0.1 Exercises on Sequences

In the following exercises find the 1) limit superior and 2) the limit inferior of the given sequence. Determine whether 3) the limit exists as $n \to \infty$ and give reasons.

1. $a_n = \dfrac{(-1)^n}{n+1}$ Answer: All limits are equal and equal to zero.

2. $a_n = n\,(-1)^n$ Answer: $\limsup_{n\to\infty} a_n = +\infty$; $\liminf_{n\to\infty} a_n = -\infty$, $\lim_{n\to\infty} a_n$ does not exist.

3. $x_n = (-1)^n + (-1)^{n+1}$, Answer: All limits are equal and equal to zero.

4. $x_n = (-1)^n + (-1)^{n+2}$, Answer: $\limsup_{n\to\infty} x_n = +2$; $\liminf_{n\to\infty} x_n = -2$, $\lim_{n\to\infty} x_n$ does not exist.

5. $b_n = n \sin\left(\dfrac{2}{n}\right)$ Answer: All limits are equal and equal to 2.

6. $x_m = \sin m\pi + \cos m\pi$ Answer: $\limsup_{m\to\infty} x_m = +1$; $\liminf_{m\to\infty} x_m = -1$, $\lim_{m\to\infty} x_m$ does not exist.

7. $a_n = 2(-1)^n + \dfrac{n}{n+1}$. Answer: $\limsup_{n\to\infty} a_n = +3$; $\liminf_{n\to\infty} a_n = -1$, $\lim_{n\to\infty} a_n$ does not exist.

8. $a_n = 2(-1)^n + \dfrac{n}{n+1}$. Answer: $\limsup_{n\to\infty} a_n = +3$; $\liminf_{n\to\infty} a_n = -1$, $\lim_{n\to\infty} a_n$ does not exist.

9. $x_n = \sin n$ Answer: **Hard!** $\limsup_{n\to\infty} x_n = +1$; $\liminf_{n\to\infty} x_n = -1$, $\lim_{n\to\infty} x_n$ does not exist. The point here is that π can be approximated arbitrarily closely by rational numbers (of the form $\frac{n}{m}$) so that if we choose n, m carefully we can make $\sin n$ as close to 1 and -1 as we want [1]

[1] To see this we approximate the number $\pi/2$ by rational numbers m/p where p is of the form $4m+1$ (and m is an integer). To do this we have to use a result from Number Theory (e.g., G.H. Hardy's, *Introduction to the Theory of Numbers*) that states: If x is irrational there exists numbers m, p such that

$$\left|\frac{m}{p} - x\right| < \frac{1}{p^2}.$$

So, for each given integer n we choose a pair of numbers m_n, p_n (with $p_n \to \infty$ as $n \to \infty$) such that

$$\left|\frac{m_n}{p_n} - \frac{\pi}{2}\right| < \frac{1}{{p_n}^2}.$$

10. $x_n = 2\cos n$ Answer: **Hard!** $\limsup_{n\to\infty} x_n = +2$; $\liminf_{n\to\infty} x_n = -2$, $\lim_{n\to\infty} x_n$ does not exist.

11. $x_n = 3x_{n-1}$ where $n = 1, 2, 3, \ldots$ and $x_0 = -2$.
Answer: All the limits are equal to $-\infty$.

12. $x_n = -x_{n-1}$ where $n = 1, 2, 3, \ldots$ and $x_0 = -1$.
Answer: $\limsup_{n\to\infty} x_n = +1$; $\liminf_{n\to\infty} x_n = -1$, $\lim_{n\to\infty} x_n$ does not exist.

Prove that the following sequences defined iteratively (or by recursion) are Cauchy sequences and find their limits. Prove the existence of the limit using an ε argument.

13. $x_n = x_{n-1}$ where $n = 1, 2, 3, \ldots$ and $x_0 = A$.
Answer: The limit is A.

14. $x_n = \frac{1}{2}x_{n-1}$, where $n = 1, 2, 3, \ldots$ and $x_0 = 1$.
Answer: The limit is zero.

15. $x_n = \frac{2}{5}x_{n-1}$ where $n = 1, 2, 3, \ldots$ and $x_0 = -2$.
Answer: The limit is zero.

Miscellaneous problems

16. Prove that the sequence (also known as an *arithmetic progression*) defined iteratively by $x_n = a + x_{n-1}$, $n \geq 1$, where a is a fixed real number, x_0 is given, is a Cauchy sequence if and only if $a = 0$.
Answer: Note that $|x_n - x_m| = |a||n - m|$ which can be made arbitrarily small if and only if $a = 0$.

17. Show that the sequence defined iteratively by $x_n = a \cdot x_{n-1}$ where $n = 1, 2, 3, \ldots$ and x_0 is a given real number, has the following properties:
    - It converges to 0 if $|a| < 1$
    - It converges to x_0 if $a = 1$
    - It converges to 0 if $a \in \mathbf{R}$ and $x_0 = 0$
    - The limit does not exist if $a = -1$ and $x_0 \neq 0$
    - It converges to ∞ if $a > 1$ and $x_0 > 0$

From this we get

$$\left| m_n - \frac{\pi p_n}{2} \right| < \frac{1}{p_n}.$$

We claim that the integers m_n form a subsequence of the natural numbers such that $\sin m_n \to 1$ as $n \to \infty$. This is because $\sin m_n = \sin(m_n + p_n\pi/2 + p_n\pi/2) = \sin(m_n - p_n\pi/2)\cos(p_n\pi/2) + \cos(m_n - p_n\pi/2)\sin(p_n\pi/2) = \cos(m_n - p_n\pi/2)$, since p_n is of the form $4m + 1$. Now use the Mean Value Theorem: $|\sin m_n - 1| = |\cos(m_n - p_n\pi/2) - \cos 0| = |\sin\xi|\,|m_n - p_n\pi/2| \leq |m_n - p_n\pi/2| < 1/p_n$. The result follows upon letting $n \to \infty$. A similar argument is used for the $\liminf$ result.

- It converges to $-\infty$ if $a > 1$ and $x_0 < 0$
- The limit does not exist if $a \leq -1$ and $x_0 \neq 0$

Answer: Show that $x_n = a^n\, x_0$ for every $n \geq 1$ after which all the results follow.

18. Determine whether or not the sequence, x_n defined for each integer n, by the finite series

$$x_n = \arctan 1 + \arctan 2 + \arctan 3 + \ldots + \arctan n,$$

for $n \geq 1$, converges as $n \to \infty$!?
Answer: No, the limit cannot exist since $\arctan n \to \pi/2$ as $n \to \infty$ and so the Divergence Test for infinite series applies and this result follows.