# Polar Coordinates

Let $r$ denote the distance of a point $P$ from the origin (an arbitrary fixed point denoted by the symbol $O$).

**Figure 1**

Next, let $\theta$ =angle between the radial line from $P$ to $O$ and the given line "$\theta = 0$", a kind of *positive axis* for our polar coordinate system. Polar coordinates are defined in terms of ordinary cartesian coordinates via the transformations

**Figure 2**

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

where

**Figure 3**

$$r \geq 0 \quad 0 \leq \theta < 2\pi.$$

That every point $P(x, y)$ in the ordinary $xy-$plane can be written in this new $(r, \theta)$-form form is a consequence of the fact that $P$ lies on the circumference of some circle centered at the origin and radius $R$ where $R$ is the distance from the point $P$ to the origin.

From these relationships we see that the coordinates of our point $P$ satisfy the relation $x^2 + y^2 = r^2 \underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} \Rightarrow x^2 + y^2 = r^2$ (so that, as we indicated, the point $P(x, y)$ is on a circle of radius $r$ centered at $O$). On the other hand, we can find $\theta$ by solving the equation

$$\tan\theta = \frac{y}{x} \quad \Longrightarrow \quad \theta = \arctan\left(\frac{y}{x}\right),$$

for $\theta$ in the interval $0 \leq \theta < 2\pi$.

Denoting the principal part (see Chapter 3) of the **arctangent** function by Arctan we see that

$$\theta = \arctan\left(\frac{y}{x}\right) = \begin{cases} \text{Arctan}\left(\frac{y}{x}\right) & \text{if} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \\ \text{Arctan}\left(\frac{y}{x}\right) + \pi & \text{if} \quad \frac{\pi}{2} < \theta \leq \frac{3\pi}{2}, \end{cases}$$

with the interpretation that $\theta = \pm\frac{\pi}{2}$ corresponds to points on the real $y-$axis and $\theta = 0$ corresponds to points on the real $x-$axis.

**Example 1** *Write the following points given in cartesian coordinates, in polar form: a)* $P(-1, 1)$, *b)* $Q(-2, -2\sqrt{3})$, *c)* $R(1, -\sqrt{3})$.

*Solution:* a) Here $P(-1, 1)$ is such that $x = -1$ and $y = 1$. So, $r^2 = (-1)^2 + (1)^2 = 2$ and $r = \sqrt{2}$. On the other hand, the angle $\theta$ subtended by the radial line from $P$ to $O$ and the line $\theta = 0$ is between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Thus,

$$\theta = \text{Arctan}\left(\frac{y}{x}\right) + \pi = \text{Arctan}(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}.$$

Thus, $(-1,1)$ corresponds to $(r,\theta) = (\sqrt{2}, \frac{3\pi}{4})$.

Note that, in fact, this pair $(r,\theta) = (\sqrt{2}, \frac{3\pi}{4})$ also represents the real and imaginary part of the **complex number** $(-1,1) = -1+\imath$, where $\imath^2 = -1$, by definition. Thus,

$$-1+\imath = \sqrt{2}e^{\frac{3\pi}{4}\imath}$$

is the *polar decomposition* of the given complex number. In general, *the point* $(a,b)$ *in the cartesian plane corresponds to the complex number* $re^{\theta\imath}$ where $r$ and $\theta$ are defined above.

b) In this case $x = -2$ and $y = -2\sqrt{3}$ so that $r^2 = (-2)^2 + (-2\sqrt{3})^2 = 16$ or $r = 4$. Note that the angle $\theta$ here lies in the *third* quadrant or that $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Hence,

$$\theta = \text{Arctan}\left(\frac{y}{x}\right) + \pi = \text{Arctan}\left(\frac{-2\sqrt{3}}{-2}\right) + \pi = \text{Arctan}\sqrt{3} + \pi = \frac{\pi}{3} + \pi = \frac{4\pi}{3},$$

since the only angle $\theta$ such that $\tan\theta = \sqrt{3}$ is $\theta = \pi/3$. It follows that the polar form of the point $Q$ is given by $(4, \frac{4\pi}{3})$ or, equivalently, the polar decomposition of the complex number $-2-2\sqrt{3}\imath$ is given by

$$-2-2\sqrt{3}\imath = 4e^{\frac{4\pi}{3}\imath}.$$

Another terminology for this is

$$4e^{\frac{4\pi}{3}\imath} = 4\text{cis}(4\pi/3).$$

c) Here we see easily that $r^2 = (1)^2 + (-\sqrt{3})^2 = 4$ so that $r = 2$. Next, since the point $R$ is in the fourth quadrant, its "angle", $\theta$, here is in the fourth quadrant and so must be an angle in the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Thus,

$$\theta = \text{Arctan}\left(\frac{y}{x}\right) = \text{Arctan}\left(\frac{-\sqrt{3}}{1}\right) = -\text{Arctan}\sqrt{3} = -\frac{\pi}{3}.$$

Hence,

$$1-\sqrt{3}\imath = 2e^{-\frac{\pi}{3}\imath}.$$

**NOTE:** Some texts use a different principal part for the arctangent function in the sense that their principal part is defined by

$$y = \text{Arctan}(x) \iff 0 \le y < \pi,\ y \ne \frac{\pi}{2}.$$

In this case our definition for the polar angle $\theta$ becomes

$$\theta = \arctan\left(\frac{y}{x}\right) = \begin{cases} \text{Arctan}\left(\frac{y}{x}\right) & \text{if} \quad 0 \le \theta < \pi, \\ \text{Arctan}\left(\frac{y}{x}\right) + \pi & \text{if} \quad \pi \le \theta < 2\pi, \end{cases}$$

In this case, the answers for a), b) and c) in Example 1 would be rewritten as (of course, they are all equal to the corresponding answers, just written differently),

$$\begin{aligned} -1+\imath &= \sqrt{2}e^{\frac{3\pi}{4}\imath} \\ -2-2\sqrt{3}\imath &= 4e^{\frac{4\pi}{3}\imath} \\ 1-\sqrt{3}\imath &= 2e^{\frac{5\pi}{3}\imath}. \end{aligned}$$

Why use a polar coordinate system? To attempt to simplify the problems we are faced with: Basically, in the section on double integrals, we will need to describe plane regions as easily as possible using "slices" in various coordinate systems, and if the region consists of sections of circles, then it is most suitable for description by a polar coordinate system.

How?

1. By adopting a coordinate system to the given region (when, say, we wish to integrate over a region using *double integration*)
2. If the region has circles or sections of such circles or is made up of a combination of arcs of circles and rays through the origin (see the margin).

Before we proceed we need to describe a typical "slice" in polar coordinates (see Chapter 9 for details on ordinary *slices*). In cartesian coordinates we recall that a slice of a region is composed of either a vertical line segment (we then call it a vertical slice) or a horizontal line segment (which we then call a horizontal slice). These can be written compactly as

$$x = constant, \quad \Longrightarrow \quad \text{Vertical slice},$$

or

$$y = constant, \quad \Longrightarrow \quad \text{Horizontal slice}.$$

In polar coordinates our "slices" are defined similarly except that we now set

$$r = constant \quad \Longrightarrow \quad \text{Circular slice},$$

or

$$\theta = constant, \quad \Longrightarrow \quad \text{Radial slice}.$$

The reason for this terminology is as follows:

**Example 2** *The slice $\theta = constant$ corresponds to a ray through the origin $O$ of the polar plane.*

*Why?* This is because the $r-$variable is unrestricted here and so, since $\theta$ is fixed the point $(r, \theta)$ is on the straight line that subtends an angle equal to $\theta$ with the polar axis. Since this line is a "ray" we can call this a "radial slice" although our slices tend to be finite line segments.

**Figure 4**

**Example 3** *The slice $r = constant$ corresponds to a circle centered at the origin $O$ of the polar plane with radius equal to the constant in question.*

*Why?* This is because the $\theta-$variable is unrestricted here and so, since $\theta$ is fixed $(0 \leq \theta < 2\pi)$ the point $(r, \theta)$ is on the indicated circle. A part (or sector) of this circle can be called a "circular slice".

**Figure 5**

In terms of sets we can describe these slices as follows:

$$\text{Radial slice} = \{(r, \theta) : \theta = \text{constant}, 0 \le r < +\infty\}.$$

On the other hand a "circular slice" can be described by

$$\text{Circular slice} = \{(r, \theta) : r = \text{constant}, 0 \le \theta < 2\pi\}.$$

Generally, our **slices will be assumed to have finite length**, just as they do when we are dealing with area problems or volumes of solids of revolution (see Chapter 9).

**Example 4** *The following are typical "slices" in polar coordinates (see the margin):*

- *Radial slice* $= \{(r, \theta) : \theta = \frac{\pi}{4},\ 1 \le r \le 2\}$
- *Radial slice* $= \{(r, \theta) : \theta = \frac{3\pi}{2},\ 0.5 \le r \le 0.8\}$
- *Circular slice* $= \{(r, \theta) : r = 1.2,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{2}\}$
- *Circular slice* $= \{(r, \theta) : r = 3,\ \frac{3\pi}{4} \le \theta \le \pi\}$

Now we can start describing regions using slices ...:

**Example 5** *Give a set-theoretic description of the disk, $\mathcal{D}$, of radius* 1 *centered at $O$ using polar coordinates.*

*Solution:* This expression "set-theoretic description" just means that we want to describe this disk as a set of points $(r, \theta)$ satisfying some inequalities. We do this by "slicing up" the disk. Take a typical radial slice, call it $\mathcal{R}$, from the origin to the circumference of the disk (see the margin). Now pick a point, let's call it $(r, \theta)$ on $\mathcal{R}$. What can we say about these two variables? One thing is certain,- $r$ can vary from 0 to 1 (depending on where it's located) and $\theta$ is just the angle the radial slice makes with the polar axis.

So, in order to describe a typical point inside the disk, we need to specify its distance from the origin and its "angle". But this distance $r$ is some number between 0 and 1, and its angle is some number between 0 and $2\pi$! So, in order to recreate this circle using slices only, we need to take the union of all these slices (that look like $\mathcal{R}$) and then let $\theta$ vary from 0 to $2\pi$. We then get the description for $\mathcal{D}$ as

$$\mathcal{D} = \{(r, \theta)\ :\ 0 \le r \le 1,\ 0 \le \theta < 2\pi\,\}.$$

**Example 6** *Give a set-theoretic description of the disk, $\mathcal{D} = \{x^2 + y^2 \le 4\}$ using both cartesian and polar coordinates. Compare the two descriptions.*

*Solution:* Cartesian coordinates: Draw the circle of radius 2 centered at the origin and take a typical vertical slice (see Chapter 9). The coordinates of the endpoints of this slice are then $(x, \sqrt{4-x^2})$ and $(x, -\sqrt{4-x^2})$. A point $(x, y)$ on this slice (that begins at $x$) has its $y-$coordinate varying from $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$. Symbolically, we write this as $-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$ (see the margin). On the other hand $x$ itself varies from $-2$ to $+2$. So, if we put all these slices *together* we can then describe the region as follows:

$$\mathcal{D} = \{(x, y) \,:\, -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}, \text{ and } -2 \leq x \leq 2\}.$$

Polar coordinates: We use the same ideas as in the preceding example, Example 5. We'll find that

$$\mathcal{D} = \{(r, \theta) \,:\, 0 \leq r \leq 2,\ 0 \leq \theta < 2\pi\},$$

that's all. This second description is much nicer than the one in cartesian coordinates since the endpoints of the inequalities for $r$ and $\theta$ are constant numbers, in the case of polar coordinates, not functions themselves! This aspect will be very useful in the later evaluation of *double integrals*.

**Example 7** *The ideas in Example 6 show that the circumference, $\mathcal{C}$, of the circle $x^2 + y^2 = R^2$ can be described by both*

$$\mathcal{C} = \{(r, \theta) \,:\, r = R, \textit{ and } 0 \leq \theta < 2\pi\},$$

*along with the cartesian description*

$$\mathcal{C} = \{(x, y) \,:\, |y| = \sqrt{R^2 - x^2}, \textit{ and } -R \leq x \leq R\}.$$

**Example 8** *Describe an entire Pizza slice (or sector of a circle) having vetex angle equal to $\pi/3$ and "radius" equal to 6 using polar coordinates.*

*Solution:* It doesn't matter where we put the pizza slice (before we eat it) so let's put its vertex at the origin $O$ of our polar plane and adjust one of its sides so that it lands along the polar axis. Let's call this sectorial region, $\mathcal{S}$. It is now easy to see that

$$\mathcal{S} = \{(r, \theta) \,:\, 0 \leq r \leq 6, \quad 0 \leq \theta \leq \frac{\pi}{3}\}$$

and this describes the slice nicely. As a challenge, you may want to try doing this using cartesian coordinates!

**Figure 6**

**Example 9** *Describe an annulus (or ring or washer), $\mathcal{A}$, having inner radius equal to $r_1$ and outer radius equal to $r_2$ (where $r_2 > r_1$) using polar coordinates.*

*Solution:* Taking a typical radial slice we note that only the part that crosses the annulus is of concern to us. In this case it is easy to see that $r_1 \leq r \leq r_2$ for any point $(r, \theta)$ on it. Thus,

$$\mathcal{A} = \{(r, \theta) : r_1 \leq r \leq r_2, \quad 0 \leq \theta < 2\pi\}.$$

**Figure 7**

**Example 10** *A vertical line "$x = constant$" in cartesian coordinates has the polar form $\{(r, \theta) : r\cos\theta = constant\}$*

**Example 11** *A horizontal line "$y =$ constant" in cartesian coordinates has the polar form* $\{(r,\theta) : r\sin\theta = \text{constant}\}$.

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|---|
| These last two examples, Examples 10-11, show that when dealing with vertical and horizontal lines in regions we should stick with cartesian coordinates, otherwise the descriptions get messy in polar coordinates. However, as we mentioned above, any regions involving circular arcs and/or radial lines should be treated with polar coordinates. |

## Area in Polar Coordinates

The basic unit of **area in polar coordinates** is some expression involving $r$ and $\theta$. Before deriving this let's find an expression for the area of a circular sector (or pizza slice) having vertex angle $\theta$ and radius $R$. Of course, because of symmetry we can assume that the angle $\theta$ is in the interval $(0, \frac{\pi}{2})$. Setting up our region as in Example 8 and using the method of vertical slices (Chapter 9), we get an expression for the area of this sector as:

$$\text{Sectorial Area} = \frac{1}{2}R^2\cos\theta\,\sin\theta + \int_{R\cos\theta}^{R}\sqrt{R^2-x^2}\,dx,$$

where the first expression in the display is simply the area of the triangle with vertices at $(0,0)$, $(R\cos\theta, 0)$ and $(R\cos\theta, R\sin\theta)$. Use of the method of trigonometric substitutions (Chapter 8) allows one to find the value of the integral in the display, that is,

$$\int_{R\cos\theta}^{R}\sqrt{R^2-x^2}\,dx = \frac{R^2\pi}{4} - \frac{1}{2}R^2\cos\theta\,\sin\theta - \frac{1}{2}R^2\left(\frac{\pi}{2}-\theta\right).$$

Adding up the expressions in the last two displays we get the simple result that (see the margin)

$$\boxed{\text{Sectorial Area} = \frac{1}{2}R^2\theta,}$$

where $\theta$, as usual, is in radians! Using this expression it is a simple matter to derive an expression for the area of a region defined by a combination of expressions such as $r = f(\theta)$ in polar coordinates. Basically, if the region $\mathcal{R} = \{(r,\theta) : f_1(\theta) \le r \le f_2(\theta),\ \theta_1 \le \theta \le \theta_2\}$, where $f_1$ and $f_2$ are given, and $\theta = \theta_1$, $\theta = \theta_2$ are its "extremities", then its area is given by a definite integral like

$$\text{Area of the region} = \frac{1}{2}\int_{\theta_1}^{\theta_2}\left(f_2(\theta)^2 - f_1(\theta)^2\right)\,d\theta.$$

**Figure 8**

We now find an expression for the **fundamental unit of area** in polar coordinates. In cartesian coordinates a "fundamental unit of area" is made up of a closed four-sided figure (like a box) of slices (see the margin) that we call a *fundamental region*. If the side lengths of two of these four sides are $dx$ and

$dy$ the fundamental unit of area will be the area of this box, that is $dx\,dy$. We want a similar expression for polar coordinates ....

In this case, the fundamental region looks like the one in the margin. Note that the area of the shaded part is simply given as

$$\begin{aligned}\text{Fundamental unit of area} &= \frac{1}{2}(r+dr)^2\,d\theta - \frac{1}{2}r^2 d\theta \\ &= r\,dr\,d\theta + \frac{1}{2}(dr)^2\,d\theta, \\ &= r\,dr\,d\theta,\end{aligned}$$

once we ignore second order terms in expressions like $(dr)^2 d\theta/2$ (see the treatment of volumes of solids of revolution for a similar technique).

So the area of a fundamental figure in polar coordinates is $r\Delta r\Delta\theta$ (approximately) or $r\,dr\,d\theta$ as we see it in practice. So we relate these fundamental units of area for cartesian and polar coordinates by means of the relation

$$dx\,dy = \underbrace{\boxed{\text{r}}}_{\text{Jacobian}}\; dr\,d\theta,$$

where this factor, $r$, appearing in front of the $dr\,d\theta$ terms is called the **Jacobian** of the transformation from cartesian to polar coordinates and we use this relationship in evaluating *double and multiple integrals*. The Jacobian is generally given by a determinant as we see here,

$$\text{Jacobian} = \det\begin{pmatrix} \frac{\delta x}{\delta r} & \frac{\delta x}{\delta\theta} \\ \frac{\delta y}{\delta r} & \frac{\delta y}{\delta\theta}\end{pmatrix}$$

where $x = r\cos\theta$ and $y = r\sin\theta$, so that

$$\text{Jacobian} = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix}$$

$$\text{Jacobian} = r\cos^2\theta - (-r\sin^2\theta) = r\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} = \boxed{\text{r}}$$

## Elliptical Coordinates

$x = ar\cos\theta$
$y = br\sin\theta$

$$Jacobian = \det\begin{pmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta\end{pmatrix}$$

$$= a\cdot b\cdot r\cdot(\cos^2\theta + sin^2\theta) = a\cdot b\cdot r$$

$$\boxed{dxdy = a\cdot b\cdot r\cdot drd\theta}$$

## Cylindrical Coordinates

$x = r \cdot \cos\theta$
$y = r \cdot \sin\theta$
$z = z$

**Figure 9**

coordinates $(r, \theta, z)$, where $(r, \theta)$ represent the polar coodinates of the projection of $p$ onto the $xy$-plane (with polar-coodinates) and $z$ is the "signed" height of $p$ above or below this plane.

$$\begin{aligned} \text{volume of a parallelepiped} &= dxdydz \\ \text{in cartesian coordinates} &= (r \cdot drd\theta)dz \end{aligned}$$

**Figure 10**

**Figure 11**

**Example 12** $\{r = 2, \quad 0 \leq z \leq 6\} = ?$

*Remark: this can is empty, in that case the surface described does not include "top","bottom" or "interior" of our can.*

*But the set $\{0 \leq r \leq 2, \quad 0 \leq z \leq 6\}$ does contain "top","bottom" and inside of can.*

**Figure 12**

**Example 13** $\{0 \leq r \leq 1, \quad 0 \leq \theta \leq \frac{\pi}{4}, \quad 0 \leq z \leq 5\}$

**Example 14** *The cylinder $x^2 + y^2 = 1$ $(z \geq 0)$ is cut by a plane $z = x + 2$ (in 3D). Describe the solid by means of:*

*1) Cartesian/ rectangular coordinates*

*2) Cylindrical coordinates*

**Figure 13**

*1) The base of this region is given by:*

$$\{(x, y, z) : 0 \leq x^2 + y^2 \leq 1, \quad z = 0\}$$

1. *With base point $P(x, y, z)$ (because $z = 0$ on base)*
   *{This turns out to be the projection of our solid onto the xy-plane.}*
   *Erect a toothpick vertically up until it intersects the surface $z = x + 2$.*
2. *Find points of intersection of this toothpick with our solid.*
3. *Now write down the inequalities which describe this toothpick.*

$$T_{x,y} = \{(x, y, z) : 0 \leq z \leq x + 2, \quad 0 \leq x^2 + y^2 \leq 1\}$$

4. *Put these T's together to reconstruct the solid!*

$$\begin{aligned}
\textit{The solid} &= \{T_{x,y} : (x, y, 0) \ \textit{"lie on base"}\} \\
&= \{T_{x,y} : 0 \leq x^2 + y^2 \leq 1\} \\
&= \{(x, y, z) : 0 \leq z \leq x + 2, \quad 0 \leq x^2 + y^2 \leq 1\} \\
&= \{(x, y, z) : 0 \leq z \leq x + 2, \quad -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}, \quad -1 \leq x \leq 1\}
\end{aligned}$$

*2)*

$$\{(r, \theta, z) : 0 \leq z \leq x + 2, \quad 0 \leq z \leq r\cos\theta + 2, \quad 0 \leq r \leq 1, \ 0 \leq \theta < 2\pi\}$$