Exercises: Describing regions as sets in different coordinate systems

Exercise Set 1

Sketch and describe the given regions as sets in the given coordinate system(s): Assume all regions are given in the xy-plane.

1. The rollerblade slide, $\mathcal{S}$, bounded by the curves $y = x^2$, $x = 2$ and $y = 0$ (in Cartesian coordinates).

2. The triangular region, $\mathcal{T}$, bounded by the curves $y = 2x$, $y = 2$ and $x = 0$ (in Cartesian coordinates).

3. The rectangular region $\mathcal{R}$, bounded by the lines $x = -2$, $x = 3$, $y = -1$ and $y = 4$ (in Cartesian coordinates).

4. The walnut shaped region, $\mathcal{W}$, bounded by the two parabolae $y = 4 - x^2$, and $y = x^2 - 4$ (in Cartesian coordinates).

5. The inclined plane, $\mathcal{T}$, bounded by the lines $y = -1$, $y = x$ and $x = 3$ (in Cartesian coordinates).

6. The kite shaped region, $\mathcal{K}$, bounded by the lines $y = x + 1$, $y = 1 - x$, $y = 2x - 2$ and $y = -2x - 2$.

7. The wing shaped region,$\mathcal{W}$, bounded by the curves $y = x^2$ and $y^2 = x$ (in Cartesian coordinates).

8. **(Hard)** The egg shaped region $\mathcal{E}$, bounded by the curves $y = \sin x$ and the lower semi circle
$$\left(x - \frac{\pi}{2}\right)^2 + y^2 = \frac{\pi^2}{4},$$
where $y \leq 0$ (in Cartesian coordinates).

9. The tent shaped region $\mathcal{T}$, bounded by the curves $y = 0$, $y = 1$, $y = 2 - x$ and $y = x + 2$ (in Cartesian coordinates).

10. An ancient Roman gate, $\mathcal{G}$, bounded by the curves $y = 0$, $x = -1$, $x = 1$ and the semi circle
$$x^2 + (y - 3)^2 = 1,$$
where $y \geq 3$ (in Cartesian coordinates).

11. The pizza slice, $\mathcal{S}$, bounded by the rays $\theta = \frac{\pi}{6}$, $\theta = \frac{\pi}{3}$ and the arc $r = 3$ (in polar coordinates).

12. The solid washer, $\mathcal{W}$, bounded by the circles $r = 2$ and $r = 3$ (in polar coordinates).

13. The sports car logo, $\mathcal{L}$, bounded by the curves $r = 1$, $\theta = 0$, $\theta = \frac{\pi}{2}$, and $r = 1$, $\theta = \pi$, $\theta = \frac{3\pi}{2}$ in (polar coordinates).

14. One arm of the First Aid logo or the Maltese cross, $\mathcal{M}$, with vertex at O bounded by the rays $\theta = \pm\frac{\pi}{6}$ and the arc of the circle

$$r^2 - 4r\cos\theta + 3 = 0$$

(in polar coordinates).

15. The closed region bounded by that part of the cardioid, $\mathcal{C}$, $r = 1 + \cos\theta$ for $0 \leq \theta \leq \pi$ and the ray $\theta = 0$ (in polar coordinates).

16. One leaf, $\mathcal{L}$, of the four leaved flower bounded by the curve $r = \cos 3\theta$ where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ (in polar coordinates).

17. The cigar shaped region, $\mathcal{C}$, bounded by the curves $y = -1$, $y = 1$, $x = 0$ and the arc of the circle $x^2 + y^2 = 4$ where $x \geq 0$ (in polar coordinates).

18. The saucer shaped region, $\mathcal{R}$, common to both the circles $x^2 + y^2 = 2y$ and $x^2 + y^2 = 1$ (in polar **and** Cartesian coordinates).

19. The tipped mailbox shaped region, $\mathcal{R}$, bounded by the curves $x = -1$, $y = 0$, $y = 1$ and the quarter circle corresponding to $x^2 + y^2 = 1$ lying in the first quadrant (in polar **and** Cartesian coordinates).

20. The spear shaped triangular region, $\mathcal{S}$, bounded by the curves $y = \frac{\sqrt{3}}{3}x$, $y = \sqrt{3}\,x$ and the curve $xy = 2$ (in polar **and** Cartesian coordinates, using vertical slices).

Answers:

1. a) Using vertical slices:

$$\mathcal{S} = \left\{(x, y) : 0 \le y \le x^2,\ 0 \le x \le 2\right\},$$

or,

b) Using horizontal slices:

$$\mathcal{S} = \{(x, y) : \sqrt{y} \le x \le y,\ 0 \le y \le 4\}.$$

Both descriptions are correct.

2. a) Using vertical slices:

$$\mathcal{T} = \{(x, y) : 2x \le y \le 2,\ 0 \le x \le 1\},$$

or,

b) Using horizontal slices:

$$\mathcal{T} = \left\{(x, y) : 0 \le x \le \frac{y}{2},\ 0 \le y \le 2\right\}.$$

Both descriptions are correct.

3. Using either vertical or horizontal slices we find:

$$\mathcal{R} = \{(x, y) : -2 \le x \le 3,\ -1 \le y \le 4\}.$$

Both descriptions are given by the same set of inequalities.

4. a) Using vertical slices:

$$\mathcal{S} = \left\{(x, y) : x^2 - 4 \le y \le 4 - x^2,\ -2 \le x \le 2\right\}$$

or,

b) Using horizontal slices: $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ where

$$\mathcal{S}_1 = \left\{(x, y) : -\sqrt{4 - y} \le x \le \sqrt{4 - y},\ 0 \le y \le 4\right\},$$

and

$$\mathcal{S}_2 = \left\{(x, y) : -\sqrt{4 + y} \le x \le \sqrt{4 + y},\ -4 \le y \le 0\right\}.$$

5. a) Using vertical slices:

$$\mathcal{T} = \{(x, y) : -1 \le y \le x,\ -1 \le x \le 3\},$$

or,

b) Using horizontal slices:

$$\mathcal{T} = \{(x, y) : y \le x \le 3,\ -1 \le y \le 3\}.$$

6. a) Using vertical slices: $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$ where

$$\mathcal{K}_1 = \{(x, y) : 2x - 2 \le y \le 1 - x,\ 0 \le x \le 1\},$$

and

$$\mathcal{K}_2 = \{(x, y) : -2x - 2 \le y \le x + 1,\ -1 \le x \le 0\},$$

or, using horizontal slices,

b) $\mathcal{K} = \mathcal{K}_3 \cup \mathcal{K}_4$ where

$$\mathcal{K}_3 = \{(x, y) : y - 1 \le x \le 1 - y,\ 0 \le y \le 1\},$$

and

$$\mathcal{K}_4 = \left\{(x, y) : -\frac{y+2}{2} \le y \le \frac{y+2}{2},\ -2 \le y \le 0\right\}.$$

7. a) Using vertical slices:

$$\mathcal{W} = \left\{(x, y) : x^2 \le y \le \sqrt{x},\ 0 \le x \le 1\right\},$$

b) or, using horizontal slices:

$$\mathcal{W} = \left\{(x, y) : y^2 \le x \le \sqrt{y},\ 0 \le y \le 1\right\}.$$

8. a) Using vertical slices:

$$\mathcal{E} = \left\{(x, y) : -\sqrt{\frac{\pi^2}{4} - \left(x - \frac{\pi}{2}\right)^2} \le y \le \sin x,\ 0 \le x \le \pi\right\},$$

or (this part is hard),

b) Using horizontal slices: $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ where

$$\mathcal{E}_1 = \{(x, y) : \operatorname{Arcsin} y \le x \le \pi - \operatorname{Arcsin} y,\ 0 \le y \le 1\},$$

and

$$\mathcal{E}_2 = \left\{(x, y) : -\frac{\pi}{2} - \sqrt{\frac{\pi^2}{4} - y^2} \le x \le \frac{\pi}{2} + \sqrt{\frac{\pi^2}{4} - y^2},\ -\frac{\pi}{2} \le y \le 0\right\}.$$

9. a) Using vertical slices: $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ where:

$$\mathcal{T}_1 = \{(x, y) : 0 \le y \le x + 2,\ -2 \le x \le -1\},$$

$$\mathcal{T}_2 = \{(x, y) : 0 \le y \le 1,\ -1 \le x \le 1\},$$

$$\mathcal{T}_3 = \{(x, y) : 0 \le y \le 2 - x,\ 1 \le x \le 2\},$$

or, using horizontal slices,

b)

$$\mathcal{T} = \{(x, y) : y - 2 \le x \le 2 - y,\ 0 \le y \le 1\}.$$

10. a) Using vertical slices:

$$\mathcal{G} = \left\{(x, y) : 3 - \sqrt{1 - x^2} \le y \le 3 + \sqrt{1 - x^2},\ \ -1 \le x \le 1\right\},$$

or, using horizontal slices,

b) $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ where

$$\mathcal{G}_1 = \{(x, y) : -1 \le x \le 1,\ 0 \le y \le 3\},$$

$$\mathcal{G}_2 = \left\{(x, y) : -\sqrt{1 - (y - 3)^2} \le x \le \sqrt{1 - (y - 3)^2},\ 3 \le y \le 4\right\},$$

11.

$$\mathcal{S} = \left\{(r, \theta) : 0 \le r \le 3,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\right\}.$$

12.

$$\mathcal{W} = \{(r, \theta) : 2 \le r \le 3,\ 0 \le \theta < 2\pi\}.$$

13. $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ where

$$\mathcal{L}_1 = \left\{(r, \theta) : 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2}\right\},$$

$$\mathcal{L}_2 = \left\{(r, \theta) : 0 \le r \le 1,\ \pi \le \theta \le \frac{3\pi}{2}\right\}.$$

14.

$$\mathcal{M} = \left\{(r, \theta) : 0 \le r \le \frac{4\cos\theta - \sqrt{16\cos^2\theta - 12}}{2},\ -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}\right\}.$$

15.

$$\mathcal{C} = \{(r, \theta) : 0 \le r \le 1 + \cos\theta,\ 0 \le \theta < \pi\}.$$

16.

$$\mathcal{L} = \left\{(r, \theta) : 0 \le r \le \cos 3\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}.$$

17. $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ where:

$$\mathcal{C}_1 = \left\{(r, \theta) : 0 \le r \le 4,\ -\text{Arcsin}\frac{1}{4} \le \theta \le \text{Arcsin}\frac{1}{4}\right\},$$

$$\mathcal{C}_2 = \left\{(r, \theta) : 0 \le r \le \csc\theta,\ \text{Arcsin}\frac{1}{4} \le \theta \le \frac{\pi}{2}\right\},$$

$$\mathcal{C}_3 = \left\{(r, \theta) : 0 \le r \le -\csc\theta,\ -\frac{\pi}{2} \le \theta \le -\text{Arcsin}\frac{1}{4}\right\}.$$

18. In polar coordinates: $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ where:

$$\mathcal{R}_1 = \left\{(r, \theta) : 0 \le r \le 2\sin\theta,\ 0 \le \theta \le \frac{\pi}{6}\right\},$$

$$\mathcal{R}_2 = \left\{(r, \theta) : 0 \le r \le 1,\ \frac{\pi}{6} \le \theta \le \frac{5\pi}{6}\right\},$$

$$\mathcal{R}_3 = \left\{(r, \theta) : 0 \le r \le 2\sin\theta,\ \frac{5\pi}{6} \le \theta \le \pi\right\},$$

or, in Cartesian coordinates (vertical slices):

$$\mathcal{R} = \left\{ (x, y) : 1 - \sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2},\ -\frac{\sqrt{3}}{2} \leq x \leq \frac{\sqrt{3}}{2} \right\},$$

or, using horizontal slices: $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ where

$$\mathcal{R}_1 = \left\{ (x, y) : -\sqrt{1 - y^2} \leq x \leq \sqrt{1 - y^2},\ \frac{1}{2} \leq y \leq 1 \right\},$$

$$\mathcal{R}_2 = \left\{ (x, y) : -\sqrt{2y - y^2} \leq x \leq \sqrt{2y - y^2},\ 0 \leq y \leq \frac{1}{2} \right\}.$$

19. In polar coordinates: $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ where:

$$\mathcal{R}_1 = \left\{ (r, \theta) : 0 \leq r \leq 1,\ 0 \leq \theta \leq \frac{\pi}{2} \right\},$$

$$\mathcal{R}_2 = \left\{ (r, \theta) : \csc\theta \leq r \leq 1,\ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4} \right\},$$

$$\mathcal{R}_3 = \left\{ (r, \theta) : 1 \leq r \leq -\sec\theta,\ \frac{3\pi}{4} \leq \theta \leq \pi \right\},$$

or, in Cartesian coordinates (vertical slices): $\mathcal{R} = \mathcal{R}_1 \cup (\mathcal{R}_2 \cup \mathcal{R}_3)$ where

$$\mathcal{R}_1 = \left\{ (x, y) : 0 \leq y \leq \sqrt{1 - x^2},\ 0 \leq x \leq 1 \right\},$$

$$\mathcal{R}_2 \cup \mathcal{R}_3 = \{ (x, y) : 0 \leq y \leq 1,\ -1 \leq x \leq 0 \},$$

or, using horizontal slices (simplest of all):

$$\mathcal{R} = \left\{ (x, y) : -1 \leq x \leq \sqrt{1 - y^2},\ 0 \leq y \leq 1 \right\}.$$

20. In polar coordinates:

$$\mathcal{S} = \left\{ (r, \theta) : 0 \leq r \leq 2\sqrt{\csc\theta},\ \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3} \right\},$$

or, in Cartesian coordinates (vertical slices): $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ where:

$$\mathcal{S}_1 = \left\{ (x, y) : \frac{\sqrt{3}x}{3} \leq y \leq \sqrt{3}x, \quad 0 \leq x \leq \frac{\sqrt{2}}{\sqrt[4]{3}} \right\},$$

$$\mathcal{S}_2 = \left\{ (x, y) : \frac{\sqrt{3}x}{3} \leq y \leq \frac{2}{x}, \quad \frac{\sqrt{2}}{\sqrt[4]{3}} \leq x \leq \sqrt{2}\sqrt[4]{3} \right\}.$$