Chapter 1

Fourier Series

1.1 Motivation

The motivation behind this topic is as follows, **Joseph-Louis Fourier**, (1768-1830), a French engineer (and mathematician) discussed heat flow through a bar which gives rise to the so-called **Heat Diffusion Problem**,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{K}\frac{\partial u}{\partial t} \tag{1.1}$$

where $u = u(x,t)$, $K > 0$ is a constant depending on the thermal properties of the bar, $u(0,t) = 0 = u(L,t)$, and $u(x,0) = f(x)$, where f is given at the outset. Think of f as being the initial state of the bar at time $t = 0$, and $u(x,t)$ as being the temperature distribution along the bar at the point x in time t. The **boundary conditions** or conditions at the end-points are given in such a way that the bar's "ends" are kept at a fixed temperature, say 0 degrees (whatever) and we can assume that most of the bar is at a temperature close to room temperature, for simplicity.

We apply the method of **Separation of Variables** first. Like Daniel Bernoulli before him, Fourier assumed that the solution he was looking for had the form,

$$u(x,t) = f(x)g(t), \tag{1.2}$$

where we need to find these two functions f, g of one variable. Substituting this expression into the diffusion equation (1.1) we find,

$$\begin{aligned} 0 &= \frac{\partial^2 u}{\partial x^2} - \frac{1}{K}\frac{\partial u}{\partial t} \\ &= f''(x)g(t) - \frac{1}{K}f(x)g'(t). \end{aligned}$$

Now, it must be the case that $f(x) \neq 0$ and $g(t) \neq 0$, otherwise $u(x,t) = 0$ for such x and t. This, however is not a sustainable conclusion on physical grounds (since the rod can't have its temperature equal to 0 except at the end-points). So, dividing both sides by the product $f(x)g(t)$ and separating out terms in x

from terms in t we get

$$\frac{f''(x)}{f(x)} = \frac{1}{K}\frac{g'(t)}{g(t)}. \tag{1.3}$$

Now this equality, (1.3), is valid for any value of x where $0 < x < L$ and any value of t where $t > 0$. So we can let $x = x_0$ where this number x_0 is any fixed number in the interval $[0, L]$. The left side of (1.3) is a constant while the right hand side is a function of t. On the other hand, we can do the same thing with t, that is, we can set $t = t_0$ in the right of (1.3) and leave the x alone. Then the right side becomes a constant and the left side is a function of x. The point is that these two *constants* must be equal because of the equality (1.3). We'll denote the common value of these constants by the symbol $-\lambda$ so that (1.3) can be rewritten as

$$\frac{f''(x)}{f(x)} = \frac{1}{K}\frac{g'(t)}{g(t)} = -\lambda.$$

These equalities can be recast as two equations, namely,

$$\frac{f''(x)}{f(x)} = -\lambda \quad \Rightarrow \quad f''(x) + \lambda\, f(x) = 0, \tag{1.4}$$

and

$$\frac{1}{K}\frac{g'(t)}{g(t)} = -\lambda \quad \Rightarrow \quad g'(t) + \lambda K\; g(t) = 0. \tag{1.5}$$

At this point we still don't know the value of this mysterious number λ but this will become clearer later. Now (1.5) is a constant coefficient first order linear differential equation. We also know that its general solution is given by

$$g(t) = c_1\, e^{-K\,\lambda\,t}.$$

Arguing on physical grounds, the bar should reach a *steady state* as $t \to \infty$ (*i.e.,* the whole bar should ultimately be at a temperature of 0 as $t \to \infty$.) This means that $\lambda > 0$, or else $g(t)$ is exponentially large (since $K > 0$ too). Okay, now that we know that $\lambda > 0$ we can write down the general solution of the constant coefficient second order linear differential equation, (1.4), as

$$f(x) = c_2\,\sin(\sqrt{\lambda}x) + c_3\cos(\sqrt{\lambda}x). \tag{1.6}$$

where c_2, c_3 are constants. Combining these expressions for f and g we get,

$$u(x,t) = (c_2 \sin\sqrt{\lambda}x + c_3 \cos\sqrt{\lambda}x)\, c_1\, e^{-K\,\lambda\,t}. \tag{1.7}$$

At this point in the analysis all we know is that *if* $u(x,t)$ looks like (1.2) then it must be expressible as (1.7). But we still don't know λ or the $c's$! So, Fourier figures the solution looks like (1.7) and in order to get some values for the constants therein he must use the **boundary conditions** given at the ends of the rod, $u(0,t) = 0 = u(L,t)$, "b.c.", for short. We note that these b.c. are really saying that

$$f(0)g(t) = 0 = f(L)g(t)$$

for all $t > 0$. Since $g(t) \neq 0$ we must have,

$$f(0) = 0 \text{ and } f(L) = 0,$$

But these two conditions on f now determine λ, but the λ is not unique. Why? The only solutions of (1.4) that satisfy $f(0) = f(L) = 0$, are those for which the constants c_2, c_3 satisfy (see (1.6))

$$\begin{aligned} 0 &= f(0) = c_2 \sin(\sqrt{\lambda} \cdot 0) + c_3 \cos(\sqrt{\lambda} \cdot 0) \\ &= 0 + c_3 \\ &= c_3 \end{aligned}$$

i.e, $c_3 = 0$ must occur. But if $c_3 = 0$ then $c_2 \neq 0$ (otherwise, $c_2 = 0$ would imply that $u(x,t) = 0$ for all x and t, (see (1.7)), an impossible conclusion! Since $c_3 = 0$ it follows from (1.6) that $f(x)$ must now look like

$$f(x) = c_2 \sin \sqrt{\lambda} x$$

with $c_2 \neq 0$. On the other hand, $c_2 \neq 0$ along with the b.c. $f(L) = 0$ gives

$$0 \quad = \quad f(L) = c_2 \sin(\sqrt{\lambda} L)$$

and this can happen only if $\sqrt{\lambda}L$ is a zero of the sine function. Since the zeros of the sine function occur at numbers of the form $n\pi$ where n is an integer, we see that $\sqrt{\lambda}L = n\pi$ is necessary, that is,

$$\lambda \quad = \quad \frac{n^2\pi^2}{L^2} = \lambda_n,$$

where n is an integer and we show the dependence of λ upon n by the symbol on the right, λ_n. So there are *infinitely many* possibilities for λ, as each one of these $\lambda = \lambda_n$ (called **eigenvalues**) generates a solution

$$f_n(x) = c_2 \sin \sqrt{\lambda_n} x$$

of the Sturm-Liouville equation

$$f''(x) + \lambda_n \ f(x) = 0, \quad f(0) = 0 = f(L).$$

These special solutions $f_n(x)$ that satisfy both the equation and the boundary conditions are called **eigenfunctions of the boundary value problem.**

Using these $f_n(x)$ we can construct $u = u_n(x,t)$, where (by (1.7) and since $c_3 = 0$ and $\lambda = \lambda_n$)

$$u_n(x,t) = c_2 \sin\left(\frac{n\pi x}{L}\right) \, e^{-\frac{Kn^2\pi^2 t}{L^2}}.$$

We emphasize that, for each integer n, these functions $u_n(x,t)$ **all** satisfy (1.1) along with the b.c. $u(0,t) = u(L,t) = 0$.

But do they also satisfy $u(x,0) = f(x)$? **Not necessarily**, that is, not unless the initial configuration of the rod at time $t = 0$ resembles that of a sine function and there is no reason why this should be so!

So Fourier probably thought ... " What if one writes ...

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{N} b_n u_n(x,t) \\ &= \sum_{n=1}^{N} b_n \sin \frac{n\pi x}{L} e^{-\frac{n^2\pi^2}{L^2} K t}, \end{aligned}$$

then this new function $u(x,t)$ also satisfies (1.1) along with the b.c. $u(0,t) = u(L,t) = 0$ (the numbers b_n also have to be determined somehow and we will see how this is done below).

But, once again, if $f(x)$ is basically *arbitrary* it is not necessarily true that

$$f(x) = u(x,0) = \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi x}{L}\right).$$

So, the great insight was the query, "What if $f(x)$ can be represented as an *infinite series* of such sine functions?", that is, what if

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \tag{1.8}$$

in the sense of convergence of the series on the right to $f(x)$ for most x in $[0,L]$? It turned out that this could be done and the representation of f given by (1.8) (where the constants b_n depend on f) would eventually be an example of a **Fourier Series**! The solution of the original problem of heat conduction in a bar would then be solved analytically by the infinite series

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{L^2}Kt},$$

where the b_n are called the **Fourier coefficients** of f on the interval $[0,L]$.

Fourier actually gave a proof of the convergence of the series he developed (in his book on the theory of heat) yet it must be emphasized that D. Bernoulli before him solved the problem of the vibrating string by wrting down the solution in terms of a "Fourier series" too!

1.2 The General Fourier Series Representation

If we proceed with the idea of Section 1.1 and instead use a bar of length $2L$ stretching from $-L$ to L and we assume that the temperature at its ends satisfy the boundary conditions

$$u(-L,t) = u(L,t), \quad \text{and} \quad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t),$$

then one can show that the assumption of separation of variables, (1.2), will lead to the trial form

$$u_n(x,t) = \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) e^{-\frac{n^2\pi^2}{L^2}Kt}$$

from which we obtain the general representation of u as

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) e^{-\frac{n^2\pi^2}{L^2}Kt},$$

valid for all x,t under consideration. In this case, the initial configuration of the rod given by $u(x,0) = f(x)$ will force the representation of f in the more general form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right). \tag{1.9}$$

The series given in (1.9) will be called the **general Fourier series representation** of the function f on the interval $[-L, L]$ having the Fourier coefficients given by a_n and b_n. The a_n will be called the **Fourier cosine coefficients** while the b_n will be called the **Fourier sine coefficients.** We'll give the main idea on how to find the value of these Fourier coefficients ....

Assume that the sum appearing in (1.9) is actually *finite* and proceeds from 1 to N and let $0 \leq m \leq N$ be a fixed integer. Multiplying both sides of (1.9) by the quantity $\cos\left(\frac{m\pi x}{L}\right)$ and integrating the resulting sum over the interval $[-L, L]$ we get

$$\int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right)\, dx =$$

$$\int_{-L}^{L} \left\{ \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)\right\} \cdot \cos\left(\frac{m\pi x}{L}\right)\, dx =$$

$$\frac{a_0}{2} \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\, dx + \sum_{n=1}^{N} a_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cdot \cos\left(\frac{m\pi x}{L}\right)\, dx +$$

$$\sum_{n=1}^{N} b_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cdot \cos\left(\frac{m\pi x}{L}\right)\, dx.$$

If $m = 0$, the last two integrals are zero (why?) while the first integral is simply equal to La_0. The last line then gives

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx. \tag{1.10}$$

a quantity equal to twice the **mean-value of the function** f over the interval $[-L, L]$. On the other hand, if $0 < m \leq N$ then (by integrating by parts, or using the identities in the margin)

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right)\, dx = \begin{cases} 0 & m \neq n\ , \\ L & m = n\ , \end{cases}$$

and

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right)\, dx = \begin{cases} 0 & m \neq n\ , \\ 0 & m = n\ , \end{cases}$$

From these relations we find that a_m is given by

$$a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right)\, dx \tag{1.11}$$

Since m is an arbitrary number, (1.11) must be true for any subscript, $m \geq 0$. A similar calculation shows that for any subscript $m \geq 0$ we also have

$$b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right)\, dx. \tag{1.12}$$

The following identities, valid for any angles A, B, including functions, are very useful:

$$\cos(A)\cos(B) = \frac{\cos(A+B) - \cos(A-B)}{2}$$

$$\sin(A)\cos(B) = \frac{\sin(A+B) + \sin(A-B)}{2}$$

The road to (1.12) is found by multiplying both sides of (1.9) by the quantity $\sin\left(\frac{m\pi x}{L}\right)$, integrating the resulting sum over the interval $[-L, L]$ and using the integral evaluation

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & m \neq n\ , \\ L & m = n\ , \end{cases}$$

The following identity is also valid for any angles A, B, including functions:

$$\sin(A)\sin(B) = \frac{\cos(A-B) - \cos(A+B)}{2}$$

as before. Since the discussion is similar to the one just presented we leave out the details.

A function f is said to be **periodic of period** P if:

$$f(x+P) = f(x) \text{ for all } x \text{ in domain of } f$$

The **period of** f is the smallest P such that $f(x+P) = f(x)$ holds, for all x. Note that in order for (1.9) to hold on any interval outside $[-L, L]$ the function f must be periodic of period $P = 2L$. This is because the functions $g(x) = \cos\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{n\pi x}{L}\right)$ are each periodic of period $2L$. For example,

$$g(x+2L) = \cos\left(\frac{n\pi(x+2L)}{L}\right) = \cos\left(\frac{n\pi x}{L} + 2n\pi\right) = \cos\left(\frac{n\pi x}{L}\right) = g(x).$$

In the case of Fourier series, we always choose $\boldsymbol{P = 2L}$.

If f is periodic with period P then, for any real number x,

$$f(x+nP) = f(x)$$

for any integer n, positive or negative. This means that the graph of f *repeats* itself on any interval of length P (see Figure 1).

A periodic function's graph repeats itself over any interval of length equal to its period, P. The display above is the graph of the periodic function $f(x) = \sin x + \cos 2x$ having period $P = 2\pi$.

Figure 1

Example 1 *a)* $f(x) = \sin x$ *is periodic with period* 2π.

b) $f(x) = \tan x$ *is periodic with period* π.

c) $f(x) = \sin 4x$ *is periodic with period* $\dfrac{\pi}{2}$.

d) $f(x) = \sin x + \cos 2x$ *is periodic with period* 2π *(Fig. 1)*

Solution: a) This follows from trigonometry; b) Note that

$$\tan(x+\pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{-\sin x}{-\cos x} = \tan x.$$

Of course, it is true that $\tan(x+2\pi) = \tan x$ but $P = \pi$ is really the period. c) The smallest multiple of π that gives $\sin 4(x+P) = \sin 4x$ is $P = \frac{\pi}{2}$. d) The reasoning for this one is the same as c).

The Dirichlet Test: A Theorem on the Convergence of a Fourier Series:

1. Let f be a function that is defined and finite on $(-L, L)$, except possibly at a finite number of points inside this interval.

2. Let f be periodic of period $2L$ outside $(-L, L)$.
3. Assume that f, f' are **piecewise continuous** in $(-L, L)$ (this means that f and its derivative are each continuous except possibly at a finite number of points).

Then the **Fourier series of f converges to:**

(a) $f(x)$, if f is continuous at x.

(b) $\dfrac{f(x+0)+f(x-0)}{2}$ if f is not continuous at x (*i.e.,* It converges to the **average value of f at x**).

We omit the proof as it requires ideas that are beyond the scope of this text. Anyhow, we'll give a few examples a little later on on how to apply this theorem. Until then we will the symbol "$\sim$" to denote the Fourier series representation of a function f.

SUMMARY

If $f(x)$ is defined on $[-L, L]$ and f has the Fourier series representation given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) \tag{1.13}$$

then the Fourier coefficients are all given by

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \quad \text{(double the "mean value" of } f \text{ over } [-L, L]) \\
a_m &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right) dx, \quad m = 1, 2, ... \\
b_m &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right) dx, \quad m = 1, 2, ...
\end{aligned}$$

Whether the Fourier series given above actually converges to $f(x)$ is another question, and one that we will answer below.

1.3 Some Remarks on Fourier Coefficients

Let f be a piecewise continuous periodic function of period $2L$, so that

$$f(x+2L) = f(x)$$

for every value of x. Then f is Riemann integrable over any finite interval and if we let $\mathcal{F}$ denote an antiderivative of f then, the F defined by

$$\begin{aligned}
F(x) &= \int_{x}^{x+2L} f(t)\ dt \\
&= \mathcal{F}(x+2L) - \mathcal{F}(x).
\end{aligned}$$

From **Leibniz's Rule** (see Chapter 7.4), we have that F is differentiable and $F'(x)$ is given by

$$\begin{aligned}
F'(x) &= \frac{d}{dx}\int_x^{x+2L} f(t)\ dt \\
&= \frac{d}{dx}\left(\mathcal{F}(x+2L) - \mathcal{F}(x)\right) \\
&= \mathcal{F}'(x+2L)\cdot(1) - \mathcal{F}'(x)\cdot(1), \\
&= f(x+2L) - f(x) \\
&= 0,
\end{aligned}$$

by the assumption on f. So, $F(x)$ is a constant function and we can deduce that

$$\begin{aligned}
F(x) &= F(0) \\
&= F(-L),
\end{aligned}$$

or

$$\begin{aligned}
\int_x^{x+2L} f(t)\ dt &= \int_0^{2L} f(t)\ dt \\
&= \int_{-L}^{L} f(t)\ dt,
\end{aligned}$$

is valid for any value of x.

It follows, that, for example, if f is periodic of period $2L$, then its **Fourier cosine coefficients** can be written as

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\ dt &\quad (1.14)\\
&= \frac{1}{L}\int_{0}^{2L} f(t)\cos\left(\frac{n\pi t}{L}\right)\ dt &\quad (1.15)\\
&= \frac{1}{L}\int_{x}^{x+2L} f(t)\cos\left(\frac{n\pi t}{L}\right)\ dt, &\quad (1.16)
\end{aligned}$$

where x can be ANY fixed point (you choose it) on the real line. A similar formula is valid for the **Fourier sine coefficients** of f which are given by

$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\ dt &\quad (1.17)\\
&= \frac{1}{L}\int_{0}^{2L} f(t)\sin\left(\frac{n\pi t}{L}\right)\ dt &\quad (1.18)\\
&= \frac{1}{L}\int_{x}^{x+2L} f(t)\sin\left(\frac{n\pi t}{L}\right)\ dt, &\quad (1.19)
\end{aligned}$$

Example The function f defined by $f(x) = x - 1$, for $0 < x < 2$ and having period 2 has an odd periodic extension on $(-L, L) = (-1, 1)$ so that its Fourier series is a pure sine series. It follows that its $a_n = 0$ for all n, including $n = 0$ (do this one separately) and that b_n are given by

$$\begin{aligned}
b_n &= \int_{-1}^{1} f(t)\sin\left(\frac{n\pi t}{L}\right)\ dt \\
&= \int_{0}^{2} (t-1)\sin\left(\frac{n\pi t}{L}\right)\ dt,
\end{aligned}$$

and so **you don't have to break up the original integral into 2 parts**, one on the interval $(-1, 0)$ and one on the interval $(0, 1)$. Just calculate the Fourier coefficient as if the interval were the interval $(0, 2)$! In other words, the three integrals,

$$\begin{aligned} b_n &= \int_{-1}^{1} f(t) \sin\left(\frac{n\pi t}{L}\right)\, dt \\ &= \int_{-1}^{0} (t+1) \sin\left(\frac{n\pi t}{L}\right)\, dt + \int_{0}^{1} (t-1) \sin\left(\frac{n\pi t}{L}\right)\, dt \\ &= \int_{0}^{2} (t-1) \sin\left(\frac{n\pi t}{L}\right)\, dt, \end{aligned}$$

are all equal! The last integral can be integrated 'by parts' using Table integration as in Chapter 8.

1.4 Even and Odd Functions

But how do we find these *Fourier coefficients*? Before we proceed to an actual calculation of these quantities let's look at some special classes of functions defined on a symmetric interval $[-L, L]$. We say that a function f is an **even function** on $[-L, L]$ if for any symbols $\pm x$ in the domain of f we have

$$f(-x) = f(x). \tag{1.20}$$

Similarly, we say that a function f is an **odd function** on $[-L, L]$ if for any symbols $\pm x$ in the domain of f we have

$$f(-x) = -f(x). \tag{1.21}$$

Geometrically these notions of even and odd functions can be interpreted as follows: An <u>even function</u> is a function f whose graph is symmetric with respect to the y-axis, while an <u>odd function</u> is a function f whose graph is symmetric with respect to the origin, O, (we call this a *central reflection*). See the margin for some examples.

Example 2 *Determine whether the given functions are even, odd or neither:*

a) $f(x) = \sin\left(\frac{n\pi x}{L}\right)$ *on* $-L \le x \le L$,

b) $f(x) = \cos\left(\frac{n\pi x}{L}\right)$ *on* $-L \le x \le L$,

c) $f(x) = \tan x$ *on* $-\frac{\pi}{2} < x < \frac{\pi}{2}$,

d) $f(x) = x^2$ *on* $-1.4 \le x \le 1.4$,

e) $f(x) = x^3$ *on* $-3 \le x \le 3$,

f) $f(x) = 1 + x$ *on* $-10 \le x \le 10$.

Solution

a) Since $\sin(-x) = -\sin x$ from trigonometry, the same must be true for any symbol $\Box$, *i.e.,* $\sin(-\Box) = -\sin\Box$. Since we can put *any* quantity inside the box, we can use $\boxed{\dfrac{n\pi x}{L}}$. This will show that $\sin\left(-\boxed{\dfrac{n\pi x}{L}}\right) = -\sin\left(\boxed{\dfrac{n\pi x}{L}}\right)$. But this last equality means that $f(-x) = -f(x)$ (by definition of f). So, f is odd by definition.

b) $f(x) = \cos\left(\dfrac{n\pi x}{L}\right)$ is even because $\cos(-x) = \cos x$ by trigonometry. The rest of the argument is similar to part (a), above, and so is omitted.

c) $f(x) = \tan x$ is odd because $\tan(-x) = -\tan(x)$ from trigonometry. You can also see this by realizing that this is a consequence of the fact that $\sin x$ is odd and $\cos x$ is even. In other words,

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\frac{\sin x}{\cos x} = -\tan x.$$

d) $f(x) = x^2$ is even. This is easy since $f(-x) = (-x)^2 = x^2 = f(x)$.

e) $f(x) = x^3$ is odd. This is true too since $f(-x) = (-x)^3 = -x^3 = -f(x)$.

f) $f(x) = 1 + x$ is <u>neither odd nor even!!</u>. This is because $f(-x) = 1 - x$, $f(x) = 1 + x$ and so $f(-x) \neq \pm f(x)$ for all x. So this function is neither even nor odd.

NOTE: Functions can be even, odd or NEITHER even nor odd as the preceding example shows.

Example 3 *Show that* **if f is an even function** *then*

$$\int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx. \tag{1.22}$$

On the other hand, **if f is an odd function** *then*

$$\int_{-L}^{L} f(x)\,dx = 0. \tag{1.23}$$

Solution Why? Well this is just a simple substitution in a definite integral. Let's prove the result for the odd case, the case given by (1.23), the other case being similar. Using the substitution $x = -t$, $dx = -dt$, in the integral in (1.23) we get

$$\int_{-L}^{L} f(x)\,dx = \int_{L}^{-L} f(-t)\,(-dt).$$

Note that we have to change the limits of integration in the integral on the right to reflect the change of variable: When $x = -t$ and $x = -L$ then $t = -x = L$.

On the other hand, when $x = L$ then $t = -L$ and so the order of the limits is merely interchanged. Next,

$$\begin{aligned}
\int_{-L}^{L} f(x)\,dx &= \int_{L}^{-L} f(-t)\,(-dt) \\
&= -\int_{L}^{-L} f(-t)\,dt \\
&= (-1)^2 \int_{L}^{-L} f(t)\,dt, \quad (\text{since f}(-\text{t}) = -\text{f}(\text{t})) \\
&= (-1)^3 \int_{-L}^{L} f(t)\,dt, \ (\text{interchanging the limits reverses the sign}) \\
&= -\int_{-L}^{L} f(t)\,dt.
\end{aligned}$$

Since t is a *free variable* in the last display, it follows that

$$I = \int_{-L}^{L} f(x)\,dx = -\int_{-L}^{L} f(t)\,dt = -I,$$

and so $I = 0$. The other result about even functions is similar.

Example 4 *Show that*

- *a) The product (or quotient) of any two even functions is even,*
- *b) The product (or quotient) of any even and any odd function is odd,*
- *c) The product (or quotient) of any two odd functions is even.*

Operations between even and odd functions behave much like operations between *plus* and *minus* signs! For example,

$$\underbrace{\text{even}}_{+} \cdot \underbrace{\text{odd}}_{-} = \underbrace{\text{odd}}_{-},$$

$$\underbrace{\text{even}}_{+} \cdot \underbrace{\text{odd}}_{-} \cdot \underbrace{\text{even}}_{+} = \underbrace{\text{odd}}_{-},$$

$$\underbrace{\text{odd}}_{-} \cdot \underbrace{\text{odd}}_{-} = \underbrace{\text{even}}_{+}$$

Solution a) Let f, g be two even functions and denote their product by h. Then $h(-x) = (fg)(-x) = f(-x)g(-x)$ by definition of their product. Since f is even we have that $f(-x) = f(x)$. Similarly, since g is even we have $g(-x) = g(x)$. Thus, $h(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x) = h(x)$. Hence h is even.

b) Let f be an even function, g be an odd function and denote their product by h. Then $f(-x) = f(x)$. Similarly, since g is odd we have $g(-x) = -g(x)$. Thus, $h(-x) = f(-x)g(-x) = f(x)(-g(x)) = -(fg)(x) = -h(x)$. Hence h is odd.

c) Let f be an odd function, g be an odd function and denote their product by h. Then $f(-x) = -f(x)$. Similarly, since g is odd we have $g(-x) = -g(x)$. Thus, $h(-x) = f(-x)g(-x) = (-f(x))\,(-g(x)) = (-)(-)(fg)(x) = +h(x)$. Hence h is even.

The main results in this section deal with the Fourier series of even and odd functions. We will see that it is a simple matter to show that **if f is an even function on the interval $[-L, L]$ then its Fourier series must be a pure cosine series** in the sense that

$$f \ \text{ is even } \iff f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \tag{1.24}$$

The reason for this is that since f is even and $\sin\left(\frac{n\pi x}{L}\right)$ is odd (see Example 2 (a)) then their product must be odd and so (by Example 1.22) we see that

$$f \text{ is even} \iff \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx = 0,$$

for $n = 1, 2, \ldots$ or equivalently, $b_n = 0$ for each $n = 1, 2, \ldots$ in (1.13). From this we get (1.24). The formulae for the a_0 and a_n are still given by (1.10) and (1.11) but they take the simpler form

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx = \frac{2}{L} \int_{0}^{L} f(x)\, dx, \tag{1.25}$$

since f is even (see 1.22), and

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx = \frac{2}{L} \int_{0}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx \tag{1.26}$$

since $f(x)\cos(\frac{n\pi x}{L})$ is itself even (as each one is even, and (1.22)). Conversely, it is also true that if $f(x)$ has a representation as a pure cosine series on $[-L, L]$ then f must be an even function on that interval.

We also show that **if f is an odd function on the interval $[-L, L]$ then its Fourier series must be a pure sine series** in the sense that

$$f \text{ is odd} \iff f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{1.27}$$

The reason for this is that since f is odd and $\cos\left(\frac{n\pi x}{L}\right)$ is even (see Example 2 (b)) then their product must be odd and so (by Example 1.22) we see that

$$f \text{ is odd} \iff \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx = 0,$$

for $n = 1, 2, \ldots$ or equivalently, $a_n = 0$ for each $n = 1, 2, \ldots$ in (1.13). This gives (1.27). The formulae for the b_n are still given by (1.12) but they take the simpler form

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx \tag{1.28}$$

since $f(x)\sin(\frac{n\pi x}{L})$ is itself even (as each one is odd, and (1.22)).

Conversely, it is also true that if $f(x)$ has a representation as a pure sine series on $[-L, L]$ then f must be an odd function on that interval.

Example 5 *Find the Fourier series of the function f defined by $f(x) = x^2$ on the interval $[-2, 2]$. What does the series converge to when $x = 0$?*

Solution: Since this function is an even function (why?) on the given interval (where $L = 2$) it follows that its Fourier series is a pure cosine series (see (1.24)).

Thus, its Fourier coefficients a_0 and a_n are given by (1.25) and (1.26). In this case these integrals become

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \int_0^2 x^2\,dx = \frac{8}{3},$$

and (recall that $L = 2$),

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx = \int_0^2 x^2\cos\left(\frac{n\pi x}{2}\right)\,dx = 16\,\frac{(-1)^n}{n^2\pi^2}$$

for any $n \geq 1$, where we use the Table Method (Integration by Parts) of Chapter 8 to get a value for the integral that gives the a_n and we also use the fact that $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ whenever n is an integer. From these observations we get that the form of the Fourier series of x^2 is given by (see (1.24))

$$x^2 \sim \frac{4}{3} + \sum_{n=1}^{\infty} 16\,\frac{(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)$$

for $-2 \leq x \leq 2$. The symbol $\sim$ means that the quantity on the right is the representation of the function on the left. Now let's use the Dirichlet Test to see what f converges to ....

Note that $f(x) = x^2$ is continuous and has a continuous derivative on the interval $[-2, 2]$. We can define f outside this interval by setting $f(x+4) = f(x)$. Why 4? Because $P = 2L$ and since $L = 2$ we must have the period of f equal to 4. Now this function f is continuous at $x = 0$. Thus, by the Dirichlet Test we get the equality

$$x^2 = \frac{4}{3} + \sum_{n=1}^{\infty} 16\,\frac{(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)$$

and at $x = 0$ the series gives the equality

$$0 = \frac{4}{3} + \sum_{n=1}^{\infty} 16\,\frac{(-1)^n}{n^2\pi^2},$$

or (multiplying both sides by (-1)),

$$\sum_{n=1}^{\infty} 16\,\frac{(-1)^{n-1}}{n^2\pi^2} = \frac{4}{3},$$

i.e.,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

Exercise Convince yourself that the Fourier series of the preceding example converges to 4 when $x = 2$ and that this gives the following result about series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Figure 2

Example 6 *Find the Fourier series of the function defined in pieces (sometimes called a* <u>*piecewise constant function*</u>*) by*

$$f(x) = \begin{cases} 8, & 0 < x < 2, \\ -8, & 2 < x < 4, \end{cases}$$

where f is periodic with period 4. What does the series converge to at $x = 2$? $x = 3$?

Solution Since the function has period 4, the graph of f on the interval $[-4, 0]$ must be the same as (or a translate of) the one on $[0, 4]$. In other words, we must have,

$$f(x) = \begin{cases} 8, & -4 < x < -2, \\ -8, & -2 < x < 0, \end{cases}$$

Therefore $f(x)$ is an **odd** function (why?) on $[-4, 4]$ and so its Fourier series is a pure <u>sine</u> series. Since the period is $P = 4$ here, we get that $P = 2L$ implies that $L = 2$. The Fourier series looks like,

$$f(x) \underbrace{\sim}_{\text{not necessarily equal}} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

where

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} f(x) \sin\left(\frac{n\pi x}{2}\right)\, dx \\ &= \underbrace{\frac{1}{2}\int_{-2}^{0} (-8) \sin\left(\frac{n\pi x}{2}\right)\, dx}_{\frac{8}{n\pi}(1-\cos n\pi)} + \underbrace{\frac{1}{2}\int_{0}^{2} (+8) \sin\left(\frac{n\pi x}{2}\right)\, dx}_{\frac{8}{n\pi}(1-\cos n\pi)} \\ &= \frac{16}{n\pi}(1 - \cos n\pi). \end{aligned}$$

Therefore,

$$f(x) \sim \sum_{n=1}^{\infty} \frac{16}{n\pi}(1 - \cos n\pi)\, \sin\left(\frac{n\pi x}{2}\right).$$

Next, we need to use the Dirichlet Test: Now, this function f is NOT continuous at $x = 2$ (since its left limit is 8 while its right limit is -8). It follows that when $x = 2$ the Fourier series converges to

$$\frac{f(x+0) + f(x-0)}{2} = \frac{(-8 + (8))}{2} = 0.$$

This result is easy to verify directly since at $x = 2$ the sine term in the Fourier series is $\sin n\pi = 0$ since n is always an integer!

However, at $x = 3$ the function IS continuous and its value there is $f(3) = -8$. Thus, we find

$$\sum_{n=1}^{\infty} \frac{16}{n\pi}(1 - \cos n\pi)\, \sin\left(\frac{3n\pi}{2}\right) = -8.$$

Example 7 *Calculate the Fourier series of the function defined by*

$$f(x) = \begin{cases} \cos x, & 0 < x < \pi, \\ 0, & \pi < x < 2\pi, \end{cases}$$

where f is periodic with period 2π.

Solution: Since $P = 2L = 2\pi$ it follows that $L = \pi$. So the Fourier cosine terms of this function are given by (1.16) with $x = 0$ and $L = \pi$. Thus,

$$\begin{aligned}
a_n &= \frac{1}{L}\int_x^{x+2L} f(t)\cos\left(\frac{n\pi t}{L}\right)\,dt \\
&= \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\,dt \\
&= \frac{1}{\pi}\int_0^{\pi} \cos t\cos(nt)\,dt, \quad (\text{since } f(t) = 0 \text{ on } (\pi, 2\pi)), \\
&= \frac{1}{2\pi}\int_0^{\pi} (\cos(n+1)t + \cos(1-n)t)\,dt, \\
&= \frac{1}{2\pi}\left\{\frac{\sin(n+1)t}{n+1} + \frac{\sin(1-n)t}{1-n}\right\}\Bigg|_0^{\pi} \\
&= 0 \quad \text{if} \quad n \neq 1.
\end{aligned}$$

So $a_n = 0$ whenever $n \neq 1$. If $n = 1$, then

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \cos^2 t\,dt = \frac{1}{2}.$$

The b_n are obtained similarly. Thus,

$$\begin{aligned}
b_n &= \frac{1}{L}\int_x^{x+2L} f(t)\sin\left(\frac{n\pi t}{L}\right)\,dt \\
&= \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\,dt \\
&= \frac{1}{\pi}\int_0^{\pi} \cos t\sin(nt)\,dt, \quad (\text{since } f(t) = 0 \text{ on } (\pi, 2\pi)), \\
&= \frac{1}{2\pi}\int_0^{\pi} (\sin(n+1)t + \sin(n-1)t)\,dt, \\
&= \frac{1}{2\pi}\left\{-\frac{\cos(n+1)t}{n+1} - \frac{\cos(n-1)t}{n-1}\right\}\Bigg|_0^{\pi} \\
&= n\frac{(1+\cos n\pi)}{\pi(n^2-1)} \quad \text{if} \quad n \neq 1.
\end{aligned}$$

Note that $\cos n\pi = 0$ whenever n is odd. On the other hand, if $n = 1$ then

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \cos t\sin t\,dt = 0.$$

It follows that the Fourier series of f is given by

$$f(x) = \frac{1}{2}\cos x + \sum_{n=2}^{\infty} n\frac{(1+\cos n\pi)}{\pi(n^2-1)}\sin nx = \frac{1}{2}\cos x + \frac{4}{3\pi}\sin 2x + \frac{8}{15\pi}\sin 4x + \ldots$$

Using Dirichlet's Test we see that this series converges to $f(x)$ except at the points of discontinuity (namely, $x = 0, \pm\pi, \pm 2\pi, \ldots$). For instance, when $x = 0$

the series converges to 1/2 and this is half the average value of the right and left limits of f at $x = 0$ (as required by the Test).

NOTE: The preceding is an example of a Fourier series containing *both* sine and cosine terms.

1.5 Even and Odd Extensions of Functions

If $f(x)$ is any function defined on an interval of the form $[0, L]$ we define its **even extension** to $[-L, L]$ by setting $f(x) = f(-x)$ for x in $[-L, 0]$ (or by reflecting its graph about the y-axis). Similarly, we define its **odd extension** to $[-L, L]$ by setting $f(x) = -f(-x)$ for x in $[-L, 0]$, or by reflecting its graph about the origin.

The reason for these definitions is that is that we can create an even function (over $(-L, L)$) out of a function that is given only on half-the-range, *i.e.,* $(0, L)$. Similarly, we can create an odd function (over $(-L, L)$) out of a function that is given only on half-the-range, *i.e.,* $(0, L)$.

We do this because we may want to expand a function in terms of a **pure cosine series** only (in which case we use the even extension since we don't want any sine terms) or in terms a **pure sine series** (in which case we use the odd extension since we don't want any cosine terms).

Example 8

- *a) Find the even extension of the function f defined by $f(x) = x(\pi - x)$ for $0 \le x \le \pi$.*
- *b) Find the odd extension of the function f defined by $f(x) = x(\pi - x)$ for $0 \le x \le \pi$.*
- *c) Find the odd extension of the function f defined by $f(x) = \cos x$ for $0 \le x \le \pi$.*
- *d) Find the even extension of the function f defined by*

$$f(x) = \sin\left(\frac{2\pi x}{L}\right) + 3\cos\left(\frac{2\pi x}{L}\right)$$

  for $0 \le x \le L$.

Solution: a) We recall the definition of an even function: For f to be even on $[-\pi, \pi]$ we must have $f(x) = f(-x)$. To get the form of f on the part $[-\pi, 0]$ we replace x by "$-x$" in the definition of $f(x)$: This gives $f(-x) = -x(\pi + x)$ for x in $[-\pi, 0]$. The even extension of f is then given by

$$f(x) = \begin{cases} x(\pi - x), & 0 < x < \pi, \\ -x(\pi + x), & -\pi < x < 0, \end{cases}$$

b) This case is similar to part a). We know that f is odd only when $f(x) = -f(-x)$. So to get the form of f on the left interval $[-\pi, 0]$, we calculate the value of $-f(-x)$ using the given expression on $[0,\pi]$. Just as before we replace x by "$-x$" in the definition of $f(x)$: This gives $-f(-x) = x(\pi + x)$ for x in $[-\pi, 0]$ (note the removal of the minus sign). The odd extension of f is then given by

$$f(x) = \begin{cases} x(\pi - x), & 0 < x < \pi, \\ x(\pi + x), & -\pi < x < 0, \end{cases}$$

c) Write $f(x) = \cos x$ for x in $[0,\pi]$. Then $-f(-x) = -\cos(-x) = -\cos x$ since $\cos(-x) = \cos x$ by trigonometry. It follows that the odd extension of $\cos x$ is given by the modified function

$$f(x) = \begin{cases} \cos x, & 0 < x < \pi, \\ -\cos x, & -\pi < x < 0, \end{cases}$$

Note that this odd extension is not continuous at $x = 0$.

d) In this case

$$f(-x) = -\sin\left(\frac{2\pi x}{L}\right) + 3\cos\left(\frac{2\pi x}{L}\right)$$

for $-L \le x \le 0$ (on account of Example 2a), b)). So, the even extension of this function f looks like,

$$f(x) = \begin{cases} \sin\left(\frac{2\pi x}{L}\right) + 3\cos\left(\frac{2\pi x}{L}\right), & 0 < x < L, \\ \\ -\sin\left(\frac{2\pi x}{L}\right) + 3\cos\left(\frac{2\pi x}{L}\right), & -L < x < 0, \end{cases}$$

Example 9 *Expand* $f(x) = \cos x, \ \ 0 < x < \pi$, *in a pure Fourier "sine" series on* $(0, \pi)$.

Solution: IDEA:

1) Note that "$\cos x$" is an <u>even</u> function while only odd functions can have pure sine series expansions.

2) So we <u>must extend</u> $\cos x$ to be an <u>odd function</u> on $(-\pi, \pi)$ by taking its odd extension to $(-\pi, 0)$ (see Example 8c)).

3) The resulting extended $f(x)$ is now an **odd periodic function of period $\boldsymbol{\pi}$**, (not 2π as one may think!) *i.e.,* $f(x + \pi) = f(x)$. Since $P = 2L$ it follows that $L = \frac{\pi}{2}$.

Figure 3

Our extended function is given by

$$f(x) = \begin{cases} \cos x, & 0 < x < \pi, \\ -\cos x, & -\pi < x < 0, \end{cases}$$

Furthermore, $f(x)$ is defined on $(-L, L) = (-\frac{\pi}{2}, \frac{\pi}{2})$ and formally, its Fourier series representation looks like

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n \sin(2nx),$$

(there can be no a_n's since $f(x)$ is odd on $(-\frac{\pi}{2}, \frac{\pi}{2})$). The Fourier sine coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x) \sin\left(\frac{n\pi x}{\pi/2}\right) dx \\ &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x) \sin(2nx)\, dx \\ &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{0} -(\cos x)\sin(2nx) dx + \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} (\cos x)\sin(2nx)\, dx \\ &= \frac{4n}{\pi(4n^2-1)} + \frac{4n}{\pi(4n^2-1)} \\ &= \frac{8n}{\pi(4n^2-1)}. \end{aligned}$$

The Fourier series of this extended cosine function is therefore of the form

$$\cos x = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{n}{(4n^2-1)}\, sin(2nx)$$

for any x in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and outside this interval by periodicity (or periodically repeating the graph). In particular we see that at $x = \frac{\pi}{4}$ we get the result

$$\frac{\pi\sqrt{2}}{16} = \sum_{n=1}^{\infty} \frac{n}{(4n^2-1)}\, sin\left(\frac{n\pi}{2}\right).$$

Note: At $x = \pi$ the Fourier series converges to $\frac{f(\pi+0)+f(\pi-0)}{2} = \frac{1+(-1)}{2} = \frac{0}{2} = 0$ (O.K. by the Dirichlet Test) so, in order to get convergence at this point, we need to define $f(\pi) = 0$.

Example 10 *Find the Fourier "cosine" series of the function defined by* $f(x) = x(\pi - x)$*, for* x *in* $(0, \pi)$.

Solution: Since we want a cosine series for $f(x)$ the extension of f to $(-\pi, 0)$ must be even (no sine terms allowed in the Fourier series expansion). Now refer to Example 8a). We know that the even extension of $f(x)$ looks like

$$f(x) = \begin{cases} x(\pi - x), & 0 < x < \pi, \\ -x(\pi + x), & -\pi < x < 0, \end{cases}$$

This extended function is *even and periodic with period* π (look at its graph, the bumps are the same right?), so $L = \frac{\pi}{2}$. The Fourier cosine coefficients are now given by (see (1.25), (1.26), and use the Table Method of Chapter 8))

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)\,dx \\ &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} x(\pi - x)\,dx \\ &= \frac{\pi^2}{3}, \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} x(\pi - x)\,\cos 2nx\ \,dx \\ &= -\frac{1}{n^2}. \end{aligned}$$

So, the Fourier cosine series of this function $f(x)$ is given by

$$f(x) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{1}{n^2}\,\cos 2nx,$$

with convergence properties according to the Dirichlet Test. In particular, since f is continuous at $x = 0$ and $f(0) = 0$, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

1.6 Parseval's Equality

If the function f satisfies the three conditions of the Dirichlet Test for convergence, then its Fourier coefficients have the property that

$$\frac{a_0^2}{2} + \sum(a_n^2 + b_n^2) = \frac{1}{L}\int_{-L}^{L} f(x)^2 dx$$

In general there always holds Bessel's Inequality, that is,

$$\frac{a_0^2}{2} + \sum(a_n^2 + b_n^2) \le \frac{1}{L}\int_{-L}^{L} f(x)^2\,dx.$$

This result is valid for any function that is piecewise continuous on $(-L, L)$ (whether or not its Fourier series actually equals $f(x)$!)

Example 11 *Using Example 6 and Parseval's Equality, show that*

$$\sum_{n\ odd}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

Solution: We know the b_n's and $f(x)$. So we can conclude that:

$$\sum_{n=1}^{\infty} \frac{16^2}{n^2\pi^2}(1-\cos n\pi)^2 = \sum_{n=1}^{\infty} b_n^2 = \frac{1}{L}\int_{-L}^{L} f(x)^2 dx = \frac{1}{2}\int_{-2}^{2} 64 \cdot dx = 64 \cdot 2 = 128$$

i.e.,

$$\sum_{n=1}^{\infty} \frac{(1-\cos n\pi)^2}{n^2\pi^2} = \frac{128}{256} = \frac{1}{2}.$$

Since $1-\cos n\pi = 0$ whenever n is even and $1-\cos n\pi = 2$ whenever n is odd, the last display becomes

$$\sum_{n \ odd}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

Example 12 *Use Example 10 and Parseval's Equality to show that*

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Solution: We know a_0, $a_n = -1/n^2$, $b_n = 0$. Note that ${a_0}^2 = \dfrac{\pi^4}{18}$. So, by Parseval's Equality we get

$$\frac{\pi^4}{18} + \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{L}\int_{-L}^{L} f(x)^2 dx = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)^2 \ dx.$$

However,

$$\frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)^2 \ dx = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{0} (-x(\pi+x))^2 \ dx + \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} (x(\pi-x))^2 \ dx = \frac{\pi^4}{15}.$$

Combining these results we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{15} - \frac{\pi^4}{18} = \frac{\pi^4}{90}.$$

1.7 Integrating and Differentiating Fourier Series

We can integrate a Fourier series term by term provided the conditions of the Dirichlet Test hold. Indeed, if the function f is piecewise continuous on $[-L, L]$ and the points a, x are in $[-L, L]$ and $f(x)$ has the expansion given by (1.13), then

$$\int_a^x f(t)\,dt = \frac{a_0}{2}\int_a^t dt + \sum_{n=1}^{\infty}\left(a_n \int_a^t \cos\left(\frac{n\pi t}{L}\right) \ dt + b_n \int_a^t \sin\left(\frac{n\pi t}{L}\right) \ dt\right),$$

or we can integrate the series term by term after which the new series will converge to the integral of the original series given on the left.

Example 13 *Use Example 9 to show that the Fourier cosine series of the sine function on* $(0, \pi)$ *is given by*

$$\sin x = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(4n^2-1)} - \sum_{n=1}^{\infty} \frac{\cos 2nx}{(4n^2-1)}.$$

Solution: We know that

$$\cos t = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{(4n^2-1)} \, sin(2nt)$$

is valid for t in $(-\pi, \pi)$. Since this function satisfies all the conditions of Dirichlet's Test we can choose $a = 0$ and fix a value of x in $(-\pi, \pi)$. We now integrate both sides of this last display over $[0, x]$ and find

$$\sin x = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{(4n^2-1)} \int_0^t sin(2nt)\, dt = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1-\cos 2nx}{(4n^2-1)}.$$

Since we can split the sum into two convergent parts the result follows.

Example 14 *Use Example 5 to calculate the Fourier series of the function* $f(x) = x^3$ *defined on* $[-2, 2]$.

Solution: First, we extend f to the whole line by periodicity (with period 4). We know from Example 5 that

$$t^2 = \frac{4}{3} + \sum_{n=1}^{\infty} 16 \, \frac{(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi t}{2}\right)$$

for t in $[-2, 2]$ with the convergence properties specified by the Dirichlet Test. Integrating both sides over the interval $[0, x]$ we find

$$\frac{x^3}{3} = 16 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2} \cdot \frac{2}{n\pi} \cdot \sin\left(\frac{n\pi x}{2}\right)$$

or

$$x^3 = \frac{96}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin\left(\frac{n\pi x}{2}\right).$$

This Fourier series also converges according to the conclusion of the Dirichlet Test. For example, at $x = 0$ the series converges to $f(0) = 0$ since the function x^3 is continuous at $x = 0$ and its value is 0 there.

Since the function x^3 is periodic with period 4 (because the original one is) we see that $\lim_{x\to 2^+} f(x) = (-2)^3 = -8$ while $\lim_{x\to 2^-} f(x) = 2^3 = 8$. Thus, according to the Dirichlet Test, the series must converge to the average of these two values, namely $(-8+8)/2 = 0$, when $x = 2$. That this is correct can be seen by inserting the value $x = 2$ into the series for x^3 and noting that $\sin n\pi = 0$ for any integer n by trigonometry.

Differentiating a Fourier series can be a risky business! This is because the differentiated series may not converge at all (let alone to the function it is supposed to represent) as we will see in the next example.

Example 15 *Calculate the Fourier sine series of the function $f(x) = x$ for x in $(0, 2)$ and show that its differentiated series does not converge at all except for $x = 0$.*

Solution: Since we want a Fourier sine series we must extend this f to the interval $(-2, 0)$ using its odd extension. This means that $f(x) = -f(-x) = -(-x) = x$ for x in $(-2, 0)$. But this means that $f(x) = x$ for all x in $(-2, 2)$. Of course, this means that $f(x) = x$ is *already* an odd function at the outset (but we may not have noticed this). A simple calculation (we omit the details) shows that, since $L = 2$,

$$b_n = \int_0^2 x \sin\left(\frac{n\pi x}{2}\right)\, dx = \frac{-4\,\cos n\pi}{n\pi} = \frac{-4\,(-1)^n}{n\pi}.$$

Since $(-1)\cdot(-1)^n = (-1)^{n+1}$, it follows that

$$x = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\,\sin\left(\frac{n\pi x}{2}\right).$$

Differentiating this series *formally* (this means "without paying any attention to the details") we find the "equality"

$$1 = 2\sum_{n=1}^{\infty} (-1)^{n+1} \sin\left(\frac{n\pi x}{2}\right).$$

Unfortunately, the series on the right CANNOT converge since

$$\lim_{n\to\infty} \left|(-1)^{n+1} \sin\left(\frac{n\pi x}{2}\right)\right| \;\; \text{does not exist!}$$

So, how does one handle the differentiation of Fourier series? There is a test we can cite that can be used without too much effort.

Test for Differentiating a Fourier Series Let f be a continuous function for all x, $-L \le x \le L$ and assume that $f(-L) = f(L)$. Extend f to a periodic function of period $2L$ outside $[-L, L]$ by periodicity. Assume that f is piecewise differentiable in $(-L, L)$ having finite left- and right-derivatives at $\pm L$. Then the differentiated Fourier series converges to $f'(x)$ on $[-L, L]$.

Example 16 *Find the Fourier sine series of the function $\pi\, x(\pi - x)/8$ valid on $(0, \pi)$ and find the value of its differentiated series.*

Solution: Note that here, $L = \pi$. The odd extension of this function is given by Example 8b), that is,

$$f(x) = \begin{cases} \frac{1}{8}\pi x(\pi - x), & 0 < x < \pi, \\ \\ \frac{1}{8}\pi x(\pi + x), & -\pi < x < 0, \end{cases}$$

The Fourier coefficients are given by

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\, \sin nx\, dx = \frac{(1 + (-1)^{n+1})}{2n^3}$$

and so the Fourier series is

$$\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots = \sum_{n=1}^{\infty} \frac{(1+(-1)^{n+1})}{2n^3} \sin nx.$$

Note that f satisfies the conditions of the Test, above. It is continuous everywhere and it fails to have a derivative at points of the form $\pm n\pi$, where n is an integer. The differentiated series looks like

$$\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots = \sum_{n=1}^{\infty} \frac{(1+(-1)^{n+1})}{2n^2} \cos nx$$

and so we can conclude that

$$\frac{\pi^2}{8} - \frac{\pi x}{4} = \sum_{n=1}^{\infty} \frac{(1+(-1)^{n+1})}{2n^2} \cos nx,$$

holds for x in the range $[-\pi, \pi]$. When $x = 0$ we recover the result of Example 11 using a different method.

Remark: Basically, the expansion of a function into a Fourier series is essentially an example of an **eigenfunction expansion**. This means that we can expand very large classes of functions into the eigenfunctions of boundary value problems associated with Sturm-Liouville equations. It just happens that the eigenfunctions that we use are the simplest ones and they are of historical and practical value.