}\right) = -\frac{1}{12}.$ Similarly, $\overline{y} = \frac{1}{2}\int_{-1}^{1}\overline{y}_{slice}\ \delta\ dA = \frac{1}{2}\int_{-1}^{1}\left(\frac{1+|x|}{2}\right)(1-|x|)\ (1-x)\ dx =$ $\frac{1}{4}\int_{-1}^{1}(1-x^2)\ (1-x)\ dx = \frac{1}{3}.$

9. $\left(\frac{3}{5}, \frac{3}{8}\right)$. The total mass, $m = \delta\int_{0}^{1}\sqrt{x}\ dx = \frac{2\delta}{3}$. So, $\overline{x} = \frac{3}{2\delta}\int_{0}^{1}\overline{x}_{slice}\ \delta\ dA = \frac{3}{2}\int_{0}^{1} x\ \sqrt{x}\ dx = \frac{3}{2}\int_{0}^{1} x^{3/2}\ dx = \frac{3}{5}.$ Furthermore,

$$\overline{y} = \frac{3}{2\delta}\int_{0}^{1}\overline{y}_{slice}\ \delta\ dA = \frac{3}{2}\int_{0}^{1}\frac{\sqrt{x}}{2}\ \sqrt{x}\ dx = \frac{3}{4}\int_{0}^{1} x\ dx = \frac{3}{8}.$$

10. $\left(\frac{3}{2}, \frac{3}{10}\right)$. The total mass, $m = \delta\int_{0}^{2}\frac{x^2}{4}\ dx = \frac{2\delta}{3}$. So, $\overline{x} = \frac{3}{2\delta}\int_{0}^{2}\overline{x}_{slice}\ \delta\ dA = \frac{3}{2\delta}\int_{0}^{2} x\ \delta\ \frac{x^2}{4}\ dx = \frac{3}{8}\int_{0}^{2} x^3\ dx = \frac{3}{2}.$ Similarly, $\overline{y} = \frac{3}{2\delta}\int_{0}^{2}\overline{y}_{slice}\ \delta\ dA = \frac{3}{2\delta}\int_{0}^{2}\delta\ \frac{x^2}{8}\ \frac{x^2}{4}\ dx = \frac{3}{2\cdot 32}\int_{0}^{2} x^4\ dx = \frac{3}{10}.$

11. $\left(\frac{1}{2}, \frac{\pi}{4}\right)$. The graph of this function is positive on $[0,1]$. The total mass, $m = \delta\int_{0}^{1} 2\ \sin(\pi x)\ dx = 2\delta\left(-\frac{\cos(\pi x)}{\pi}\right)\Big|_{0}^{1} = \frac{4\delta}{\pi}$. So, $\overline{x} = \frac{\pi}{4\delta}\int_{0}^{1}\overline{x}_{slice}\ \delta\ dA = \frac{\pi}{4\delta}\int_{0}^{1} x\ 2\sin(\pi x)\ \delta\ dx$ $= \frac{\pi}{2}\int_{0}^{1} x\ \sin(\pi x)\ \ dx = \frac{1}{2}.$ Similarly, $\overline{y} = \frac{\pi}{4\delta}\int_{0}^{1}\overline{y}_{slice}\ \delta\ dA = \frac{\pi}{4\delta}\int_{0}^{1}\frac{2\sin(\pi x)}{2}\ \delta\ 2\sin(\pi x)\ dx = \frac{\pi}{2}\int_{0}^{1}\sin^2(\pi x)\ dx =$ $\frac{\pi}{2}\int_{0}^{1}\frac{1-\cos(2\pi x)}{2}\ dx = \frac{\pi}{4}$. Note the symmetry about the line $x = 1/2$ so that the center of mass must lie along this line.

12. $\left(\frac{2(e^2-1)}{1+e^2}, \frac{3e^4+1}{8(1+e^2)}\right)$. The total mass is $m = \int_{0}^{2} x\ \delta\ e^x\ dx = \delta\int_{0}^{2} x\ e^x\ dx =$ $(e^2+1)\delta$. Next,

$\overline{x} = \frac{1}{1+e^2}\int_{0}^{2} x^2\ e^x\ dx = \frac{2e^2-2}{1+e^2}$. Finally, one more application of

Integration by Parts (or the Table Method)
$\overline{y} = \dfrac{1}{2(1+e^2)} \displaystyle\int_0^2 x\ e^{2x}\ dx = \dfrac{3e^4+1}{8(1+e^2)}.$

13. $\left(0, \dfrac{11}{3\sqrt{3}+2\pi}\right)$. Since $\delta = 2$, the total mass is $m = \displaystyle\int_{-1}^{1} \sqrt{4-x^2}\ \delta\ dx = 2\int_{-1}^{1} \sqrt{4-x^2}\ dx = 2\sqrt{3} + \dfrac{4\pi}{3}$, where we used the trig. substitution $x = 2\sin\theta$, etc.

The geometric area is not so easy to calculate in this case, so we return to the integral definition. Now, because of symmetry about the line $x = 0$ and since δ is constant, we must have

$\overline{x} = 0.$

Furthermore,
$\overline{y} = \dfrac{3}{6\sqrt{3}+4\pi} \displaystyle\int_{-1}^{1} \dfrac{\sqrt{4-x^2}}{2} \sqrt{4-x^2}\ 2\ dx = \dfrac{3}{6\sqrt{3}+4\pi} \int_{-1}^{1} (4-x^2)\ dx = \dfrac{3}{6\sqrt{3}+4\pi} \cdot \dfrac{22}{3} = \dfrac{11}{3\sqrt{3}+2\pi}.$

14. $\left(\dfrac{6}{5}, -\dfrac{2}{5}\right)$, see the solved example. The total mass is
$m = \displaystyle\int_0^2 (6x-3x^2)\ 2x\ dx = 8$. Next, $\overline{x} = \dfrac{1}{8} \displaystyle\int_0^2 x\ (6x-3x^2)\ 2x\ dx = \dfrac{6}{5}.$
Similarly,

$\overline{y} = \dfrac{1}{8} \displaystyle\int_0^2 \dfrac{x^2-2x}{2}\ (6x-3x^2)\ 2x\ dx = -\dfrac{2}{5}.$

9.6 Chapter Exercises

1. $\dfrac{\pi}{3} = \displaystyle\int_0^1 \pi\ y^2\ dy = \int_0^1 2\pi\ x(1-x)\ dx.$
2. $\dfrac{\pi}{3} = \displaystyle\int_0^1 \pi\ x^2\ dx = \int_0^1 2\pi\ y(1-y)\ dy.$
3. $\dfrac{\pi}{3} = \displaystyle\int_0^1 \pi\ y^2\ dy = \int_0^{1/2} \pi\ (1-4x^2)\ dx.$
4. $\dfrac{4\pi}{3} = \displaystyle\int_0^1 4\pi\ x^2\ dx = \int_0^2 \pi \left(1 - \dfrac{y^2}{4}\right)\ dy.$
5. $\frac{16\pi}{3}$ (you'll need two terms if you use horizontal slices here).

$$\begin{aligned} \frac{16\pi}{3} &= \int_0^2 2\pi\ x^2\ dx \\ &= \frac{3\pi}{4} \int_0^2 y^2\ dy + \pi \int_2^4 \left(4 - \frac{y^2}{4}\right)\ dy. \end{aligned}$$

6. $\dfrac{8\pi}{5} = 4\pi \displaystyle\int_0^1 y^{3/2}\ dy = \int_{-1}^1 \left(1-x^4\right)\ dx.$
7. $\dfrac{\pi^2}{2} = \pi \displaystyle\int_0^{\pi} \sin^2 x\ dx = 4\pi \int_0^1 y\ \text{Arcsin}\, y\ dy.$

8. $\pi^2 - 2\pi = 2\pi \int_0^{\pi/2} x \cos x \, dx = \pi \int_0^1 \text{Arccos}^2 y \, dy$. This last integral is very hard to evaluate! Try the substitution $y = \cos u$, $dy = -\sin u \, du$. Then use the Table method and then back-substitute. The first intergal in x is evaluated using the Table method.

9. $\frac{\pi}{4}(e^2 - 1) = \pi \int_0^1 x^2 e^{2x} \, dx$. You can't use horizontal slices here because it is almost impossible to solve for x in terms of y in the expression for $y = xe^x$. Use the Table method to evaluate the integral.

10. $2\pi \left(1 - \frac{5}{e^2}\right) = 2\pi \int_0^2 x^2 e^{-x} \, dx$. Use the Table method to evaluate the integral.

11. $\frac{3\pi}{10} = \pi \int_0^1 \left(x - x^4\right) \, dx = 2\pi \int_0^1 \left(y^{3/2} - y^3\right) \, dy$.

12. $\frac{4223\pi}{5670} = \pi \int_0^{1/3} \left(1 - (x^3 - 3x + 1)^2\right) \, dx + \pi \int_{1/3}^1 \left(1 - x^6\right) \, dx$.

Chapter 10

Solutions

10.1 Exercise Set 52

1. $y(x) = 3e^x$ means $y'(x) = 3e^x$ and $y''(x) = 3e^x$. So, all these derivatives are equal which means that
$$(1-x)y''(x) + xy'(x) - y(x) = 3e^x((1-x) + x - 1) = 3e^x(0) = 0.$$
2. From $y = 2e^x - 0.652e^{-x}$ we have
$$y' = 2e^x + 0.652e^{-x} \quad \text{and} \quad y'' = 2e^x - 0.652e^{-x}.$$
Clearly $y'' = y$. So $y'' - y = 0$.
3. From $y = 2\sin x$ we have $y' = 2\cos x$ and $y'' = -2\sin x$. Next, $y'''(x) = -2\cos x$, and finally $y^{(4)}(x) = 2\sin x = y(x)$.
Alternately, note that $y'' = -y$. Taking the second derivative of both sides of this last identity, we have $y^{(4)} = -y''$. But $-y'' = -(-y) = y$. So $y^{(4)} = y$, or $y^{(4)} - y = 0$.
4. $y(x) = ce^{3x} - e^{2x}$, $y'(x) = 3ce^{3x} - 2e^{2x}$, so
$$\begin{aligned} y'(x) - 3y(x) &= (3ce^{3x} - 2e^{2x}) - 3(ce^{3x} - e^{2x}) \\ &= 3ce^{3x} - 2e^{2x} - 3ce^{3x} + 3e^{2x} = e^{2x}. \end{aligned}$$
5. Differentiating $y = c_1x^2 + c_2x + c_3$ three times, we see that $y' = 2c_1x + c_2$, $y'' = 2c_1$ and $y''' = 0$ which is what we wanted to show.
6. No. Even though the function $y = e^x - e^{-x}$ satisfies the equation $y'' - y = 0$, its derivative $y' = e^x + e^{-x}$ is equal to $e^0 + e^{-0} = 2$ at $x = 0$. So the initial condition $y'(0) = 1$ fails for this function.
7. No, because this function does not satisfy the initial condition, $y(0) = 1$. In this case, $y(0) = 2$, so it cannot be a solution of the stated initial value problem. On the other hand, $y(x) = e^x + e^{-x}$, $y'(x) = e^x - e^{-x}$ and so, $y'(x) \neq y(x)$, so again it isn't a solution (because it doesn't even satisfy the equation).
8. From $y = e^{2x} - e^{-2x}$ we have $y' = 2e^{2x} + 2e^{-2x}$ and hence $y'(0) = 2e^0 + 2e^0 = 4$, violating the second initial condition $y'(0) = 2$.
9. No. The value of the function $\sin x + \cos x$ is 1 at $x = 0$. So the initial condition $y(0) = 0$ is not satisfied.
10. No, because although $y(x) = x^2$ does satisfy the equation, (since $y'''(x) = 0$) and it does satisfy $y(0) = 0$ and $y'(0) = 0$ it is NOT the case that $y''(0) = 3$, since, in fact, $y''(0) = 2$.

11. From $y = (c_1 + c_2x)e^x$ we have $y' = (c_1 + c_2x)e^x + c_2e^x$. Hence the intial condtions $y(0) = 1$ and $y'(0) = 0$ gives $c_1 = 1$ and $c_2 + c_1 = 0$, from which it is easy to get $c_1 = 0$ and $c_2 = -1$. So the solution satisfying the required initial conditions is $y = (1 - x)e^x$.

12. The initial condtion $y(1) = -1$ gives $\frac{1}{1-C} = -1$, from which we obtain $C = 2$. Thus the answer is $y(t) = \dfrac{t}{1-2t}$.

13. The general solution is given by $y(x) = x^4 + c_1x^2/2 + c_2x + c_3$. But $y(0) = 0$ means that $c_3 = 0$. Next, $y'(0) = 0$, means that $c_2 = 0$ and finally $y''(0) = 0$ means that $c_1 = 0$. Combining all this we get that the solution of the initial value problem is given by $y(x) = x^4$.

14. $y = -4e^{t^2} + 2$.

15. $y = -5e^{-1} - 1 + (1 + 2e^{-1})x + (x + 2)e^{-x}$.

10.2 Exercise Set 53

1. Let $f(x) = (1 - x^2)^{-1}$ and $g(y) = 4 + y^2$. We separate the variables and use Table 10.1, with $a = 0$, $y(a) = 1$, to find

$$\int_1^{y(x)} \frac{1}{u^2+4}\, du = \int_0^x \frac{1}{1-t^2}\, dt,$$

as the form of the required solution. Now use the trigonometric substitution $u = 2\tan\theta$ on the left and partial fractions on the right to find the special antiderivatives,

$$\frac{1}{2}\text{Arctan}\,\frac{y(x)}{2} - \frac{1}{2}\text{Arctan}\,\frac{1}{2} = \frac{1}{2}\ln\left|\frac{1-x}{1+x}\right|$$

as the required solution. We don't need to solve for $y(x)$.

2. $e^y = x^2e^x - 2xe^x + 2e^x + e^{-1} - 2$, or $y = \ln((x^2 - 2x + 2)e^x + e^{-1} - 2)$.

3. We can rewrite the equation as $\frac{y'}{y} = 1 + \frac{1}{x}$. Taking integrals, we have $\ln|y| = x + \ln|x| + C$, which gives the general solution $|y| = |x|e^{x+C}$. So it looks like: $y = |x|e^{x+C}$ or $y = -|x|e^{x+C}$, depending on whether $y(x) > 0$ or $y(x) < 0$, respectively. Now, since $y(1) = 1 > 0$ we must use the form $y = |x|e^{x+C}$ of the general solution (otherwise $y(1)$ cannot be equal to 1). This gives $1 = e^{1+C}$ from which we get $C = -1$. So the required solution is $y = xe^{x-1}$.

4. Let $f(x) = \cos x$ and $g(y) = y^3$. We separate the variables and use Table 10.1, with $a = 0$, $y(a) = -2$, to find

$$\int_{-2}^{y(x)} \frac{1}{u^3}\, du = \int_0^x \cos t\, dt,$$

as the form of the required solution. Both sides are easily integrated to give

$$\begin{aligned} \left(-\frac{1}{2u^2}\right)\Bigg|_{-2}^{y(x)} &= \sin x, \\ -\frac{1}{2y(x)^2} + \frac{1}{8} &= \sin x. \end{aligned}$$

as the required solution. We don't need to solve for $y(x)$.

5. $y = e^{\frac{1}{2}e^{x^2}}$. Let $x^2 = u$ in the integral on the right.

6. $y = \left(\dfrac{2}{3}x + 2\right)^3$.

7. Let $f(x) = x \sin x^2$ and $g(y) = 1$. We separate the variables and use Table 10.1, with $a = \sqrt{\pi}$, $y(a) = 0$, to find (using the FTC)

$$y(x) - y(\sqrt{\pi}) = \int_{\sqrt{\pi}}^{x} t \sin t^2 \, dt,$$

as the form of the required solution. The right-side needs a substitution only, namely, $u = t^2$, etc. Then,

$$\begin{aligned} y(x) - y(\sqrt{\pi}) &= \int_{\sqrt{\pi}}^{x} t \sin t^2 \, dt, \\ &= \frac{1}{2}\int_{\pi}^{x^2} \sin u \, du, \\ &= \frac{1}{2}\left((-\cos x^2) - (-\cos \pi)\right), \\ y(x) &= \frac{1}{2}\left((-\cos x^2) - 1\right), \\ &= -\frac{1}{2}\left(1 + \cos x^2\right), \end{aligned}$$

as the required solution.

8. $y = 6x$. Since $x \neq 0$, near the intial condition, we can divide both sides of the diffferential equation $xy' = 6x$ by x, and find $y'(x) = 6$ and the result follows.

10.3 Exercise Set 54

1. Let t be in hours, $N(t) = N(0)e^{kt}$. We are given $N(0) = 6000$ and $N(2.5) = 18000$. We need to find a formula for $N(t)$ and then find the value of $N(5)$. We set $t = 2.5$ into the general Growth Law above and find $N(2.5) = N(0)e^{2.5k}$, or $18000 = 6000e^{2.5k}$ from which we get $3 = e^{2.5k}$ or $k = (\ln 3)/2.5$. Substituting this value back into the original equation $N(t) = N(0)e^{kt}$ and simplifying, we find $N(t) = 6000\ 3^{\frac{t}{2.5}}$ as the virus population after t hours. Thus, after $t = 5$ hours we have, $N(5) = 6000\ 3^{\frac{5}{2.5}} = 54,000$ viruses.

2. Here we may apply the half-life formula $N(t) = N(0)/e^{t/T}$ to get $20 = 50/2^{t/1600}$ which gives $2^{t/1600} = 2.5$. Taking natural logarithm on both sides and rearranging the identity, we have

$$t = 1600\frac{\ln 2.5}{\ln 2} \approx 2115.$$

So it takes about 2115 years for the original sample of 50 micrograms to be reduced to 20 micrograms.

3. a) Divide both sides of the differential equation by $T_0 - T$. Then the left-side depends only on $T(t)$ while the right side is a constant. So the equation is separable since it takes the form

$$T'(t) = f(t)g(T),$$

where $f(t) = c$ and $g(T) = T_0 - T$. Now use Table 10.1 in the form

$$\int_{T(0)}^{T(t)} \frac{du}{T_0 - u} = \int_0^t c \, dt.$$

Integration shows that $-\ln|T_0 - u|\Big|_{u=T(0)}^{u=T(t)} = ct$ from which we can derive that

$$T(t) = T_0 + (T(0) - T_0)e^{-ct}.$$

b) Here we have $T_0 = 20$, $T_0 = 90$ and $T(4) = 80$. So, letting $t = 4$ in the above identity, we have $80 = 20 + (90 - 20)e^{-4c}$. So $e^{-4c} = 6/7$. Now we have to find t such that $T(t) = 70$. Applying the same identity again, we have $70 = 20 + (90 - 20)e^{-ct}$, or

$$5/7 = e^{-ct} = \left(e^{-4c}\right)^{t/4} = (6/7)^{t/4}.$$

Taking the natural logarithm on both sides and rearranging, we have

$$t = 4\frac{\ln 5 - \ln 7}{\ln 6 - \ln 7} \approx 8.731.$$

Thus it takes about nine minutes to reach a drinkable temperature.

4. We assume a Law of Growth, $N(t) = N(0)e^{kt}$. After 5,700 years there will be exactly one-half of the original amount, $N(0)$, which translates into $N(5700) = N(0)/2$.Thus,$N(0)/2 = N(0)e^{5700k}$, and the $N(0)$ cancel out (they always do!). So, $1/2 = e^{5700k}$ which means that $k = -(\ln 2)/5700$. This gives the value of k. Next, if 90% decays then only 10% remains, right? But we want a value of t such that $N(t) = (0.1)N(0)$ (which translates as "90% of the original amount has decayed"). But whatever this value of t may be, it is also given by $(0.1)N(0) = N(t) = N(0)\left(e^{(-\ln 2)}\right)^{\frac{t}{5700}} = \frac{1}{2^{\frac{t}{5700}}}$. So, $(0.1) = \frac{1}{2^{\frac{t}{5700}}}$ which, when solved for t, gives $t = 5700\dfrac{\ln 10}{\ln 2} \approx 18,935$ years.

5. a) Denote by $N(t)$ the number of bacteria at time t. Then we can write $T(t) = 4000e^{kt}$ where k is the rate of growth per bacteria. By assumption, $T(0.5) = 12000$ and so $12000 = 4000e^{0.5k}$, which gives $e^{0.5k} = 3$, or $0.5k = \ln 3$, that is $k = 2\ln 3 = \ln 9$. Thus

$$N(t) = 4000e^{(\ln 9)t} = 4000 \cdot 9^t.$$

b) After 20 minutes, the population of the bacteria is

$$N(1/3) = 4000 \times 9^{1/3} \approx 8320.$$

c) Suppose that $N(t) = 50000$. Then $50000 = 4000 \times 9^t$ and hence $9^t = 12.5$. So

$$t = \frac{\ln 12.5}{\ln 9} \approx 1.15.$$

Thus the bacteria population reach 50000 in an hour and 9 minutes.

6. Since the population satisfies a Law of Growth we can write $P(t) = P(0)e^{kt}$, where t is in years. We are given that $P(0) = 6 \times 10^8$ and that 300 years "later" (from A.D. 1650) the population was 2.8×10^9, that is, $P(300) = 2.8 \times 10^9$. We want a value of t such that $P(t) = 25\times 10^9$. So, $2.8\times 10^9 = P(300) = P(0)e^{300k} = 6 \times 10^8\ e^{300k}$ and we can solve for k giving $0.467 \times 10 = e^{300k}$, or $k = \frac{\ln 4.67}{300} \approx 0.00513$. Thus, $P(t) = (6\times 10^8)e^{0.00513t}$ is the Law of Growth, at time t in years. But we want $P(t) = 25 \times 10^9$ so this means that $25 \times 10^9 = (6 \times 10^8)e^{0.00513t}$ and we can solve for t, using logarithms. This gives, $41.667 = e^{0.00513t}$ or $t = \frac{\ln 41.667}{0.00513} \approx 727$ years. Thus, the population of the earth will reach 25 billion around the year $1650 + 727 = 2377$.

7. The differential equation for this learning model is $P' = c(1 - P)$ which is of the form $P' = f(t)g(P)$ where $f(t) = c$ and $g(P) = 1 - P$. Using Table 10.1 we see that the general solution is given by evaluating the integrals

$$\begin{aligned}
\int_{P(0)}^{P(t)} \frac{du}{1-u} &= \int_0^t c\ dt \\
-\ln|1-u|\Big|_{u=P(0)}^{u=P(t)} &= ct \\
\ln|1-P(t)| &= -ct \quad (\text{since } P(0)=0), \\
P(t) &= 1 - e^{-ct}.
\end{aligned}$$

b) By assumption, we have $P(3) = 0.4$ and hence $0.4 = 1 - e^{-3c}$, which gives $e^{-3c} = 0.6$. Now suppose $P(t) = 0.95$, that is $1 - e^{-ct} = 0.95$, which can be rewritten as $(e^{-3c})^{t/3} = 0.05$, or $0.6^{t/3} = 0.05$. Taking natural logarithm on both sides and rearranging, we have

$$t = 3\frac{\ln 0.05}{\ln 0.6} \approx 17.6.$$

Thus the student has to do at least 18 exercises in order to achieve a 0.95 (or 95 percent) chance of mastering the subject.

8. As suggested, we use the "integrating factor" $e^{kt/m}$ and let $z(t) = v(t)e^{kt/m}$. Then, by the product rule,

$$z'(t) = v'(t)e^{kt/m} + v(t) \cdot \frac{k}{m}e^{kt/m} = \left(v'(t) + \frac{k}{m}v(t)\right) e^{kt/m} = ge^{kt/m},$$

in view of the fact that the equation $mv' = mg - kv$ can be rewritten as $v' + \frac{k}{m}v = g$. Integrating, we have

$$z(t) = z(0) + \int_0^t ge^{ks/m}\, ds = v(0) + \frac{mg}{k}\left(e^{kt/m} - 1\right),$$

where $v(0)$ is just the initial speed v_o. Thus

$$v(t) = z(t) \cdot e^{-kt/m} = \frac{mg}{k}\left(1 - e^{-kt/m}\right) + v_o e^{-kt/k}.$$

9. In this case we assume a Law of Decay of the usual form $N(t) = N(0)e^{kt}$. We can use the Half-Life Formula and find $N(t) = N(0)/2^{t/T}$ where T is the half-life of the radionuclide. In this case, $T = 29.1$, so the amount left at time t is given by $N(t) = N(0)/2^{t/29.1}$. The initial sample of 5 g means that $N(0) = 5$. If we want to find out when 90% of the sample has decayed, then there is only 10% of it left, that is we want to find t such that $N(t) = (0.1)N(0) = (0.1)(5) = 0.5$. This means that $0.5 = 5/2^{t/29.1}$, or using logarithms, $t = \frac{\ln 100}{\ln 2} \times 29.1 \approx 193.33$ years.

Chapter 11

Solutions

11.1

11.2

11.3

11.4 Exercise Set 55

1. $\frac{\partial f}{\partial x} = 3x^2 + 2y, \quad \frac{\partial f}{\partial y} = 2x - 2y$
2. $f(x,y) = \frac{y^2}{x} = y^2x^{-1}$, so $\frac{\partial f}{\partial x} = -y^2x^{-2} = -\frac{y^2}{x^2}, \quad \frac{\partial f}{\partial y} = \frac{2y}{x}$
3. $\frac{\partial f}{\partial x} = \frac{\partial(2x^2)}{\partial x}e^{xy} + 2x^2\frac{\partial(e^{xy})}{\partial x} = 4x\ e^{xy} + 2x^2y\ e^{xy}, \quad \frac{\partial f}{\partial y} = 2x^3e^{xy}$
4. $\frac{\partial f}{\partial x} = e^{3y}\frac{\partial(x+ln\ y)}{\partial x} = e^{3y}(1) = e^{3y}$
   $\frac{\partial f}{\partial y} = \frac{\partial(x+ln\ y)}{\partial y}e^{3y} + (x + ln\ y)\frac{\partial(e^{3y})}{\partial y} = \frac{1}{y}e^{3y} + (x + ln\ y)(3)e^{3y}$
5. $\frac{\partial f}{\partial x} = 2(x - 2y + 3)\frac{\partial(x-2y+3)}{\partial x} = 2(x - 2y + 3$
   $\frac{\partial f}{\partial y} = 2(x - 2y + 3)\frac{\partial(x-2y+3)}{\partial y} = 2(x - 2y + 3)(-2) = -4(x - 2y + 3)$
6. $$\frac{\partial f}{\partial x} = \frac{(y + x)\frac{\partial x}{\partial x} - x\frac{\partial(y+x)}{\partial x}}{(y + x)^2} = \frac{y + x - x(1)}{(y + x)^2} = \frac{y}{(y + x)^2}$$
   Thus $\frac{\partial f}{\partial x} = \frac{2}{(2-3)^2} = 2$ at $(-3, 2)$.
   $f(x, y) = x(y + x)^{-1}$, so
   $\frac{\partial f}{\partial y} = -x(y + x)^{-2}\frac{\partial(y+x)}{\partial y} = -\frac{x}{(y+x)^2} = -\frac{-3}{(2-3)^2} = 3$ at $(-3, 2)$.
7. $f(x, y) = (y^2 + 2x)^{\frac{1}{2}}$
   $\frac{\partial f}{\partial x} = \frac{1}{2}(y^2 + 2x)^{-\frac{1}{2}}\ \frac{\partial(y^2+2x)}{\partial x} = \frac{1}{2}(y^2 + 2x)^{-\frac{1}{2}}(2) = \frac{1}{\sqrt{y^2+2x}} = \frac{1}{\sqrt{(-1)^2+8}} = \frac{1}{3}$
   $\frac{\partial f}{\partial y} = \frac{1}{2}(y^2 + 2x)^{\frac{-1}{2}}\ \frac{\partial(y^2+2x)}{\partial y} = \frac{1}{2}(y^2 + 2x)^{\frac{-1}{2}}(2y) = \frac{y}{\sqrt{y^2+2x}} = -\frac{1}{3}$
8. $\frac{\partial f}{\partial x} = \frac{\partial(xy)}{\partial x}e^{2x-y} + xy\frac{\partial e^{2x-y}}{\partial x} = ye^{2x-y} + 2xye^{2x-y} = 2e^{2-2} + 2(1)(2)e^{2-2} = 6.$
   $\frac{\partial f}{\partial y} = \frac{\partial(xy)}{\partial y}e^{2x-y} + xy\frac{\partial e^{2x-y}}{\partial y} = xe^{2x-y} + xy(-1)e^{2x-y} = e^0 - 2e^0 = -1$

9. $\frac{\partial f}{\partial x} = 3x^2y - 2y^2, \quad \frac{\partial f}{\partial y} = x^3 - 4xy$
$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2y - 2y^2) = 6xy$
$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(x^3 - 4xy) = -4x$
$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x^3 - 4xy) = 3x^2 - 4y$
$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(3x^2y - 2y^2) = 3x^2 - 4y$

10. $\frac{\partial f}{\partial x} = \frac{2x}{x^2-y^2}, \quad \frac{\partial f}{\partial y} = \frac{-2y}{x^2-y^2}$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\frac{\partial(2x)}{\partial x}(x^2 - y^2) - 2x\frac{\partial(x^2-y^2)}{\partial x}}{(x^2 - y^2)^2} = \frac{2(x^2 - y^2) - 2x(2x)}{(x^2 - y^2)^2} = \frac{-2(x^2 + y^2)}{(x^2 - y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\frac{\partial(-2y)}{\partial y}(x^2 - y^2) - (-2y)\frac{\partial(x^2-y^2)}{\partial y}}{(x^2 - y^2)^2} = \frac{-2(x^2 - y^2) + 2y(-2y)}{(x^2 - y^2)^2} = \frac{-2(x^2 + y^2)}{(x^2 - y^2)^2}$$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{-2y}{x^2-y^2}\right) = -2y(-1)(x^2 - y^2)^{-2}(2x) = \frac{4xy}{(x^2-y^2)^2}$
$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{2x}{x^2-y^2}\right) = 2x(-1)(x^2 - y^2)^{-2}(-2y) = \frac{4xy}{(x^2-y^2)^2}$

11. $\frac{\partial f}{\partial x} = e^{y-x} - xe^{y-x}, \quad \frac{\partial f}{\partial y} = xe^{y-x}$
$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(e^{y-x} - xe^{y-x}\right) = -e^{y-x} - [e^{y-x} - xe^{y-x}] = xe^{y-x} - 2e^{y-x}$
$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(xe^{y-x}) = xe^{y-x}$
$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(xe^{y-x}) = e^{y-x} - xe^{y-x}$
$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(e^{y-x} - xe^{y-x}) = e^{y-x} - xe^{y-x}$

12. $\frac{\partial f}{\partial x} = 8x - 3y^2, \quad \frac{\partial f}{\partial y} = -6xy + 3y^2$
$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(8x - 3y^2) = 8, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(-6xy + 3y^2) = -6x + 6y$
$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(-6xy + 3y^2) = -6y, \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(8x - 3y^2) = -6y$

11.5

11.6 Exercise Set 56

1. f is continuous whenever the denominator is not zero, that is f is continuous at (x, y) where $x + y \neq 0$. f is discontinuous whenever $x + y = 0$.
2. Yes, it is continuous because we have seen that $\lim_{\square \to 0} \frac{\sin \square}{\square} = 1 = f(0, 0)$ here.
3. $\frac{\partial z}{\partial x} = 2, \quad \frac{\partial f}{\partial y} = 5$
4. $\frac{\partial z}{\partial x} = 2xy^3, \frac{\partial z}{\partial y} = 3x^2y^2 + 5, \frac{\partial^2 z}{\partial y^2} = 6x^2y$
5. $\frac{\partial z}{\partial x}(3, 4) = \frac{2}{5}, \quad \frac{\partial z}{\partial y}(3, 4) = \frac{1}{5}$
6. $\frac{\partial f}{\partial x} = yz(\sin x)^{yz-1} \cos x, \quad \frac{\partial f}{\partial y} = z(\sin x)^{yz} \ln(\sin x), \quad \frac{\partial f}{\partial z} = y(\sin x)^{yz} \ln(\sin x)$
7. $\frac{\partial A}{\partial b} = h$ and $\frac{\partial A}{\partial h} = b$. The first gives the rate of change of the area as a function of the base and the second gives the rate of change of the area as a function of the height.
8. $\sin 2t + 2e^{2t} + e^t(\sin t + \cos t)$. Use the Chain Rule.

9. $\frac{\partial z}{\partial x} = \frac{e^x}{e^x + e^y}, \quad \frac{dz}{dx} = \frac{3x^2 e^y + e^x}{e^x + e^y}.$

10. $(0, 0)$ is the only critical point. The Second Derivative Test shows that it is a minimum with a minimum value equal to -1 there. There is no global maximum since $\lim_{x\to\infty} f(x, 0) = +\infty$.

11. $(2, 1)$ is the only critical point. The Second Derivative Test shows that it is a maximum with a maximum value equal to $+2$ there. There is no minimum since $\lim_{x\to\infty} g(x, 0) = -\infty$.

12. Use Lagrange's method on the function $f(x, y, z, \lambda) = xyz - \lambda(x + y + z - 6)$ and find its critical points. The only critical point with positive coordinates that satisfies $x + y + z = 6$ is $(2, 2, 2)$ (and $\lambda = 4$). The maximum value is thus $xyz = 2 \cdot 2 \cdot 2 = 8$.

13.

$$\frac{\partial C}{\partial x} = \frac{1}{2}\ 10y(xy)^{\frac{-1}{2}} + 149 = 5\sqrt{\frac{y}{x}} + 149$$
$$\frac{\partial C}{\partial y} = 5x(xy)^{\frac{-1}{2}} + 189 = 5\sqrt{\frac{x}{y}} + 189$$

Thus, when $x = 120$ and $y = 160$, we have $\frac{\partial C}{\partial x} = 5\sqrt{\frac{160}{120}} + 149 = 154.77$, and $\frac{\partial C}{\partial y} = 5\sqrt{\frac{120}{160}} + 189 = 193.33$.

14. (a) $\frac{\partial R}{\partial x_1} = 200 - 8x_1 - 8x_2 = 200 - 8(4) - 8(12) = 72$

(b) $\frac{\partial R}{\partial x_2} = 200 - 8x_1 - 8x_2 = 72$.

15. (a) $\frac{\partial U}{\partial x} = -10x + y$

(b) $\frac{\partial U}{\partial y} = x - 6y$

(c) When $x = 2$ and $y = 3$,

$$\frac{\partial U}{\partial x} = -20 + 3 = -17, \quad \frac{\partial U}{\partial y} = 2 - 18 = -16$$

The purchase of one more unit would not result in an increase in satisfaction for either product since U is decreasing with both x and y. But, satisfaction would decrease less for product B (U decreases less with respect to y than x.

16. (a) $\frac{\partial f}{\partial x} = 3x^2 - 3 = 0$ for $x = \pm 1$, and $\frac{\partial f}{\partial y} = -2y + 4 = 0$ for $y = 2$. Thus the critical points of f are: (1, 2), (-1, 2)

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial y^2} = -2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\text{Thus}\ \ D = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left[\frac{\partial^2 f}{\partial x \partial y}\right]^2 = -12x$$

For (1, 2): $D = -12 < 0$, so (1, 2) is a saddle point. For (-1, 2): $D = 12 > 0$ and $\frac{\partial^2 f}{\partial x^2} = 6(-1) = -6 < 0$, so (-1, 2) is a local maximum.

(b) For critical points

$$\frac{\partial f}{\partial x} = 2x + 4y = 0 \quad (1), \quad \text{and} \quad \frac{\partial f}{\partial y} = 4x + 8y^3 = 0 \quad (2)$$

From (1), $x = -2y$. Substitute in (2), so $4(-2y) + 8y^3 = 0$. Thus $8y^3 - 8y = 0$, so $y(y^2 - 1) = 0$ and $y = 0, -1, +1$. The corresponding x values are: $x = 0, 2, -2$. Critical points are: (0, 0), (2, - 1), (-2, 1).

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 24y^2, \quad \frac{\partial^2 f}{\partial x \partial y} = 4$$

$D = 2(24y^2) - 4^2 = 48y^2 - 16$.

At (0,0): $D = -16 < 0$, so (0, 0) a is saddle point. At (2, -1): $D = 48 - 16 > 0$, and $\frac{\partial^2 f}{\partial x^2} > 0$, so (2, -1) is a local minimum. At (-2, 1): $D = 48 - 16 > 0$, and $\frac{\partial^2 f}{\partial x^2} > 0$, so (-2, 1) also a local minimum.

17. Profit, $P = R - C$, where $R = x_1p_1 + x_2p_2 = 200(p_2 - p_1)p_1 + (500 + 100p_1 - 180p_2)p_2 = 200p_1p_2 - 200p_1^2 + 500p_2 + 100p_1p_2 - 180p_2^2 = 300p_1p_2 - 200p_1^2 - 180p_2^2 + 500p_2$, and $C = 0.5x_1 + 0.75x_2 = 0.5 \times 200(p_2 - p_1) + 0.75(500 + 100p_1 - 180p_2) = 375 - 25p_1 - 35p_2$. Thus,

$$P(p_1, p_2) = 300p_1p_2 - 200p_1^2 - 180p_2^2 + 535p_2 - 375$$

For a critical point:

$$\frac{\partial P}{\partial p_1} = 300p_2 - 400p_1 + 25 = 0 \quad (1)$$

$$\frac{\partial P}{\partial p_2} = 300p_1 - 360p_2 + 535 = 0 \quad (2)$$

From (1): $p_1 = (300p_2 + 25)/400$ (3). Substitute in (2): $300\left(\frac{300p_2+25}{400}\right) - 360p_2 + 535 = 0$. This reduces to $135p_2 = 553.75$, so $p_2 = \$4.10$. Substitute in (3): $p_1 = (300 \times 4.10 + 25)/400 = \$3.14$.

$$\frac{\partial^2 P}{\partial p_1^2} = -400, \quad \frac{\partial^2 P}{\partial p_2^2} = -360, \quad \frac{\partial^2 P}{\partial p_1 \partial p_2} = 300$$

Thus $D = (-400)(-360) - (300)^2 > 0$, and $\frac{\partial^2 P}{\partial p_1^2} < 0$, so (3.14, 4.10) is a local maximum, so the maximum profit is $300(3.14)(4.10) - 200(3.14)^2 - 180(4.10)^2 + 535(4.10) + 25(3.14) - 375 = \$761.48$.

18. Want to maximize $f(x, y) = 100x^{\frac{3}{4}}y^{\frac{1}{4}}$ subject to the constraint $150x + 250y = 50,000$.

$$F(x, y, \lambda) = 100x^{\frac{3}{4}}y^{\frac{1}{4}} - \lambda(150x + 250y - 50,000)$$

For a maximum:

$$\frac{\partial F}{\partial x} = 75x^{\frac{-1}{4}}y^{\frac{1}{4}} - 150\lambda = 0 \ (1), \qquad \frac{\partial F}{\partial y} = 25x^{\frac{3}{4}}y^{\frac{-3}{4}} - 250\lambda = 0 \ (2)$$

$$\text{From}(1), \quad \lambda = \frac{1}{2}\, x^{\frac{-1}{4}}y^{\frac{1}{4}} \quad \text{From}(2) \quad \lambda = \frac{1}{10}\, x^{\frac{3}{4}}y^{\frac{-3}{4}}$$

Thus, $\frac{1}{2}x^{\frac{-1}{4}}y^{\frac{1}{4}} = \frac{1}{10}x^{\frac{3}{4}}y^{\frac{-3}{4}}$. Multiplying both sides by $10x^{\frac{1}{4}}y^{\frac{3}{4}}$, gives $5y = x$. Substituting $x = 5y$ in the constraint equation gives $150(5y) + 250y = 50,000$, so $y = 50$, and thus $x = 5y = 250$. Therefore, the maximum production level is:

$$f(250, 50) = 100(250)^{\frac{3}{4}}(50)^{\frac{1}{4}} \approx 16,719$$

19.

$$F(x, y, \lambda) = 2ln\ x + ln\ y - \lambda(2x + 4y - 48)$$

For a maximum:

$$\frac{\partial F}{\partial x} = \frac{2}{x} - 2\lambda = 0, \quad \text{so} \quad \lambda = \frac{1}{x} \quad (1)$$

$$\frac{\partial F}{\partial y} = \frac{1}{y} - 4\lambda = 0, \quad \text{so} \quad \lambda = \frac{1}{4y} \quad (2)$$

Thus $\frac{1}{x} = \frac{1}{4y}$, so $4y = x$. Substituting in constraint equation gives $8y + 4y - 48 = 0$, so $y = 4$, and $x = 16$ maximize the utility function subject to the constraint.

11.7 Chapter Exercises

1. For critical points, $\frac{\partial f}{\partial x} = 3x^2 - 3 = 0$, thus $x = \pm 1$, and $\frac{\partial f}{\partial y} = 2y + 6 = 0$, so $y = -3$. Thus critical points are: $(1, -3), (-1, -3)$. Now $\frac{\partial^2 f}{\partial x^2} = 6x$, $\frac{\partial^2 f}{\partial y^2} = 2$, $\frac{\partial^2 f}{\partial x \partial y} = 0$. Thus

$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2}{\partial x \partial y} = 12x.$$

At $(1, -3)$, $D = 12 > 0$ and $\frac{\partial^2 f}{\partial y^2} = 2 > 0$. Thus a local minimum at $(1, -3)$.
At $(-1, -3)$, $D = -12 < 0$, so a saddle point.

2. $\frac{\partial f}{\partial x} = x^2 - 4 = 0$ gives $x = \pm 2$. $\frac{\partial f}{\partial y} = -6y^2 + 6 = 0$ gives $y = \pm 1$. So critical points are at $(-2,\ -1), (-2,\ 1), (2,\ -1), (2,\ 1)$.

$$\frac{\partial^2 f}{\partial x^2} = 2x, \quad \frac{\partial^2 f}{\partial y^2} = -12y, \quad \frac{\partial^2 f}{\partial x \partial y} = 0.$$

$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2}{\partial x \partial y} = 2x(-12y) - 0 = -24xy$$

$(-2,\ -1):$ $D = -24(-2)(-1) < 0$, so saddle point
$(-2,\ 1):$ $D = -24(-2)(1) > 0$ and $\frac{\partial^2 f}{\partial x^2} = -4 < 0$, so local maximum
$(2,\ -1):$ $D = -24(2)(-1) > 0$ and $\frac{\partial^2 f}{\partial x^2} = 4 > 0$, so local minimum
$(2,\ 1):$ $D = -24(2)(1) < 0$, so saddle point.

3. For critical points,
$\frac{\partial f}{\partial x} = 4x^3 - 8y = 0$ (1), $\frac{\partial f}{\partial y} = -8x + 4y = 0$ (2)
From (2), $y = 2x$ (3), substitute in (1), so
$4x^3 - 16x = 0$, $4x(x^2 - 4) = 0$, $x = 0, -2, 2$.
From (3) the corresponding values of y are: $y = 0, -4, 4$.
Thus $(0,\ 0), (-2,\ -4), (2,\ 4)$ give critical points.
$\frac{\partial f}{\partial x^2} = 12x^2$, $\frac{\partial^2 f}{\partial y^2} = 4$, $\frac{\partial^2 f}{\partial x \partial y} = -8$.
Thus $D = (12x^2)(4) - (-8)^2 = 48x^2 - 64$.
$(0,\ 0):$ $D < 0$, therefore saddle point.
$(-2,\ -4):$ $D = 48(4) - 64 > 0$, and $\frac{\partial^2 f}{\partial y^2} = 4 > 0$, so local minimum.
$(2,\ 4):$ $D = 48(4) - 64 > 0$, and $\frac{\partial^2 f}{\partial y^2} = 4 > 0$, so local minimum.

4. For critical points,
$\frac{\partial f}{\partial x} = 6x - 6y = 0$ (1), $\frac{\partial f}{\partial y} = -6x + 3y^2 - 9 = 0$ (2)
From (2), $y^2 - 2y - 3 = 0$, so $(y-3)(y+1) = 0$, so $y = -1, 3$.
From (1), $x = y$, thus critical points at: $(-1,\ -1) and (3,\ 3)$.
$\frac{\partial^2 f}{\partial x^2} = 6$, $\frac{\partial^2 f}{\partial y^2} = 6y$, $\frac{\partial^2 f}{\partial x \partial y} = -6$
$D = 6(6y) - 36 = 36y - 36$
$(-1,\ -1):$ $D = -72 < 0$, so a saddle point.
$(3,\ 3):$ $D = 36(3) - 36 > 0$, and $\frac{\partial^2 f}{\partial x^2} = 6 > 0$, so a local minimum.

5. For critical points:
$\frac{\partial f}{\partial x} = 6y^2 - 6x^2 = 0$ (1) $\frac{\partial f}{\partial y} = 12xy - 12y^3 = 0$ (2)
From (1), $x^2 = y^2$, so $x = \pm y$. Substitute in (2).

If $x = y$		If $x = -y$	
$12y^2 - 12y^3 = 0$		$-12y^2 - 12y^3 = 0$	
$12y^2(1-y) = 0$		$-12y^2(1+y) = 0$	
so $y = 0$	or $y = 1$	so $y = 0$	or $y = -1$
so $x = 0$	so $x = 1$	so $x = 0$	so $x = 1$

Thus critical points at: $(0,\ 0), (1,\ -1), (1,\ 1)$
$\frac{\partial^2 f}{\partial x^2} = -12x$, $\frac{\partial^2 f}{\partial y^2} = 12x - 36y^2$, $\frac{\partial^2 f}{\partial x \partial y} = 12y$
$D = -12x(12x - 36y^2) - (12y)^2 = 144[-x^2 + 3xy^2 - y^2]$
$(0,\ 0):$ $D = 0$, therefore test inconclusive
$(1,\ -1):$ $D = 144[-1 + 3 - 1] = 144 > 0$, and $\frac{\partial^2 f}{\partial x^2} = -12 < 0$, so local maximum
$(1,\ 1):$ $D = 144 > 0$, and $\frac{\partial^2 f}{\partial x^2} = -12 < 0$, so local maximum

6. For critical points: $\frac{\partial f}{\partial x} = 4x - 4x^3 = 0$, so $4x(1-x^2) = 0$, and $x = 0,\ \pm 1$.
Also $\frac{\partial f}{\partial y} = -2y = 0$, so $y = 0$. Thus critical points at $(-1,\ 0)$, $(0,\ 0)$, $(1,\ 0)$
$\frac{\partial^2 f}{\partial x^2} = 4 - 12x^2$, $\frac{\partial^2 f}{\partial y^2} = -2$, $\frac{\partial^2 f}{\partial x \partial y} = 0$

$D = (4 - 12x^2)(-2) - 0 = 8(3x^2 - 1)$

$(0,\ 0):\quad D = -8 < 0,\quad$ so saddle point

$(-1,\ 0):\quad D = 16 > 0,\quad$ and $\frac{\partial^2 f}{\partial y^2} < 0,\quad$ so local maximum

$(1,\ 0):\quad D = 16 > 0,\quad$ and $\frac{\partial^2 f}{\partial y^2} < 0,\quad$ so local maximum

7. $\frac{\partial f}{\partial x} = y\ e^x - 3 = 0,$ so $ye^x = 3$ (1), $\frac{\partial f}{\partial y} = e^x - 1 = 0,$ so $e^x = 1,$ so $x = 0$. Substituting in (1) $y = 3$. Thus critical point at $(0,\ 3)$

   $\frac{\partial^2 f}{\partial x^2} = ye^x,\quad \frac{\partial^2 f}{\partial y^2} = 0,\quad \frac{\partial^2}{\partial x \partial y} = e^x$

   Thus $D = 0 - (e^x)^2 = -e^{2x}$ At $(0,\ 3):\quad D = -1 < 0,\quad$ so a saddle point.

8. For critical points:

   $\frac{\partial f}{\partial x} = 2x + 4y = 0,\quad (1),\quad \frac{\partial f}{\partial y} = 4x + 8y^3 = 0,\quad (2)$

   From (1), $x = -2y$. Substitute in (2). Thus

   $-8y + 8y^3 = 0,\quad$ so $\quad y(-1 + y^2) = 0,\quad$ so $\quad y = 0, \pm 1$

   If $y = 0,\quad x = -2y = 0.$ If $y = -1,\quad x = -2y = 2.$ If $y = 1,\quad x = -2$ So critical points at: $(0,\ 0),\ (2,\ -1),\ (-2,\ 1)$

   $\frac{\partial^2 f}{\partial x^2} = 2,\quad \frac{\partial^2 f}{\partial y^2} = 24y^2,\quad \frac{\partial^2 f}{\partial x \partial y} = 4.$ So $D = 48y^2 - 16$

   $(0,\ 0):\quad D < 0,\quad$ so saddle point

   $(-2,\ 1):\quad D = 48 - 16 > 0,\quad$ and $\frac{\partial^f}{\partial x^2} = 2 > 0,\quad$ so a local minimum

   $(2,\ -1):\quad D = 48 - 16 > 0,\quad$ and $\frac{\partial^2 f}{\partial x^2} > 0,\quad$ so a local minimum.

9. Objective function is $f(x, y) = x^2 + 3y^2 + 10$

   Constraint is $g(x, y) = x + y - 4 = 0$ Thus

   $$F(x, y\lambda) = x^2 + 3y^2 + 10 - \lambda(x + y - 4)$$

   $\frac{\partial F}{\partial x} = 2x - \lambda = 0,\quad$ so $\quad \lambda = 2x$

   $\frac{\partial F}{\partial y} = 6y - \lambda = 0,\quad$ so $\quad \lambda = 6y$

   Thus $2x = 6y$ so $x = 3y$. Substitute in constraint. So $3y + y - 4 = 0$, so $y = 1$ Therefore $x = 3(1) = 3$ Thus the minimum occurs for $x = 3,\quad y = 1$ and the minimum is $f(3, 1) = 3^2 + 3(1) + 10 = 22$

10. Objective function: $f(x, y) = x^2 + xy - 3y^2$

    Constraint: $g(x, y) = x + 2y - 2 = 0$ Thus

    $F(x, y, \lambda) = x^2 + xy - 3y^2 - \lambda(x + 2y - 2)$

    $\frac{\partial F}{\partial x} = 2x + y - \lambda = 0,$

    $\frac{\partial F}{\partial y} = x - 6y - 2\lambda = 0,\quad$ so $\quad \lambda = \frac{x}{2} - 3y$

    Thus $2x + y = \frac{x}{2} - 3y,$ so $\frac{3}{2}x = -4y,$ so $x = -\frac{8y}{3}$ Substitute in constraint, so $-\frac{8y}{3} + 2y = 2,$ so $-8y + 6y = 6,$ so $y = -3$ and thus $x = 8$

    Therefore the constrained maximum value occurs at $(8, -3)$, and is $x^2 + xy - 3y^2 = 8^2 + 8(-3) - 3(-3)^2 = 13$

11. Let x be length of east and west sides, y be length of north and south sides.

    Objective function = area of garden = xy

    Constraint function is $10(2y) + 15(2x) = 480$ Thus

    $F(x, y, \lambda) = xy - \lambda(20y + 30x - 480)$

    $\frac{\partial F}{\partial x} = y - 30\lambda = 0,\quad$ so $\quad \lambda = y/30$

    $\frac{\partial F}{\partial y} = x - 20\lambda = 0,\quad$ so $\quad \lambda = x/20$

    Thus $\frac{y}{30} = \frac{x}{20},$ giving $y = \frac{3}{2}x$ Substitute in constraint.

    So $20\left(\frac{3x}{2}\right) + 30x = 480,\quad 30x + 30x = 480,\quad$ so $\quad x = 8$ and $y = \frac{3}{2}(8) = 12$

    The dimensions of the largest possible garden are: 8 ft for east/west sides and 12 ft for north/south sides.

12. Objective function: $5x_1^2 + 500x_1 + x_2^2 + 240x_2$
constraint: $x + y = 1000$
$F(x_1, x_2, \lambda) = 5x_1^2 + 500x_1 + x_2^2 + 240x_2 - \lambda(x_1 + x_2 - 1000)$
$\frac{\partial F}{\partial x} = 10x_1 + 500 - \lambda$, so $\lambda = 10x_1 + 500$
$\frac{\partial F}{\partial y} = 2x_2 + 240 - \lambda$, so $\lambda = 2x_2 + 240$
Thus $10x_1 + 500 = 2x_2 + 240$, so $x_2 = 5x_1 + 130$ Substitute in constraint.
Thus $x_1 + 5x_1 + 130 = 1000$, so $x_1 = \frac{870}{6} = 145$, and $x_2 = 5(145) + 130 = 855$ Thus 145 units should be produced at plant 1 and 855 units at plant 2.

Appendix A

Review of Exponents and Radicals

In this section we review the basic laws governing exponents and radicals. This material is truly necessary for a manipulating fundamental expressions in Calculus. We recall that if $a > 0$ is any real number and r is a positive integer, the symbol a^r is shorthand for the product of a with itself r-times. That is, $a^r = a \cdot a \cdot a \cdots a$, where there appears r a's on the right. Thus, $a^3 = a \cdot a \cdot a$ while $a^5 = a \cdot a \cdot a \cdot a \cdot a$, etc. By definition we will always take it that $a^0 = 1$, regardless of the value of a, so long as it is not equal to zero, and $a^1 = a$ for any a.

Generally if $r, s \geq 0$ are any two non-negative real numbers and $a, b > 0$, then the Laws of Exponents say that

$$a^r \cdot a^s = a^{r+s} \tag{A.1}$$
$$(a^r)^s = a^{r \cdot s}, \qquad (a^r)^{-s} = a^{-r \cdot s} \tag{A.2}$$
$$(ab)^r = a^r \cdot b^r \tag{A.3}$$
$$\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r} \tag{A.4}$$
$$\frac{a^r}{a^s} = a^{r-s}. \tag{A.5}$$

The Laws of Radicals are similar. They differ only from the Laws of Exponents in their representation using *radical* symbols rather than powers. For example, if $p, q > 0$ are integers, and we interpret the symbol $a^{\frac{p}{q}}$ as the q-th root of the number a to the power of p, *i.e.,*

$$a^{\frac{p}{q}} = \sqrt[q]{\mathbf{a^p}} = \left(\sqrt[q]{\mathbf{a}}\right)^p, \tag{A.6}$$

we obtain the Laws of Radicals

$$\sqrt[p]{a^p} = a, \quad \left(\sqrt[p]{a}\right)^p = a$$
$$\sqrt[p]{ab} = \sqrt[p]{a} \cdot \sqrt[p]{b}$$
$$\sqrt[p]{\frac{a}{b}} = \frac{\sqrt[p]{a}}{\sqrt[p]{b}}$$
$$\sqrt[p]{\sqrt[r]{a}} = \sqrt[pr]{a}.$$

Note that we obtain the rule $\sqrt[p]{ab} = \sqrt[p]{a} \cdot \sqrt[p]{b}$ by setting $r = \frac{1}{p}$ in (A.3) above and using the symbol interpreter $a^{\frac{1}{p}} = \sqrt[p]{a}$. For example, by (A.6), we see that $9^{\frac{3}{2}} = \left(\sqrt{9}\right)^3 = 3^3 = 27$, while $(27)^{\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$.

We emphasize that these two laws are *completely general* in the sense that the symbols a, b appearing in them need not be single numbers only (like 3 or 1.52) but can be any abstract combination of such numbers or even other symbols and numbers together! For example, it is the case that

$$\left(2x + y\sqrt{x}\right)^{-1} = \frac{1}{2x + y\sqrt{x}}$$

and this follows from the fact that

$$a^{-1} = \frac{1}{a}$$

for any non-zero number a. Incidentally, this latest identity follows from (A.1) with $r = 1, s = -1$ and the defintion $a^0 = 1$. In order to show the power of these formulae we use the Box Method of Section 1.2 to solidify their meaning. Thus, instead of writing the Laws of Exponents and Radicals as above, we rewrite them in the form

$$\square^{\,r} \cdot \square^{\,s} = \square^{\,r+s} \tag{A.7}$$

$$(\square^{\,r})^s = \square^{\,r \cdot s}, \qquad (\square^{\,r})^{-s} = \square^{\,-r \cdot s} \tag{A.8}$$

$$(\square_{\,1}\, \square_{\,2})^r = (\square_{\,1})^r \cdot (\square_{\,2})^r \tag{A.9}$$

$$\left(\frac{\square_{\,1}}{\square_{\,2}}\right)^r = \frac{(\square_{\,1})^r}{(\square_{\,2})^r} \tag{A.10}$$

$$\frac{\square^{\,r}}{\square^{\,s}} = \square^{\,r-s}, \tag{A.11}$$

and remember that we can put any abstract combination of numbers or even other symbols and numbers together inside the Boxes in accordance with the techniques described in Section 1.2 for using the Box Method. So, for example, we can easily see that

$$\left(2x + y\sqrt{x}\right)^{-1} = \frac{1}{2x + y\sqrt{x}}$$

alluded to above since we know that

$$\square^{\,-1} = \frac{1}{\square},$$

and we can put the group of symbols $2x + y\sqrt{x}$ *inside* the Box so that we see

$$\left(\boxed{2x + y\sqrt{x}}\right)^{-1} = \frac{1}{\boxed{2x + y\sqrt{x}}}$$

and then remove the sides of the box to get the original identity.

Another example follows: Using (A.9) above, that is,

$$(\square_{\,1}\, \square_{\,2})^r = (\square_{\,1})^r \cdot (\square_{\,2})^r$$

we can put the symbol $3y$ inside box 1, *i.e.,* $\square_{\,1}$, and $x + 1$ inside box 2, *i.e.,* $\square_{\,2}$, to find that if $r = 2$ then, once we remove the sides of the boxes,

$$(3y(x + 1))^2 = (3y)^2(x + 1)^2 = 9y^2(x + 1)^2.$$

The Box Method's strength lies in assimilating large masses of symbols into one symbol (the box) for ease of calculation!

Remark Using the same ideas and (A.8) we can show that $\left(3^2\right)^{-3} = 3^{-2\cdot 3} = 3^{-6}$ and NOT equal to 3^{2-3} as some might think! You should leave the parentheses alone and not drop them when they are present!

Of course we can put anything we want inside this box so that (if we put $\sqrt{2}x - 16xy^2 + 4.1$ inside) it is still true that (use (A.8))

$$\left(\square^{2}\right)^{-3} = \left(\square^{-3}\right)^{2} = \square^{-3\cdot 2} = \square^{-6}$$

or

$$\left(\left(\sqrt{2}x - 16xy^2 + 4.1\right)^2\right)^{-3} = \left(\sqrt{2}x - 16xy^2 + 4.1\right)^{-6}.$$

Finally, don't forget that

$$\square^{0} = 1, \quad \square^{1} = \square, \quad \text{and,} \quad \square^{-1} = \frac{1}{\square}.$$

Example 532 *Simplify the product $2^3 3^2 2^{-1}$, without using your calculator.*

Solution We use the Laws of Exponents:

$$\begin{aligned} 2^3 3^2 2^{-1} &= 2^3 2^{-1} 3^2 \\ &= 2^{3-1} 3^2 && \text{by (A.7)} \\ &= 2^2 3^2 \\ &= (2\cdot 3)^2 && \text{by (A.9)} \\ &= 6^2 \\ &= 36. \end{aligned}$$

Example 533 *Simplify the expression $(2xy)^{-2} 2^3 (yx)^3$.*

Solution Use the Laws (A.7) to (A.11) in various combinations:

$$\begin{aligned} (2xy)^{-2} 2^3 (yx)^3 &= (2xy)^{-2} (2yx)^3 && \text{by (A.9)} \\ &= (2xy)^{-2+3} && \text{by (A.7)} \\ &= (2xy)^1 \\ &= 2xy. \end{aligned}$$

Example 534 *Simplify $2^8 4^{-2} (2x)^{-4} x^5$.*

Solution We use the Laws (A.7) to (A.11) once again.

$$\begin{aligned} 2^8 4^{-2} (2x)^{-4} x^5 &= 2^8 (2^2)^{-2} (2x)^{-4} x^5 \\ &= 2^8 2^{-4} (2x)^{-4} x^5 && \text{by (A.8)} \\ &= 2^{8-4} (2x)^{-4} x^5 && \text{by (A.7)} \\ &= 2^4 2^{-4} x^{-4} x^5 && \text{by (A.9)} \\ &= 2^{4-4} x^{-4+5} && \text{by (A.7)} \\ &= 2^0 x^1 \\ &= x. \end{aligned}$$

Example 535 *Simplify* $\dfrac{2^{2^3}\left(2^2\right)^{-3}}{4}$.

Solution Work out the highest powers first so that since $2^3 = 8$ it follows that $2^{2^3} = 2^8$. Thus,

$$\begin{aligned} \frac{2^{2^3}\left(2^2\right)^{-3}}{4} &= \frac{2^8 2^{-3\cdot 2}}{4} \qquad \text{by (A.8)} \\ &= \frac{2^8 2^{-6}}{2^2} = \frac{2^{8-6}}{2^2} = \frac{2^2}{2^2} \\ &= 1. \end{aligned}$$

Example 536 *Write* $\left((49)^{-\frac{1}{2}}\right)^3$ *as a rational number (ordinary fraction).*

Solution $\left((49)^{-\frac{1}{2}}\right)^3 = \left((49)^{\frac{1}{2}}\right)^{-3}$ by (A.8). Next, $(49)^{\frac{1}{2}} = \sqrt{49} = 7$, so, $\left((49)^{\frac{1}{2}}\right)^{-3} = 7^{-3} = \frac{1}{7^3} = \frac{1}{343}$.

Example 537 *Simplify as much as possible:* $(16)^{-\frac{1}{6}}4^{\frac{7}{3}}(256)^{-\frac{1}{4}}$.

Solution The idea here is to rewrite the bases 4, 16, 256, in lowest common terms, if possible. Thus,

$$\begin{aligned} (16)^{-\frac{1}{6}}4^{\frac{7}{3}}(256)^{-\frac{1}{4}} &= (4^2)^{-\frac{1}{6}}4^{\frac{7}{3}}(4^4)^{-\frac{1}{4}} \\ &= 4^{-\frac{1}{3}}4^{\frac{7}{3}}4^{-1} \qquad \text{by (A.8)} \\ &= 4^{-\frac{1}{3}+\frac{7}{3}-1} \qquad \text{by (A.7)} \\ &= 4^{\frac{6}{3}-1} \\ &= 4. \end{aligned}$$

Example 538 *Simplify to an expression with positive exponents:* $\dfrac{x^{-\frac{1}{6}}x^{\frac{2}{3}}}{x^{\frac{5}{12}}}$.

Solution Since the bases are all the same, namely, x, we only need to use a combination of (A.7) and (A.11). So,

$$\begin{aligned} \frac{x^{-\frac{1}{6}}x^{\frac{2}{3}}}{x^{\frac{5}{12}}} &= \frac{x^{-\frac{1}{6}+\frac{2}{3}}}{x^{\frac{5}{12}}} \qquad \text{by (A.7)} \\ &= \frac{x^{\frac{1}{2}}}{x^{\frac{5}{12}}} \\ &= x^{\frac{1}{2}-\frac{5}{12}} \qquad \text{by (A.11)} \\ &= x^{\frac{7}{12}}. \end{aligned}$$

Example 539 *Show that if* $r \neq 1$ *then* $1 + r + r^2 = \dfrac{1 - r^3}{1 - r}$.

Solution It suffices to show that $(1+r+r^2)(1-r) = 1-r^3$ for any value of r. Division by $1 - r$ (only valid when $r \neq 1$) then gives the required result. Now,

$$\begin{aligned} (1 + r + r^2)\cdot(1 - r) &= (1 + r + r^2)\cdot(1) + (1 + r + r^2)\cdot(-r) \\ &= (1 + r + r^2) + (1)\cdot(-r) + r\cdot(-r) + r^2\cdot(-r) \\ &= 1 + r + r^2 - r - r^2 - r^3 \qquad \text{by (A.7)} \\ &= 1 - r^3 \end{aligned}$$

and that's all.

Example 540 *For what values of a is $x^4+1=(x^2+ax+1)\cdot(x^2-ax+1)$?*

Solution We simply multiply the right side together, compare the coefficients of like powers and then find a. Thus,

$$\begin{aligned} x^4+1 &= (x^2+ax+1)\cdot(x^2-ax+1)\\ &= x^2\cdot(x^2-ax+1)+ax\cdot(x^2-ax+1)+1\cdot(x^2-ax+1)\\ &= (x^4-ax^3+x^2)+(ax^3-a^2x^2+ax)+(x^2-ax+1)\\ &= x^4-ax^3+x^2+ax^3-a^2x^2+ax+x^2-ax+1\\ &= x^4+(2-a^2)\cdot x^2+1. \end{aligned}$$

Comparing the coefficients on the left and right side of the last equation we see that $2-a^2=0$ is necessary. This means that $a^2=2$ or $a=\pm\sqrt{2}$.

Note Either value of a in Example 540 gives the same factors of the polynomials x^4+1. More material on such factorization techniques can be found in Chapter 5.

Exercise Set 57

Simplify as much as you can to an expression with positive exponents.

1. $16^2\times 8\div 4^3$
2. $(25^2)^{\frac{1}{2}}$
3. $2^4 4^2 2^{-2}$
4. $\frac{3^2 4^2}{12}$
5. $5^3 15^{-2} 3^4$
6. $(2x+y)\cdot(2x-y)$
7. $1+(x-1)(x+1)$
8. $\left((25)^{-\frac{1}{2}}\right)^2+5^{-2}$
9. $(4x^2y)^2 2^{-4}x^{-2}y$
10. $(1+r+r^2+r^3)\cdot(1-r)$
11. $(a^9b^{15})^{\frac{1}{3}}$
12. $(16a^{12})^{\frac{3}{4}}$
13. $\frac{x^{\frac{1}{4}}x^{-\frac{2}{3}}}{x^{\frac{1}{6}}}$
14. $\left(\frac{1}{16}\right)^{-\frac{3}{2}}$
15. $\left(1+5^{\frac{1}{3}}\right)\cdot\left(1-5^{\frac{1}{3}}+(25)^{\frac{1}{3}}\right)$
16. $\left(9x^{-8}\right)^{-\frac{3}{2}}$
17. $9^{-\frac{1}{6}}3^{\frac{7}{3}}(81)^{-\frac{1}{4}}$
18. $\dfrac{(12)^{\frac{3}{2}}(16)^{\frac{1}{8}}}{(27)^{\frac{1}{6}}(18)^{\frac{1}{2}}}$
19. $\dfrac{3^{n+1}9^n}{(27)^{\frac{2n}{3}}}$

20. $\dfrac{\sqrt{xy}\; x^{\frac{1}{3}}\; 2y^{\frac{1}{4}}}{(x^{10}y^9)^{\frac{1}{12}}}$

21. Show that there is **no** real number a such that $(x^2+1)=(x-a)\cdot(x+a)$.
22. Show that $(1+x^2+x^4)\cdot(1-x^2)=1-x^6$.
23. Find an expression for the quotient $\dfrac{1-x^8}{1-x}$ as a sum of powers of x only.
24. Show that $(x-1)(x+1)(1+x^2)+1=x^4$.
25. Show that $3\left(x^2yz\right)^3 \div x^4y^3-3x^2z^3=0$ for any choice of the variables x,y,z so long as $xy\neq 0$.
26. Using the identities (A.7) and $a^0=1$ only, show that $a^{-r}=\dfrac{1}{a^r}$ for any real number r.
27. Show that if r,s are any two integers and $a>0$, then $(a)^{-rs}=(a^r)^{-s}=(a^s)^{-r}$.
28. Give an example to show that

$$2^{x^y}\neq(2^x)^y\,.$$

In other words, find two numbers x,y that have this property.

29. Show that for any number $r\neq -1$ we have the identity $1-r+r^2=\dfrac{1+r^3}{1+r}$ and use this to deduce that for any value of $x\neq -2$,

$$1-\frac{x}{2}+\frac{x^2}{4}=\frac{x^3+8}{4(x+2)}.$$

30. If $a>0$ and $2x=a^{\frac{1}{2}}+a^{-\frac{1}{2}}$ show that

$$\frac{\sqrt{x^2-1}}{x-\sqrt{x^2-1}}=\frac{a-1}{2}.$$

Suggested Homework Set 37 *Do all even-numbered problems from 2 - 28.*

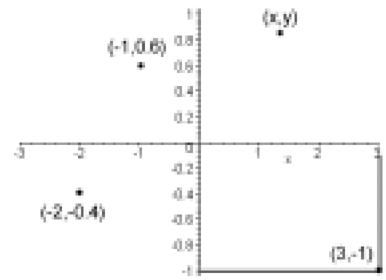
Web Links

Many more exercises may be found on the web site:

http://math.usask.ca/readin/

Appendix B

The Straight Line



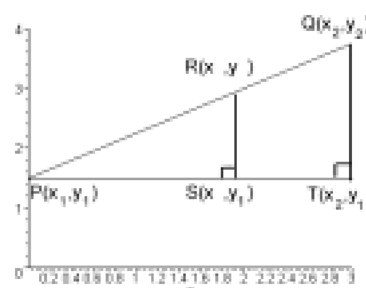
Points in the Euclidean plane

Figure 244

In this section we review one of the most fundamental topics of analytic geometry, the representation of a **straight line** with respect to a given set of coordinate axes. We recall that a point in the Euclidean plane is denoted by its two coordinates (x, y) where x, y are real numbers either positive, negative or zero, see Figure 244.

Thus, the point $(3, -1)$ is found by moving three positive units to the right along the x-axis and one unit "down" (because of the negative sign) along a line parallel to the y-axis. From the theory of plane Euclidean geometry we know that two given points determine a unique (straight) line. Its *equation* is obtained by describing every point on the straight line in the form $(x, y) = (x, f(x))$ where $y = f(x)$ is the equation of the straight line defined by some function f. To find this equation we appeal to basic Euclidean geometry and, in particular, to the result that states that *any two similar triangles in the Euclidean plane have proportional sides*, see Figure 245. This result will be used to find the equation of a straight line as we'll see.

We start off by considering two given points P and Q having coordinates (x_1, y_1) and (x_2, y_2) respectively. Normally, we'll write this briefly as $P(x_1, y_1)$ etc. Remember that the points P, Q are given ahead of time. Now, we join these two points by means of a straight line $\mathcal{L}$ and, on this line $\mathcal{L}$ we choose some point that we label as $R(x, y)$. For convenience we will assume that R is between P and Q.



The triangles PRS and PQT have proportional sides as they are similar.

Figure 245

Next, see Figure 245, we construct the two similar right-angled triangles $\triangle PQT$ and $\triangle PRS$. Since they are similar the length of their sides are proportional and so,

$$\frac{PS}{SR} = \frac{PT}{TQ}.$$

In terms of the coordinates of the points in question we note that $PS = x - x_1$, $SR = y - y_1$, $PT = x_2 - x_1$, $TQ = y_2 - y_1$. Rewriting the above proportionality relation in terms of these coordinates we get

$$\frac{x - x_1}{y - y_1} = \frac{x_2 - x_1}{y_2 - y_1},$$

or equivalently, solving for y and rewriting the equation, we see that

$$y = mx + b$$

where

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

is called the **slope of the straight line** and the number $b = y_1 - mx_1$ is called the **y-intercept** (*i.e.,* that value of y obtained by setting $x = 0$). The x-intercept is that value of x obtained by setting $y = 0$. In this case, the **x-intercept** is the complicated-looking expression

$$x = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1}.$$

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ be any two points on a line $\mathcal{L}$. The equation of $\mathcal{L}$ is given by

$$y = mx + b \tag{B.1}$$

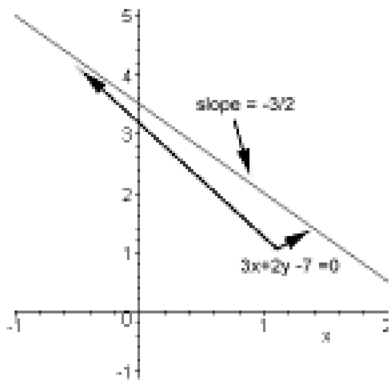
and will be called the **slope-intercept form of a line** where

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{B.2}$$

is called the **slope of the straight line** and the number

$$b = y_1 - mx_1 \tag{B.3}$$

is the **y-intercept**.



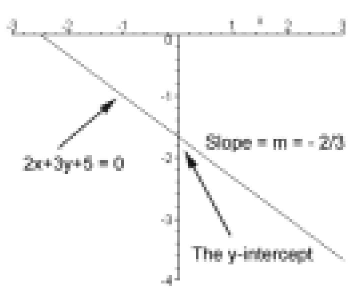
The graph of the line $3x + 2y - 7 = 0$ with a negative slope equal to $-3/2$.

Figure 246

Example 541 *Find the slope of the line whose equation is* $3x + 2y - 7 = 0$.

Solution First, let's see if we can rewrite the given equation in "slope-intercept form". To do this, we solve for y and then isolate it (by itself) and then compare the new equation with the given one. So, subtracting $3x - 7$ from both sides of the equation gives $2y = 7 - 3x$. Dividing this by 2 (and so isolating y) gives us $y = \frac{7}{2} - \frac{3}{2}x$. Comparing this last equation with the form $y = mx + b$ shows that $m = -\frac{3}{2}$ and the y-intercept is $\frac{7}{2}$. Its graph is represented in Figure 246.

Example 542 *Find the equation of the line passing through the points* $(2, -3)$ *and* $(-1, -1)$.



The line $2x + 3y + 5 = 0$ and its y-intercept.

Figure 247

Solution We use equations (B.1), (B.2) and (B.3). Thus, we label the points as follows: $(x_1, y_1) = (2, -3)$ and $(x_2, y_2) = (-1, -1)$. But the slope m is given by (B.2), *i.e.,*

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 + 3}{-1 - 2} = -\frac{2}{3}.$$

On the other hand the y-intercept is given by

$$b = y_1 - mx_1 = -3 + \frac{2}{3}(2) = -\frac{5}{3}.$$

The equation of the line is therefore $y = -\frac{2}{3}x - \frac{5}{3}$ or, equivalently, $2x + 3y + 5 = 0$, (see Figure 247).

Remark: It doesn't matter *which* point you label with the coordinates (x_1, y_1), you'll still get the same slope value and y-intercept! In other words, if we interchange the roles of (x_1, y_1) and (x_2, y_2) we get the *same* value for the slope, etc. and the *same* equation for the line.

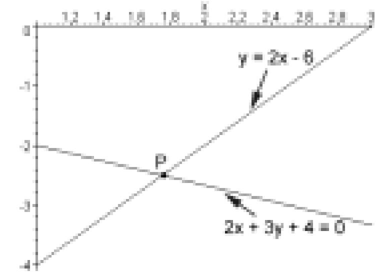
Example 543 *Find the equation of the line through* $(1, 4)$ *having slope equal to* 2.

Solution We are given that $m = 2$ in (B.1), so the equation of our line looks like $y = 2x + b$ where b is to be found. But we are given that this line goes thourhg the point $(1, 4)$. This means that we can set $x = 1$ and $y = 4$ in the equation $y = 2x + b$ and use this to find the value of b. In other words, $4 = 2 \cdot 1 + b$ and so $b = 2$. Finally, we see that $y = 2x + 2$ is the desired equation.

Example 544 *Find the equation of the line whose x-intercept is equal to* -1 *and whose y-intercept is equal to* -2.

Solution Once again we can use (B.1). Since $y = mx + b$ and the y-intercept is equal to -2 this means that $b = -2$ by definition. Our line now takes the form $y = mx - 2$. We still need to find m though. But by definition the fact that the x-intercept is equal to -1 means that when $y = 0$ then $x = -1$, *i.e.,* $0 = m \cdot (-1) - 2$ and this leads to $m = -2$. Thus, $y = -2x - 2$ is the equation of the line having the required intercepts.

Example 545 *Find the point of intersection of the two lines* $2x + 3y + 4 = 0$ *and* $y = 2x - 6$.



The two lines $2x+3y+4 = 0$ and $y = 2x-6$ and their point of intersection P$(\frac{7}{4}, -\frac{5}{2})$

Figure 248

Solution The point of intersection is necessarily a point, let's call it (x, y) once again, that belongs to *both* the lines. This means that $2x + 3y + 4 = 0$ AND $y = 2x - 6$. This gives us a system of two equations in the two unknowns (x, y). There are two ways to proceed; (1): We can isolate the y-terms, then equate the two x-terms and finally solve for the x-term, or (2): Use the method of elimination. We use the first of these methods here. Equating the two y-terms means that we have to solve for y in each equation. But we know that $y = 2x - 6$ and we also know that $3y = -2x - 4$ or $y = -\frac{2}{3}x - \frac{4}{3}$. So, equating these two y's we get

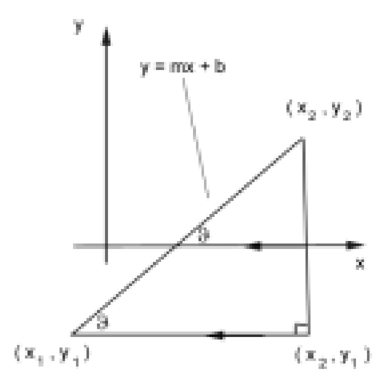
$$2x - 6 = -\frac{2}{3}x - \frac{4}{3}$$

or, equivalently,

$$6x - 18 = -2x - 4.$$

Isolating the x, gives us $8x = 14$ or $x = \frac{7}{4}$. This says that the x-coordinate of the required point of intersection is given by $x = \frac{7}{4}$. To get the y-coordinate we simply use EITHER one of the two equations, plug in $x = \frac{7}{4}$ and then solve for y. In our case, we set $x = \frac{7}{4}$ in, say, $y = 2x - 6$. This gives us $y = 2 \cdot (\frac{7}{4}) - 6 = -\frac{5}{2}$. The required point has coordinates $(\frac{7}{4}, -\frac{5}{2})$, see Figure 248.

Prior to discussing the *angle between two lines* we need to recall some basic notions from Trigonometry, see Appendix **??**. First we note that the slope m of a line whose equation is $y = mx + b$ is related to the *angle* that the line itself makes with the x-axis. A look at Figure 249 shows that, in fact,



The angle θ between the line $y = mx + b$ and the x-axis is related to the slope m of this line via the relation $m = \tan\theta$.

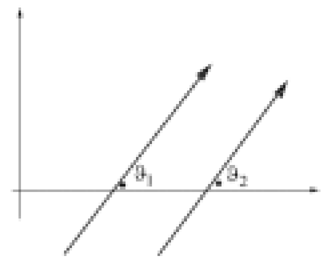
Figure 249

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{opposite}}{\text{adjacent}} = \tan\theta,$$

by definition of the tangent of this angle. So,

$$\boxed{m = \text{Slope} = \tan\theta}$$

where the angle θ is usually expressed in **radians** in accordance with the conventions of Calculus.



Parallel lines have the same slope and, conversely, if two lines have the same slope they are parallel

Figure 250

Now, if two lines are parallel their corresponding angles are equal (this is from a really old result of Euclid - sometimes called the **corresponding angle theorem**, CAT, for short). This means that the angle that each one makes with the x-axis is the same for each line (see Figure 250), that is $\theta_2 = \theta_1$. But this means that the slopes are equal too, right? Okay, it follows that if two lines are parallel, then their slopes are equal and conversely, if two lines have equal slopes then they must be parallel. If $\theta_2 = \theta_1 = \frac{\pi}{2}$, the lines are still parallel but they are now perpendicular with respect to the x-axis. In this case we say they have no slope or their slope is infinite. Conversely, if two lines have no slopes they are parallel as well (just draw a picture).

We now produce a relation that guarantees the *perpendicularity* of two given lines. For instance, a glance at Figure 251 shows that if θ_1, θ_2 are the angles of inclination of the two given lines and we assume that these two lines are perpendicular, then, by a classical result of Euclidean geometry, we know that

$$\begin{aligned} \theta_2 &= \theta_1 + \frac{\pi}{2} \\ \tan\theta_2 &= \tan\left(\theta_1 + \frac{\pi}{2}\right) \\ &= -\cot\theta_1 \\ &= -\frac{1}{\tan\theta_1}. \end{aligned}$$

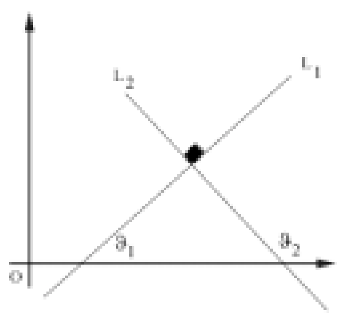


Figure 251

Since $m_2 = \tan\theta_2, m_1 = \tan\theta_1$, it follows that $m_2 = -\frac{1}{m_1}$. We have just showed that two lines having slopes m_1, m_2 are perpendicular only when

$$m_2 = -\frac{1}{m_1}$$

that is, two lines are perpendicular only when the product of their slopes is the number -1. The converse is also true, that is, if two lines have the product of their slopes equal to -1 then they are perpendicular. This relates the geometrical notion of perpendicularity to the stated relation on the slopes of the lines. It follows from this that if a line has its slope equal to zero, then it must be parallel to the x-axis while if a line has no slope (or its slope is infinite) then it must be parallel to the y-axis.

Example 546 *Find the slopes of the sides of the triangle whose vertices are* $(6, 2), (3, 5)$ *and* $(5, 7)$ *and show that this is a right-triangle.*

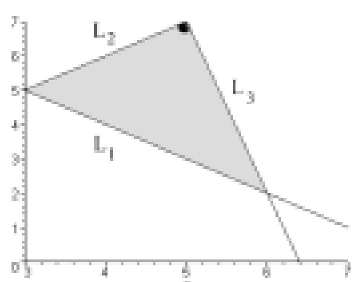


Figure 252

Solution Since three distinct points determine a unique triangle on the plane it suffices to find the slopes of the lines making up its sides and then showing that the product of the slopes of two of them is -1. This will prove that the triangle is a right-angled triangle.

Now the line, say $\mathcal{L}_1$, joining the points $(6, 2), (3, 5)$ has slope $m_1 = \frac{5-2}{3-6} = -1$ while the line, $\mathcal{L}_2$, joining the points $(3, 5)$ and $(5, 7)$ has slope $m_2 = \frac{7-5}{5-3} = 1$. Finally, the line, $\mathcal{L}_3$, joining $(6, 2)$ to $(5, 7)$ has slope $m_3 = \frac{7-2}{5-6} = -5$. Since $m_1 \cdot m_2 = (-1) \cdot (1) = -1$ it follows that those two lines are perpendicular, see Figure 252. Note that we didn't actually have to calculate the *equations* of the lines themselves, just the *slopes*!

Example 547 *Find the equation of the straight line through the point* $(6, -2)$ *that is (a) parallel to the line* $4x - 3y - 7 = 0$ *and (b) perpendicular to the line* $4x - 3y - 7 = 0$.

Solution (a) Since the line passes through $(x_1, y_1) = (6, -2)$ its equation has the form $y - y_1 = m_1(x - x_1)$ or $y = m_1(x-6) - 2$ where m_1 is its slope. On the other hand, since it is required to be parallel to the $4x - 3y - 7 = 0$ the two must have the *same* slope. But the slope of the given line is $m = \frac{4}{3}$. Thus, $m_1 = \frac{4}{3}$ as well and so the line parallel to $4x - 3y - 7 = 0$ has the equation $y = \frac{4}{3}(x-6) - 2$ or, equivalently (multiplying everything out by 3), $3y - 4x + 30 = 0$.

(b) In this case the required line must have its slope equal to the negative reciprocal of the first, that is $m_1 = -\frac{3}{4}$ since the slope of the given line is $m = \frac{4}{3}$. Since $y = m_1(x-6) - 2$, see above, it follows that its equation is $y = -\frac{3}{4}(x-6) - 2$ or, equivalently, $4y + 3x - 10 = 0$.

Exercise Set 58

1. Find the slope of the line whose equation is $2x - 3y = 8$
2. Find the slope of the line whose equation is $2x - 3y = -8$
3. Find the slope of the line whose equation is $y - 3x = 2$
4. Find the equation of the line passing through the point $(2, -4)$ and $(6, 7)$
5. Find the equation of the line passing through the point $(-4, -5)$ and $(-2, -3)$
6. Find the equation of the line passing through $(-1, -3)$ having slope -2
7. Find the equation of the line passing through $(6, -2)$ having slope $\frac{4}{3}$
8. Write the equation of the line whose $x-$intercept is 2 and whose $y-$intercept is 3
9. Write the equation of the line whose $x-$intercept is $\frac{1}{2}$ and whose $y-$intercept is $\frac{1}{3}$
10. Find the point of intersection of the two lines $y = x + 1$ and $2y + x - 1 = 0$
11. Find the points of intersection of the two lines $2y = 2x + 2$ and $3y - 3x - 3 = 0$. Explain your answer.
12. Find the point of intersection (if any) of the two lines $y - x + 1 = 0$ and $y = x$
13. Recall that the distance between two points whose coordinates are $A(x_1, y_1)$, $B(x_2, y_2)$ is given by

    $$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

    Of course, this quantity AB is also equal to the length of the line segment joining A to B. Use this information to answer the following questions about the triangle formed by the points $A(2, 0)$, $B(6, 4)$ and $C(4, -6)$: (a) Find the equation of the line through AB; (b) Find the length of the altitude from C to AB (*i.e.,* the length of the perpendicular line through C meeting AB); (c) Find the area of this triangle ABC
14. Find the equation of the straight line through $(1, 1)$ and perpendicular to the line $y = -x + 2$
15. Find the equation of the straight line through $(1, 1)$ and parallel to the line $y = -x + 2$

Appendix C

Solutions

C.1 APPENDIX A - Exercise Set

1. 2
2. $\frac{1}{25}$
3. $2^6 = 64$
4. 12
5. 45
6. $4x^2 - y^2$
7. x^2
8. $\frac{2}{25}$
9. x^2y^3
10. $1 - r^4$
11. a^3b^5
12. $8a^9$
13. $\frac{1}{x^{\frac{7}{12}}}$
14. 64
15. 6
16. $\frac{x^{12}}{27}$
17. 3
18. 8
19. 3^{n+1}
20. 2
21. Expand the right side, collect terms and compare the coefficients. You'll find that $1 = -a^2$ which is an impossibility since the right side is always negative or zero and the left side is positive.
22. See Example 539 where you set $r = x^2$.
23. $1 + x^2 + x^4 + x^6$; see the previous exercise.
24. Expand the left-side and simplify.
25. Use the Power Laws and simplify

26. Since $a^{r+s} = a^r a^s$ we can set $s = -r$. Then $a^0 = a^r a^{-r}$ and since $a^0 = 1$ we get $1 = a^r a^{-r}$ and the result follows.
27. See the Introduction to this section for a similar argument.
28. Let $x = 2$, $y = 3$. Then $2^8 \neq 2^6$.
29. Replace r by $-r$ in Example 539 and then set $r = \frac{x}{2}$ and simplify.
30. Write x as $x = \dfrac{a^{\frac{1}{2}} + a^{-\frac{1}{2}}}{2}$. Square both sides of this equality, use the Powers Laws, and then subtract 1 from the result. Simplify.

C.2 APPENDIX B - Exercise Set

1. $\frac{2}{3}$
2. $\frac{2}{3}$
3. 3
4. $y = 3x - 10$
5. $y = x - 1$
6. $y = -2x - 5$
7. $y = \frac{4}{3}x - 10$
8. $y = -\frac{3}{2}x + 3$
9. $y = -\frac{2}{3}x + \frac{1}{3}$
10. $\left(-\frac{1}{3}, \frac{2}{3}\right)$
11. There is no intersection whatsoever since the lines are parallel or have the same slope $(= 1)$
12. There is no intersection point either since the lines are parallel or have the same slope $(= 1)$
13. a) $y = x - 2$
    b) The altitude has length $\sqrt{32} = 4\sqrt{2}$. First, we find the equation of the line through $(4, -6)$ having slope -1 as it must be perpendicular to the line through $(2, 0)$ and $(6, 4)$ (*i.e.,* $y = x - 2$). This line is given by $y = -x - 2$. The find the point of intersection of this line with $y = x - 2$. We get the point $(0, -2)$. The base of the triangle has length given by the distance formula in the exercise applied to the points A and B. Its value is $\sqrt{32}$. The altitude has height given by the same distance formula, namely, the distance between the points C$(4, -6)$ and $(0, -2)$; its value is $\sqrt{32}$ as well. The rest follows.
    c) Area $= (1/2)\sqrt{(32)}\sqrt{(32)} = 16$
14. $y = x$
15. $y = -x + 4$

Credits

The design, graphics, and caricatures in this book were created and manipulated by the author. Layout and front cover design by the author. The original cover on the Modules showed a ghosted image of the original front page of Leibniz's pioneering paper on the Calculus, dated 1684, and written in Latin. The present manuscript shows a colour photograph (by the author) of a path in the Taronga Zoo in Sydney, Australia.

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Index

$\pm\infty$, 68
$e \approx 2.7182818459$, 172
e^x, 178
n^{th} term, 247
squiggle, 50
$y-$intercepts, 230
(3n+1)-problem, 248
, 205
epsilon-delta method, 272

integral, logarithmic, 428

absolute maximum value, 232
absolute value, 11, 12, 451
absolute value function, 43
absolute zero, 29
Actual value, 419
Addition identities, 17
Advanced Topics, 30, 31, 39, 41, 43, 61, 103, 141
AG- inequality, 24
air, 523
air resistance, 523
air resitance, 523
airplane, 523
Alcuin of York, 36
Alexandria, 428
Algebra, 202
Analysis, 119
Analytic Geometry, 41
anatomy, indefinite integral, 289
antiderivative, 284, 299, 505
antiderivative, geometric interpretation, 299
Arabic numerals, 38
Arccosecant, 133
Arccosine, 133
Arccotangent, 133
Archimedes, 75
Arcsecant, 133
Arcsine, 133
Arctangent, 133
area, 299, 417, 454
area between two curves, 125
area of a sector, 46
arithmetic mean, 24
arrow diagram, 2
asteroid, 505
average radius, 471
axis of rotation, 470

back substitution, 336
bacteria, 522
base, 166, 180
Berkeley, Bishop, 104
Bernoulli, 165
Bernoulli numbers, 311
Bernoulli, Johann, 150
Bernoulli, John, 293
Binomial Theorem, 8, 24, 85
biology, 511
biomathematics, 505
black hole, 295
Black-Sholes equation, 75
Bolzano's Theorem, 55, 123, 140
box method, 4
break-point, 212
break-points, 204, 210, 213

C, 144
C me Hava Pizza, 229
C++, 144
C++, program, 328
Calculus, 104, 522
Calculus, Fundamental Theorem of, 283, 318
Cantor, Georg, 68
Carbon, 522
carcinogen, 521
catenary, 179
Cauchy problem, 508
Cauchy-Kovalevski Theorem, 15
center of mass, 283, 492
central reflection, 240
centroid, 492
Chain Rule, 91–93, 339, 505, 533
change of base, 190
change of base formula, 180
change of variable formula, 333
Charlemagne, 36
chemical, 520
classical solution, 505
closed interval, 4
Cobalt, 520
Cobb-Douglas, 545
comet, 505
commutative, 125
completing the square, 402, 407

complex number, 309
composition of two functions, 125
concave down, 223, 230
concave up, 223, 230
concave up (down) at every point of the interval, 225
concavity, 223, 224
constant, 220
constant function, 120
constant of integration, 285
constraint, 538, 540, 541
continuity, 36, 264
Continuity of Various Trigonometric Functions, 46
continuous, 30, 31, 34, 36, 38, 42, 123, 264, 526
continuous, Bolzano's Theorem, 55
continuous, function properties, 39
continuous, IVT, 54
continuous, piecewise, 316, 417, 427
convergence of a sequence, 141
convergence of an infinite sequence, 172
convergence to a non-zero limit, 251
converse, 322
convexity, 223
Cooling, Newton's Law of, 522
critical point, 219, 537
critical points, 230
cubic polynomial, 426
culture, 522
curvature, 418
curvature, parabola, 424
curve sketching, 201
curvilinear trapezoids, 424
curvilinear triangle, 46, 467
cylinder, volume, 470

decay constant, 193
Decay Law, 518
decay rate, 519
Decay, Law, 293
Decay, Radioactive, 519
deck, 454
decomposition, partial fraction, 370, 374
Decreasing, 230
decreasing, 220
defined, 10
definite integral, 299
definite integral, estimating, 418
definite integral, properties, 301
demand, demand function, 545
dependent variable, 2, 506
derivative, 75, 76, 299
derivative, partial, 529, 530
derivative, trigonometric function, 111, 112, 115
Derivatives of Inverse Trigonometric Functions, 137
Descartes, René, 41
determinate, 61, 150
difference of the limits, 40
difference, forward, 314
differentiable at the point **a**, 86
differentiable everywhere, 86
differentiable on **I**, 85
differential equation, 292, 505
differential equation, separable, 104
differential equation, solution, 292
differential equations, 10, 121
differential operator, 100, 310
differentials, 104
Differentiation, 283
discontinuity, 263, 527
discontinuous, 30, 34, 36, 42, 43, 528
Dom (f), 2
domain, 2, 3, 125, 130, 133
domain of a sequence, 247
double root, 324
Double-angle identities, 17
Dusky Sparrow, 140

earth, 522
earth, mass, 497
economics, 416
Einstein, 1
Einstein, A., 26
Einstein, Albert, 273
electron, 284
ellipse, rotated ellipse, 542
Engineering, 308
entropy, 195
epsilon-delta method, 30
equally-spaced points, 418
equation, differential, 292
equation, ordinary differential, 505
equation, separable differential, 505
Error, 419, 425
error estimate, 416
Error term, Trapezoidal Rule, 420
estimate, error, 416
Euclid, 428
Euclid, of Alexandria, 312, 328
Euler's Exponential Function, 165
Euler's exponential function, 178
Euler's number, 291
Euler's number **e**, 174
Euler, Leonhard, 165
even function, 240, 306
EVEN SYMMETRY, 240
Exact value, 419
Existence Theorem, 508
explicit function, 104
explicit relation, 104
exponent, 166
exponential function, $\mathbf{e^x}$, 177
Exponents, Law of, 661
extended real number, 58, 68, 269
extrema, extremum, 537, 540

extreme point, 222

factorials, 24
Fibonacci sequence, 247
finite serties, 416
fluxions, 104
Formula, Half-Life, 521
forward difference operator, 314
Fourier Series, 308
fractions, partial, 370
free variable, 511
free-fall, 523
FTC, 320
FTC, see Fundamental Theorem of Calculus, 320
fulcrum, 493
function, 1, 2
function, even, 306
function, explicit, 104
function, Gaussian, 320
function, implicit, 105
function, inverse, 462
function, odd, 306
function, rational, 366
functions, hyperbolic, 335
functions, inverse, 283
Fundamental Theorem of Calculus, 283, 318, 345, 505

Galileo, 75
Gamma Function, 24
Gauss, 428
Gauss, C.F., 309
Gaussian function, 320
Gaussian integral, 429
general exponential function, 179
GENERAL EXTREMUM TEST, 241
general solution, 104, 507, 511
Generalized Power Rule, 93, 288, 289
geometric mean, 24
geometric series, 550
global extrema, 223
global maximum, 222
global maximum value, 222
global minimum, 222
global minimum value, 222
Grégoire de Saint Vincent, 187
graph, 30, 201
gravity, 523
greatest integer function, 249
Growth and Decay Law, 517
Growth and Decay Laws, 511
Growth Law, 518
Growth, Law, 293
guessing limits, 61

half-life, 193, 518, 521
Half-Life Formula, 521
Heaviside Function, 31
Heaviside, Oliver, 31
Higher order partial derivative, 531
higher-order derivatives, 100
hole, black, 295
horizontal asymptotes, 226, 230
horizontal line test, 126, 133
Horizontal line test, Test horizontal line, 126
hyperbolic cosine, 179
hyperbolic functions, 335

identical sequences, 247
identity, 16
implicit differentiation, 104, 114
implicit form, 512
implicit function, 105
Implicit Function Theorem, 105, 108, 512
implicit relation, 104, 512
improper integral, 430, 432
improper integrals, 24
incantations, 150
Increasing, 230
increasing, 220, 301
increasing sequence, 173
Increasing Sequence Theorem, 173
indefinite integral, 285, 505
indefinite integral, anatomy, 289
independent variable, 2, 506
indeterminate form, 61, 146
Indeterminate Forms, 150
indeterminate forms, 61, 150
Induction, Mathematical, 308
Inequalities, 20
infinite limits, 268
infinite series, 177, 309, 416
infinity, ∞ properties, 58
initial condition, 508
initial value problem, 293, 508, 512
inner radius, 471
instantaneous velocity, 7, 75
integrable, Riemann, 316
integral, 283
integral, Gaussian, 429
integral, improper, 430, 432
integral, indefinite, 285, 289
integral, Riemann, 315
integrand, 332
Integration, 283
Integration by Parts, 331, 345, 451
integration, numerical, 416
Intelligence Quotient, 428
intercepts, 230
Intermediate Value Theorem, 54, 123
intersect, 55
intersection, 55
interval, partition, 314
interval, symmetric, 306
inverse function, 462

Inverse Function Theorem, 182
Inverse Functions, 108
inverse functions, 283
inverse of a function, 125
inverse of a trigonometric function, 133
inverse of an operator, 125
inverse trigonometric functions, 120
irrational number, 165
isotope, 518
iteration, 141
IVT, 54

Javascript, 248
Johannes Müller of Königsberg, 49
Jordan content, 318

Kelvin, 29
Kepler, 75
Kepler's equation, 148
Kilowatt, 422
Kovalevskaya, Sofya, 15

L'Hospital, 165
L'Hospital's Rule, 150, 151, 423
L'Hospital's Rule for Limits at Infinity, 158
Lagrange multipliers, multipliers, 537, 540
Lagrange parameter, 540
laser, 423
Law of Decay, 518
Law of Growth, 518
Law of Growth and Decay, 511, 517
Law, Newton's Cooling, 522
Laws of Exponents, Exponent Laws, 661, 662
Laws of Growth and Decay, 293
Laws of Radicals, Radical Laws, 661
learning model, 522
Lebesgue measure, 318
left continuity, right continuity, 228
left derivative, 80
left-derivative, 83
left-hand limit, 279
left-neighborhood, 151
left-neighbourhood, 224
Legendre, 428
Leibniz, 1
Leibniz's Rule, 306, 318
Leibniz, Rule, 318
limit at infinity, 58
limit from the left, 31, 32, 41, 80, 253, 257–259, 279
limit from the right, 31, 41, 80, 253–255, 279
limit of a sequence, 248, 249
limit, Riemann sum, 315
limiting speed, 523
Limits of Indeterminate Forms, 69
Limits, properties, 40
line, 667
Linear Algebra, Algebra, 130
Linear Factor, 202
lines parallel, parallel lines, 670
local maximum, 222, 223
local maximum value, 222
local minimum, 222
local minimum value, 222
logarithm, 519
logarithmic integral, 428
Logistic equation, 11
loop, 427
Lorentz, H., 26

Main Theorem on Limits, 279, 281
Marginal Cost, 544
marginal cost function, 525
Mathematical Induction, 308
Mathzone, 248
maximum, 119
maximum value, 22, 56
Mean Value Theorem, 118, 120, 122, 299, 316
Midpoint Rule, 416
minimum, 119
minimum value, 56
minus infinity, $-\infty$, 58
model of learning, 522
moment about an axis, 494
moments, 493
monotone increasing, 172
motion, rectilinear, 451
multiple of the limit, 40
multiple of two continuous functions, 40

natural domain, 10
natural logarithm, 172, 179
neighborhood, 150
net change in position, 451
Newton, 75, 140
Newton's Law of Cooling, 196, 522
Newton's Method, 140
Newton's method, 320, 370
Newton, Sir Isaac, 1
Nieuwentijt, Bernard, 104
norm, 314
norm of a partition, 314
normal distribution, 195
normal line, 108
nuclear, 521
Number Theory, 248
Number theory, 311
number, complex, 309
number, prime, 312, 328
numerical approximations, 75
numerical integration, 416

odd function, 240, 306
ODD SYMMETRY, 240
Odd-even identities, 16
one-sided limit, 41, 278, 279
one-sided limits, 253
one-to-one, 126, 129
open interval, 4
operator, differential, 310
operator, forward difference, 314
operator, summation, 310
ordinary derivative, 75
ordinary differential equation, 75, 505
Oresme, Nicola, 41
oscillating sequence, 261
outer radius, 471

parabola, 424
parabola, curvature, 424
parachute, 523
Partial Differential Equations, 15
partial fraction decomposition, 374
partial fractions, 370
particular solution, 507
partition, 417
partition of an interval, 314
partition, norm, 314
partition, points, 426
partition, regular, 322
Parts, Integration by, 331
pattern recognition, 517
perpendicular, 670
piecewise continuous, 316, 417, 427
planet, 505
plasma field, 124
Plug-in Method, 373
plus infinity, $+\infty$, 58
Plutonium, 521
Poincaré, Henri, 5
point of inflection, 224, 226
points of inflection, 230
Polynomial, 201
polynomial, 40, 43, 212
polynomial inequalities, 209
polynomial inequality, 210
polynomial, trigonometric, 308
population biology, 511
position, net change in, 451
power, 166
Power Rule, Generalized, 288
prime number, 312, 328, 428
principal branch, 135
Principle of Mathematical Induction, 308
probability, 284, 522
Probability Theory, 320
problem, Cauchy, 508
problem, initial value, 293, 508
problem, three-row, 360
product of the limits, 40
product or quotient of two continuous functions, 40
Product Rule, 87, 345
program, C++, 328
projection, 475
proof by contradiction, 121
punctured neighborhood, 151
Pythagorean identities, 17

quadratic irreducible, 402
Quadratic Irreducible Factor, 202
Quadrature formula, 416
qualitative analysis, 122
quantum mechanics, 284
quasi-periodic function, 149
quotient of the limits, 40
Quotient Rule, 87

R-integral, 316
radian, 45, 110
radians, 5, 46
radio-isotope, 519
radioactive, 518
Radioactive Decay, 519
radioactive decay, 193
radioactivity, 518
Radiocarbon dating, 522
radiocarbon dating, 511
Radioisotopes, 194
radionuclide, 518
Radium, 522
radius, average, 471
radius, inner, 471
radius, outer, 471
Ran (f), 2
range, 2, 3, 125, 130, 133
range of a sequence, 247
rate of decay, 193
rate, decay, 519
rational function, 43, 366
rational functions, 201
rational number, 177
rationalization, 83
Real Analysis, 318
reciprocal of a number, 20
Reciprocal Rule, 93
rectilinear motion, 451
Regiomontanus, 49
regular partition, 322
relativity, 26
relativity, special, 268
removing the absolute value, 13
resistance, air, 523
revolution, solid, 470
revolution, solid of, 283
Riemann Hypothesis, 309
Riemann integrable, 316
Riemann Integral, 299
Riemann integral, 118, 315

Riemann sum, 417
Riemann sum, limit, 315
Riemann Zeta Function, 309
Riemann, B., 314
Riemann, Bernhard, 310
right derivative, 80
right-derivative, 83
right-hand limit, 279
right-neighborhood, 151
right-neighbourhood, 224
Rolle's Theorem, 119, 122
Rolle, Michel, 119
Roman numerals, 38
root, 55, 123, 145, 204, 212, 213
root of a polynomial, 55
root, double, 324
rotation, 470
Rule, Leibniz, 318
Rule, Leibniz's, 306
Rule, Product, 345
Rule, Substitution, 331, 333, 511

Saarinen, Eero, 179
Sabertooth Tiger, 522
Sandwich Theorem, 39, 58
scaling, 423
Schrödinger's equation, 11
SDT, 204, 212, 213
Seaside Dusky Sparrow, 140
second derivative, 100
Second Derivative Test, 537, 538, 540
sector of a circle, 46
semi-open interval, 4
separable differential equation, 505
separable equation, 511
separation, of variables, 293
sequence, 141, 172, 247, 417
sequence, finite, 247, 308
sequence, infinite, 247
series, finite, 309, 416
series, infinite, 309, 416
Sierpinski's Estimate, 313, 328
Sign Decomposition Table of a Polynomial, , 204
signum function, 18
simple root, 142
Simpson's Rule, 320, 416, 424
Simpson's Rule, Error term, 425
size of SDT, 205
SLANT ASYMPTOTES, 241
slice, 454
slice, horizontal, 455
slice, vertical, 455
slope, 76, 77
slope-intercept form, 668
Sociology, 283
Solar energy, 454
Solar Flare, 30
solid of revolution, 283, 470
solution, differential equation, 292
solution, general, 507
solution, particular, 507
Sophie Germain, 15
Sparrow, Dusky Seaside, 140
speed of light, 1, 26, 29, 124, 268
speed, limiting, 523
Square Root of the Square Rule, 12
square root of the square rule, 11
square, completing the, 407
star, 505
step-function, 30
stock options, 75
straight line, 667
Strontium, 523
subscripts, 247
subsequences, 253
substitution, 331
Substitution Rule, 331, 333, 451, 511
substitution, method of, 333
substitution, trigonometric, 406
sum of the limits, 40
sum or difference of two continuous functions, 39
Sum/Difference Rule, 87
summation, 308
summation operator, 310
surface, 526
symbol, 3
symmetric interval, 306
symmetry, 240

tangent line, 119
tangent line to P, 76
tangent line, slope, 77
tangent to a curve, 76
Taylor polynomial approximation, 202
Taylor series, 202
telescoping sum, 312
term of the sequence, 247
The Power Rule, 85
Theorem, Mean Value, 299
theorems, 118
Theory of Inequalities, 20
third derivative, 100
thought experiment, 273
three-row problem, 360
Tiger, Sabertooth, 522
torus, 470
total distance travel, 451
toxic, 521
Track & Field, 29
transcendental functions, 179
transformation of the independent variable, independent variable transformation, 100
trapezoid, 417
trapezoid, curvilinear, 424
Trapezoidal Rule, 416, 418

Trapezoidal Rule, Error term, 420
travel, total distance, 451
Triangle Inequalities, 21
triangle, curvilinear, 467
triangles similar, similar triangles, 667
trigonometric functions, 46
Trigonometric Identities, 16
trigonometric identities, 110
trigonometric identity, 53
trigonometric integral, 385
trigonometric polynomial, finite, 308
trigonometric substitution, 406
Trigonometry, 669
trigonometry, 45
two-sided limit, 36
two-sided limits, 278
Type I, 202
Type II, 202
Type II factor, 402

undecidable statements, 103
URL, 252
Utility, Marginal Utility, 545

variable, change of, 333
variable, dependent, 506
variable, dummy, 285
variable, free, 285, 511
variable, independent, 506
variables, separation of, 293
vertical asymptote, 213, 227, 229, 269
vertical asymptotes, 230
Vertical Line Test, 105
vertical tangent line, 229
virus, 521
volume, 454
volume of a solid of revolution, 125
volume, cylinder, 470

wall, width of, 471
wave phenomena, 110
web sites, 248, 252
Weierstrass, Karl, 30
width, wall, 471
World Wide Web, 9, 248, 252

zeros, 204, 309
Zeta Function, Riemann, 309