

Solutions Manual

1.1

1.2 Exercise Set 1 (page 10)

1. $-2, -1, 2, \frac{1}{4}$.
2. $(x+1)^3 \sin(x+1)$. Use the Box method.
3. $z - 2 + 2 \sin(z-2) - \cos z$.
4. $-2 \cos(x+ct)$.
5. $f(\pi/2) = \sin(\cos(\pi/2)) = \sin 0 = 0$.
6. $2x+h$. You get this by dividing by h since $h \neq 0$.
7. $\sin(t+3) \frac{\cos h - 1}{h} + \cos(t+3) \frac{\sin h}{h}$.
8. (a) $-\pi^2$, (b) $4\pi^2$.
9. (a) $f(0) = 1$, (b) $f(0.142857) = 0.857143$, (c) Since $0 < x < 1$ we see that $2 < 3x+2 < 5$. So, $f(3x+2) = (3x+2)^2 = 9x^2 + 12x + 4$.
10. $f(F(x)) = x$, $F(f(x)) = |x|$.
11. The Box method gives that $g(x+1) = (x+1)^2 - 2(x+1) + 1 = x^2$.
12. Again we use the Box method with the quantity $(x-1)/(2-x)$ inside the Box. Since $h(\square) = (2\square+1)/(1+\square)$, we use some simple algebra to see that the right-hand side becomes just x .
13. 8. Observe that $f(x+h) - 2f(x) + f(x-h) = 8h^2$, so that, for $h \neq 0$ the cancellation of the h^2 -terms gives the stated result.
14. The definition of the function tells us that (using the Box method), $f(x+1) = (x+1) - 1 = x$ whenever $0 \leq x+1 \leq 2$, which is equivalent to saying that $f(x+1) = x$ whenever $-1 \leq x \leq 1$. We use the same idea for the other interval. Thus, $f(x+1) = 2(x+1) = 2x+2$ whenever $2 < x+1 \leq 4$, equivalently, $f(x+1) = 2x+2$ whenever $1 < x \leq 3$. Since the interval $\{1 < x \leq 2\}$ is contained inside the interval $\{1 < x \leq 3\}$ it follows that $f(x+1) = 2x+2$ for such x .

1.3 Exercise Set 2 (page 19)

1.
$$f(x) = \begin{cases} x^2 - 1, & x \geq 1 \text{ or } x \leq -1, \\ 1 - x^2, & -1 < x < 1. \end{cases}$$
2.
$$f(x) = \begin{cases} 3x + 4, & \text{if } x \geq -4/3, \\ -3x - 4, & \text{otherwise.} \end{cases}$$
3.
$$h(x) = \begin{cases} x^2, & \text{if } x \geq 0, \\ -x^2, & \text{otherwise.} \end{cases}$$
4.
$$f(x) = \begin{cases} 1 - t, & \text{if } t \geq 0, \\ 1 + t, & \text{if } t < 0. \end{cases}$$
5.
$$g(w) = \begin{cases} \sin w & \text{for } w \text{ in any interval of the form } [2\pi n, 2\pi n + \pi], \\ -\sin w & \text{otherwise,} \end{cases}$$

where n is an integer.
6.
$$f(x) = \begin{cases} \frac{1}{x\sqrt{x^2-1}}, & \text{if } x > 1, \\ -\frac{1}{x\sqrt{x^2-1}}, & \text{if } x < -1. \end{cases}$$
7.
$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0. \end{cases}$$

$$8. \quad f(x) = \begin{cases} 2x, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

$$9. \quad f(x) = \begin{cases} 0, & \text{if } x \geq 0, \\ 2x, & \text{if } x < 0. \end{cases}$$

1.4 Exercise Set 3 (page 28)

1. Correction: If $A < 0$, then $-A < B$ implies $-1/A > 1/B$.
2. This is false. To see this, let $A = 1$ and $B = 0$.
3. Correction: $0 \leq A < B$ implies $A^2 < B^2$.
4. Correction: $A > B > 0$ implies $1/A < 1/B$.
5. Correction: $A < B$ implies $-A > -B$.
6. Correction: If $A^2 < B^2$ and $B > 0$, then $A < B$.
7. This statement is correct. There is nothing wrong!
8. $(0, \pi)$. (Note: To complete our argument we need $\sin x > 0$, which is guaranteed by $0 < x < \pi$.)
9. Its values are less than or equal to 6. Actually, its largest value occurs when $x = 2$ in which case $f(2) \approx 5.8186$.
10. g is unbounded: This means that it can be greater than (resp. less than) any given number. The problem occurs at $x = 0$.
11. From $x > 1$ we see that both x and $x - 1$ are positive. Hence we can square both sides of the inequality $x > x - 1$ to arrive at $x^2 > (x - 1)^2$. (Alternately, since both x and $x - 1$ are positive, $x^2 > x^2 - x - (x - 1) = x^2 - 2x + 1 = (x - 1)^2$.)
12. From $p \leq 1$ we see that $1 - p \geq 0$. Since $x \geq 1$ (certainly this implies the positivity of x), we have $x^{1-p} \geq 1^{1-p}$, or $x^{1-p} \geq 1$. Now $x^{1-p} = x^{-(p-1)} = \frac{1}{x^{p-1}}$. So the last inequality can be rewritten as $\frac{1}{x^{p-1}} \geq 1$. We can multiply both sides of this inequality by $\sin x$ because $1 \leq x \leq \pi$ guarantees that $\sin x$ is positive.
13. Since both x and x^2 are ≥ 0 , we can apply the AG-inequality to get $\frac{x+x^2}{2} \geq \sqrt{x \cdot x^2} = \sqrt{x^3}$. Since $x + x^2 \geq 0$, we have $x + x^2 \geq \frac{x+x^2}{2}$. So $x + x^2 \geq \sqrt{x^3}$. Yes, we can square both sides since $x \geq 0$, and so both terms in the inequality are greater than or equal to 0.
14. Yes. Under no further conditions on the symbol, since it is true that $(\square - 1)^2 \geq 0$ for any symbol, $\square$. Expanding the square and separating terms we get that $\square^2 \geq 2\square - 1$.
15. Since $1 - p \geq 0$ and $|x| \geq 1$, we have $|x|^{1-p} \geq 1^{1-p} = 1$, or $|x| |x|^{-p} \geq 1$, which gives $|x| \geq |x|^p$. Taking reciprocals, we get $\frac{1}{|x|} \leq \frac{1}{|x|^p}$. (The last step is legitimate because both $|x|^p$ and $|x|$ are positive.)
16. $|v| < c$. This is because we need $1 - v^2/c^2 > 0$. Now solve this inequality for v .
17. If $n = 2$, the result is clear, because $2 < (1.5)^2 < 3$. So let's assume that $n > 2$, now. We use (1.12) with the quantity "1/n" inside the box symbol (or replacing the box by $1/n$, if you like). We'll see that

$$\begin{aligned} \left(1 + \boxed{\frac{1}{n}}\right)^n &= 1 + n \boxed{\frac{1}{n}} + \frac{n(n-1)}{2!} \boxed{\frac{1}{n}}^2 + \cdots + \frac{n(n-1)\cdots(2)(1)}{n!} \boxed{\frac{1}{n}}^n \\ &= 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!} \left(\frac{1}{n}\right)^2 + \cdots + \frac{n(n-1)(n-2)\cdots(2)(1)}{n!} \left(\frac{1}{n}\right)^n \\ &= 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{n^2} \left(\frac{1}{2!}\right) + \cdots + \frac{n(n-1)(n-2)\cdots(2)(1)}{n^n} \left(\frac{1}{n!}\right). \end{aligned}$$

Now, we regroup all the terms in the above display in the following way Note that the following term is not apparent in the display above, but it IS there! See Equation (1.12).

$$\begin{aligned} \frac{n(n-1)(n-2)}{n^3} &= \left(\frac{n}{n}\right) \left(\frac{n-1}{n}\right) \left(\frac{n-2}{n}\right) \\ &= (1) \left(\frac{n-1}{n}\right) \left(\frac{n-2}{n}\right) \\ &= \left(\frac{n-1}{n}\right) \left(\frac{n-2}{n}\right), \\ &= \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right). \end{aligned}$$

A similar idea is used for the other terms. Okay, so using this rearrangement of terms we can rewrite $(1 + \frac{1}{n})^n$ as

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &= \dots \\ &= 1 + 1 + \left(1 - \frac{1}{n}\right) \frac{1}{2!} + \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \frac{1}{3!} + \dots \\ &\quad + \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \left(1 - \frac{3}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right) \frac{1}{n!}. \end{aligned}$$

(where there are $(n+1)$ terms in the right hand side). Now, notice that for every integer $n > 2$, each term of the form "1 - (something)/n" is less than 1 and bigger than zero, because we're subtracting something positive from 1. So,

$$\begin{aligned} \left(1 - \frac{1}{n}\right) &< 1 \\ \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) &< \left(1 - \frac{2}{n}\right) < 1, \dots \end{aligned}$$

where we have used Figure 9 with $A = 1 - 2/n$, $\square = 1 - 1/n$ (or with the symbols "1 - 1/n" inside the box), and $\triangle = 1$ (or with "1" inside the triangle). Using these estimates we can see that we can replace every term inside the "large brackets" by 1 so that

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &= \dots \\ &= 1 + 1 + \left(1 - \frac{1}{n}\right) \frac{1}{2!} + \cdots + \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \left(1 - \frac{3}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right) \frac{1}{n!} \quad (1.1) \\ &< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \end{aligned}$$

We're almost done! Now we use the following inequalities ...

$$\begin{array}{rcl}
3! = 3 \times 2 \times 1 & > & 2 \times 2 \times 1 = 2^2 \\
4! = 4 \times 3 \times 2 \times 1 & > & 2 \times 2 \times 2 \times 1 = 2^3 \\
5! = 5 \times 4 \times 3 \times 2 \times 1 & > & 2 \times 2 \times 2 \times 2 \times 1 = 2^4 \\
& \dots & \\
n! & > & 2^{n-1}
\end{array}$$

Now since we must “reverse the inequality when we take reciprocals of positive numbers” (Table 1.2, Table 1.3) we get that for every integer $n > 2$,

$$n! > 2^{n-1} \quad \text{implies} \quad \frac{1}{n!} < \frac{1}{2^{n-1}}$$

Combining this estimate with Equation (1.1) we get a new estimate, namely,

$$\begin{aligned}
\left(1 + \frac{1}{n}\right)^n &< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\
&< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}.
\end{aligned} \tag{1.2}$$

Now, the sum on the right above is a **finite geometric series** and we know that, if $n > 2$,

$$1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = \frac{1 - (\frac{1}{2})^{n+1}}{1 - \frac{1}{2}} < \frac{1}{1 - \frac{1}{2}} = 2.$$

Now you can see that, when we combine this latest estimate with (1.2) we find

$$\begin{aligned}
\left(1 + \frac{1}{n}\right)^n &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \\
&< 1 + 2 = 3
\end{aligned}$$

which is what we wanted to show. Okay, this looks a bit long, but we did include all the details, right? Eventually, you'll be able to skip many of the details and do them in your head, so to speak, and the whole thing will get shorter and faster, you'll see.

It looks tough, but we'll be using this 200 yr old inequality later on, in Chapter 4, when we define Euler's number, 2.7182818284590.

1.5 Chapter Exercises (page 30)

- 6, 1, 2, $2\frac{3}{4} = \frac{11}{4}$.
- $(x^2 + 1)^3 \cos(x^2 + 1)$.
- $z + 3 + 2 \sin(z + 3) - \cos(z + 5)$.
- $-\frac{\sin h}{h} \sin x + \frac{\cos h - 1}{h} \cos x$.
- From $\frac{3}{x} > 6$ we see that x must be positive: $x > 0$. So we can rewrite it as $3 > 6x$, which gives $x < \frac{1}{2}$. Thus the solution is $0 < x < \frac{1}{2}$.
- $x \geq -\frac{4}{3}$, since we subtract 4 from both sides ...
- $x < \frac{1}{2}$. Note that $2x - 1 < 0$ and so $2x < 1$.
- $|x| > \sqrt{5}$. In other words, either $x > \sqrt{5}$ or $x < -\sqrt{5}$.
- $|t| < \sqrt[4]{5}$. That is, $-\sqrt[4]{5} < t < \sqrt[4]{5}$.
- $-\infty < x < +\infty$. That is, x can be any real number. This is because the stated inequality implies that $\sin x \leq 1$ and this is always true!
- $z \geq 2^{1/p}$. (Note: For general p , z^p is defined only for $z > 0$.)
- $|x| \leq 3$. Or $-3 \leq x \leq 3$.

13.

$$f(x) = \begin{cases} x + 3, & \text{for } x \geq -3, \\ -x - 3, & \text{for } x < -3. \end{cases}$$

14.

$$g(x) = \begin{cases} t - 0.5, & \text{if } t \geq 0.5, \\ -t + 0.5, & \text{otherwise.} \end{cases}$$

15.

$$g(t) = \begin{cases} 1 - t, & \text{if } t \leq 1, \\ t - 1, & \text{otherwise.} \end{cases}$$

16.

$$f(x) = \begin{cases} 2x - 1, & x \geq \frac{1}{2} \\ 1 - 2x, & x < \frac{1}{2} \end{cases}$$

17.

$$f(x) = \begin{cases} 1 - 6x, & \text{if } x \leq 1/6, \\ 6x - 1, & \text{otherwise.} \end{cases}$$

18.

$$f(x) = \begin{cases} x^2 - 4, & \text{if either } x \geq 2 \text{ or } x \leq -2, \\ 4 - x^2, & \text{if } -2 < x < 2. \end{cases}$$

19.

$$f(x) = \begin{cases} 3 - x^3, & \text{if } x \leq \sqrt[3]{3}, \\ x^3 - 3, & \text{if } x > \sqrt[3]{3}. \end{cases}$$

20. $f(x) = |(x - 1)^2| = (x - 1)^2 = x^2 - 2x + 1$ for all x . (Note that $(x - 1)^2$ is always ≥ 0 for any value of x .)

21.

$$f(x) = |x(2-x)| = \begin{cases} x(2-x), & \text{if } 0 \leq x \leq 2, \\ x(x-2), & \text{otherwise.} \end{cases}$$

22. $f(x) = |x^2 + 2| = x^2 + 2$ for all x , because $f(x) = x^2 + 2 \geq 2 > 0$ to begin with.

23. From $p \leq 1$ we have $1-p \geq 0$. So $x \geq 1 > 0$ gives $x^{1-p} \geq 1^{1-p} = 1$. Now $x^{1-p} = x^{-(p-1)} = \frac{1}{x^{p-1}}$. Thus $\frac{1}{x^{p-1}} \geq 1$. On the other hand, from $0 \leq x \leq \pi/2$ we have $\cos x \geq 0$. So we can multiply $\frac{1}{x^{p-1}} \geq 1$ throughout by $\cos x$ to arrive at $\frac{\cos x}{x^{p-1}} \geq \cos x$.

24. 2, 2.25, 2.370370, 2.44141, 2.48832, 2.52163, 2.54650, 2.56578, 2.58117, 2.59374. Actually, these numbers approach the value 2.71828...

25. From $0 \leq x \leq \frac{\pi}{2}$ we have $\sin x \geq 0$ and $\cos x \geq 0$. Thus we may apply the AG-inequality to get $\frac{\sin x + \cos x}{2} \geq \sqrt{\sin x \cos x}$. Since $\sin 2x = 2 \sin x \cos x$, we see that $\sqrt{\sin x \cos x} = \sqrt{\frac{\sin 2x}{2}}$ and so $\frac{\sin x + \cos x}{2} \geq \sqrt{\frac{\sin 2x}{2}}$. Multiplying both sides by $\sqrt{2}$ we get the desired inequality.

Solutions

2.1 Exercise Set 4 (page 39)

1. 4
2. 1
3. 0
4. $+\infty$, since $t > 2$ and $t \rightarrow 2$.
5. 0
6. -1 , since $|x| = -x$ for $x < 0$.
7. 0
8. $-\frac{1}{\pi}$
9. 0
10. $+\infty$, since $|x - 1| = 1 - x > 0$ for $x < 1$.
11. 0
12. $\frac{1}{6}$
13. i) 0, ii) 1. Since the limits are different the graph must have a break at $x = 1$.
14. i) 1, ii) 1, iii) 0, iv) 1; since the one-sided limits are equal at $x = 0$ and $g(0) = 1$, the graph has no break at $x = 0$. But since these limits are different at $x = 1$, it must have a break at $x = 1$.
15. i) 1, ii) 2, iii) 1, iv) 2.

2.2 Exercise Set 5 (page 45)

1. No, because the left and right-hand limits at $x = 0$ are different, ($2 \neq 0$).
2. Yes, the value is 4, because the two one-sided limits are equal (to 4).
3. Yes, the value is 0, because the two one-sided limits are equal (to 0).
4. Yes, the value is 0, because the two one-sided limits are equal (to 0).
5. Yes, the value is 0, because the two one-sided limits are equal; remove the absolute value, first, and note that $\sin 0 = 0$.
6. No, because the left-hand limit at $x = 0$ is $-\infty$ while the right-hand limit there is $+\infty$.
7. No, because the left-hand limit at $x = 0$ is $-\infty$ and the right-hand limit there is $+\infty$.
8. Yes, the answer is $1/2$ because the two one-sided limits are equal (to $\frac{1}{2}$).
9. Yes, because the two one-sided limits are equal (to 2).
10. No, because the left-hand limit at $x = 0$ is $+3$ and the right-hand limit there is $+2$ ($3 \neq 2$).
11. a) Yes, the left and right-hand limits are equal (to 0) and $f(0) = 0$;
b) Yes, because g is a polynomial;
c) Yes, because the left and right-limits are equal to 3 and $h(0) = 3$;
d) Yes, since by Table 2.4d, the left and right-limits exist and are equal and $f(0) = 2$;
e) Yes, because f is the quotient of two continuous functions with a non-zero denominator at $x = 0$. Use Table 2.4d again.
12. Follow the hints.

2.2 Exercise Set 6 (page 49)

1. $x = 0$ only; this is because the right limit is 2 but the left-limit is 0. So, f cannot be continuous at $x = 0$.
2. $x = 0$ only; this is because the right limit is 1 but the left-limit is 0. So, f cannot be continuous at $x = 0$.
3. $x = \pm 1$ because these are the roots of the denominator, so the function is infinite there, and so it cannot be continuous there.
4. $x = 0$ only. In this case the right limit is the same as the left-limit, 1, but the value of $f(0) = 2$ is not equal to this common value, so it cannot be continuous there.
5. $x = 0$ only. This is because the right-limit at $x = 0$ is $+\infty$, so even though $f(0)$ is finite, it doesn't matter, since one of the limits is infinite. So, f cannot be continuous at $x = 0$.
6. $x = 0$ only, because the left-limit there is 1.62 while its right-limit there is 0. There are no other points of discontinuity.

2.2 Exercise Set 7 (page 56)

1. -1. Use the trigonometric identity, $\sin(\square + \pi) = -\sin \square$.
2. -1. Use the hint.
3. 2. Multiply the expression by $1 = \frac{2}{2}$ and rearrange terms.
4. 0. Let $\square = 3x$, rearrange terms and simplify.
5. 2. Multiply the whole expression by "1" or $\frac{2x}{4x} \cdot \frac{4x}{2x}$.
6. 1. Let $\square = \sqrt{x-1}$. As $x \rightarrow 1$ we have $\square \rightarrow 0$ and $\frac{\sin \square}{\square} \rightarrow 1$.

2.2 Exercise Set 8 (page 57)

1. 0. Continuity of the quotient at $x = 2$.
2. 0. Note that $\cos 0 = 1$.
3. $\frac{1}{6}$. Factor the denominator.
4. -1. Rewrite the secant function as the reciprocal of the cosine function and use the trig. identity $\cos \square = -\sin(\square - \frac{\pi}{2})$.
5. -2. Factor out the 2 from the numerator and then use the idea of Exercise 4, above.
6. 0. The function is continuous at $x = 2$, and $\sin 2\pi = 0$.
7. 3. Multiply and divide the expression by 3 and rewrite it in a more familiar form.
8. $-\infty$. Use your calculator for a test of this limit. The numerator approaches -1 and the denominator approaches 0 through positive values. So the quotient must approach the stated value.
9. $+\infty$. The denominator approaches 0 through negative values, while the numerator approaches -1 . Thus, the quotient approaches the stated value.
10. 0. The function is continuous at $x = 0$.
11. $x = \pi$. The denominator is 0 and the numerator isn't.
12. $x = 0$. Since $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \neq f(0)$, we know that f cannot be continuous there, by definition.
13. None. This is because f is a polynomial and so it is continuous everywhere.
14. $x = \pm 1$, the roots of the denominator.
15. $x = \pm 2$. For $x = 2$ the numerator is of the form $0/0$ and $f(2)$ is not defined at all, so the function is not continuous here (by definition). Next, the denominator is zero for $x = -2$, but the numerator isn't zero here. So the function is of the form $-4/0 = -\infty$ and so once again, f is not continuous here because its value here is $-\infty$.
16. $\frac{3}{2}$. Use the Hint. We know from the Hint (with $A = x, B = 2x$) that $\cos x - \cos 2x = -2 \sin(3x/2) \sin(-x/2)$. Then

$$\begin{aligned} \frac{\cos x - \cos 2x}{x^2} &= -\frac{2 \sin(3x/2)}{x} \frac{\sin(-x/2)}{x}, \\ &= -\frac{2 \left(\frac{3}{2}\right) \sin(3x/2) \left(\frac{-1}{2}\right) \sin(-x/2)}{\left(\frac{3x}{2}\right) \left(\frac{-x}{2}\right)}, \\ &= -\left(-\frac{3}{2}\right) \frac{\sin(3x/2)}{\left(\frac{3x}{2}\right)} \frac{\sin(-x/2)}{\left(\frac{-x}{2}\right)}. \end{aligned}$$

Now use the hint with $\square = \frac{3x}{2}$ and $\square = -\frac{x}{2}$, as $x \rightarrow 0$. Both limits approach 1 and so their product approaches $3/2$.

17. 0. Use the Hint. We can rewrite the expression as

$$\begin{aligned} \frac{\tan x - \sin x}{x^2} &= \frac{\tan x (1 - \cos x)}{x^2}, \\ &= \frac{\tan x}{x} \left(\frac{1 - \cos x}{x} \right), \\ &= \left(\frac{\sin x}{x} \right) \left(\frac{1}{\cos x} \right) \left(\frac{1 - \cos x}{x} \right). \end{aligned}$$

As $x \rightarrow 0$, the first term approaches 1, the second term approaches 1, while the last term approaches 0, by Table 2.12. So, their product approaches 0.

18. $+\infty$. The limit exists and is equal to $+\infty$.
19. $a = -\frac{\pi}{2}$, $b = -\pi a = \frac{\pi^2}{2}$.
20. 1. Rationalize the denominator. Note that the function is continuous at $x = 0$.

2.3

2.4 Exercise Set 9 (page 65)

0. This is a limit as $x \rightarrow \infty$, not as $x \rightarrow 0$.
0. Divide the numerator and denominator by x and simplify.
1. Divide the numerator and denominator by x and simplify.
- $\frac{1}{2}$. Rationalize the numerator first, factor out $\sqrt{x}$ out of the quotient, simplify and then take the limit.
0. Use the Sandwich Theorem.
- The graph of the function $\sin x$ isn't going anywhere definite; it just keeps oscillating between 1 and -1 forever and so it cannot have a limit. This is characteristic of periodic functions in general.

2.5

2.6 Chapter Exercises (page 76)

- Since f is a polynomial, it is continuous everywhere and so also at $x = 1$.
- g is the product of two continuous functions (continuous at 0) and so it is itself continuous at $t = 0$.
- h is the sum of three continuous functions and so it is continuous at $z = 0$.
- f is a constant multiple of a continuous function and so it is continuous too (at $x = \pi$).
- The graph of f is 'V'-shaped at $x = -1$ but it is continuous there nevertheless.
- The limit is $3 - 2 + 1 = 2$ since f is continuous at $x = 1$.
- The limit is $0 \cdot 1 = 0$ since g is continuous at $t = 0$.
- The limit is $0 + (2)(0) - \cos 2 = -\cos 2 \approx 0.416$ since h is continuous at $z = 0$.
- The limit is $2 \cdot \cos \pi = (2)(-1) = -2$ since f is continuous at $x = \pi$.
- The limit is $|-1 + 1| = |0| = 0$ since f is continuous at $x = -1$.
0. The function is continuous at $t = 2$.
- $\frac{1}{8}$. Factor the denominator first, then take the limit.
- $+\infty$. Use extended real numbers.
1. Remove the absolute value first.
- $+\infty$.
- i) 1; ii) 1; iii) 0; iv) 1; v) Since (i) and (ii) are equal we see that g is continuous at $x = 0$ as $g(0) = 1$, by definition. Since the left and right limits at $x = 1$ are different (by (iii) and (iv)), we see that g is not continuous at $x = 1$ and so the graph has a break there.
- The limit from the left is 2 and the limit from the right is 1. So the limit cannot exist.
- $|-2| = 2$. The absolute value function is continuous there.
- $0/(-1) = 0$. The quotient is continuous at $x = -2$.
0. The function is continuous at that point.
- Does not exist. The left-hand limit as $x \rightarrow 1$ is 1, but the right-hand limit as $x \rightarrow 1$ is $|1 - 1| = 0$, so the limit cannot exist.
- $x = 0$. This is because the left- and right-hand limits there are not equal. For example, the left limit is -2 while the right-limit is 0. Use the definition of the absolute value, OK?
- $x = 0$. The left-hand limit is -1 while the right-hand limit is 1.
- None. The denominator is $x^3 - 1 = (x - 1)(x^2 + x + 1)$ with $x = 1$ as its only real root. Why? By "completing the square", we have $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4} > 0$ and hence $x^2 + x + 1$ does not have real roots. The only possible point of discontinuity is $x = 1$. But both the left and right limits at $x = 1$ are $-1/3$, which is also the value of f at $x = 1$. Hence f is continuous at $x = 1$ and so everywhere.
- $x = 0$. Even though the values of the left and right limits here are 'close' they are not equal, since $-0.99 \neq -1$.
- $x = 0$. The left and right-hand limits there are both equal to $+\infty$, so f cannot be continuous there.
- $\frac{a}{b}$. Multiply the expression by $1 = \frac{bx}{ax} \frac{ax}{bx}$, simplify. Then take the limit.
- $+\infty$. This limit actually exists in the extended reals. Observe that the numerator approaches 1 regardless of the direction (left or right) because it is continuous there, while the denominator approaches 0 regardless of the direction, too, and for the same reason. The quotient must then approach $1/0 = +\infty$ in the extended reals.
0. Break up the expression into three parts, one involving only the term x , another with the term $\sin x/x$ and the remaining one with the term $x/\sin 2x$. The first term approaches 0, the next term approaches 1 while the last term approaches $1/2$, by Exercise 27, with $a = 2$, $b = 1$ and Table 2.4, (d). So, the product of these three limits must be equal to zero.
1. Let $\square = \sqrt{3 - x}$. As $x \rightarrow 3^-$, we have $\square \rightarrow 0^+$ and so $\frac{\sin \square}{\square} \rightarrow 1$.
- $\frac{b}{a}$. See Exercise 27 in this Section: Multiply the expression by ax/ax , re-arrange terms and evaluate.
0. This limit actually exists. This is because the numerator oscillates between the values of ± 1 as $x \rightarrow \infty$, while the denominator approaches ∞ . The quotient must then approach (something)/ $\infty = 0$ in the extended reals.
- Does not exist. There are many reasons that can be given for this answer. The easiest is found by studying its graph and seeing that it's not 'going anywhere'. You can also see that this function is equal to zero infinitely often as $x \rightarrow -\infty$ (at the zeros or roots of the sine function). But then it also becomes as large as you want it to when x is chosen to be anyone of the values which makes $\sin x = -1$. So, it oscillates like crazy as $x \rightarrow -\infty$, and so its limit doesn't exist.

34. 0. Hard to believe? Rationalize the numerator by multiplying and dividing by the expression $\sqrt{x^2 + 1} + x$. The numerator will look like $(x^2 + 1) - x^2 = 1$, while the denominator looks like $\sqrt{x^2 + 1} + x$. So, as $x \rightarrow +\infty$, the numerator stays at 1 while the denominator tends to ∞ . In the end you should get something like $1/\infty = 0$ in the extended reals.
35. Set $a = -5$, $b = 1$ in Bolzano's Theorem and set your calculator to radians. Now, calculate the values of $f(-5)$, $f(1)$. You should find something like $f(-5) = -4.511$ and $f(1) = 1.382$ so that their product $f(-5) \cdot f(1) < 0$. Since the function is a product of continuous functions, Bolzano's Theorem guarantees that $f(x) = 0$ somewhere inside the interval $[-5, 1]$. So, there is a root there.
36. Set $a = -3$, $b = 0$. Now, calculate the values of $f(-3)$, $f(0)$. Then $f(-3) = -9$ and $f(0) = 2$ so that their product $f(-3) \cdot f(0) < 0$. Since the function is a polynomial, it is a continuous function, so Bolzano's Theorem guarantees that $f(x) = 0$ somewhere inside the interval $[-3, 0]$. So, there is a root there.
37. Let $f(x) = x^2 - \sin x$. Write $f(a) \cdot f(b)$. Now let a, b with $a < b$ be any two numbers whatsoever. Check that your calculator is in radian mode, and calculate the values $f(a) \cdot f(b)$ like crazy! As soon as you find values of a, b where $f(a) \cdot f(b) < 0$, then STOP. You have an interval $[a, b]$ where $f(x) = 0$ somewhere inside, by Bolzano's Theorem. For example, $f(-0.3) \cdot f(2.5) = 2.179$, $f(0.3) \cdot f(1.5) = -0.257 < 0$. STOP. So we know there is a root in the interval $[0.3, 1.5]$.

Solutions

3.1 Exercise Set 10 (page 88)

1. 4. Use the binomial theorem to expand and simplify.
2. -1 . Note that $f(x) = -x$ for $x < 0$ and so for $x = -1$, too.
3. $+\infty$. The quotient is equal to $1/h^2 \rightarrow +\infty$ as $h \rightarrow 0$.
4. a) $+\infty$, b) 1. Note that $f(1+h) = 1+h$ for $h < 0$ and $f(1+h) = 2+h$ for $h > 0$.
5. $\frac{1}{2\sqrt{2}} \approx 0.3536$.
6. 4. Use the binomial theorem to expand and simplify.
7. 3.
8. -4 .
9. 6.
10. 1. Note that $f(x) = x$ near $x = 1$.
11. 0. Note that $f(x) = x^2$ for $x > 0$ and $f(x) = -x^2$ for $x < 0$.
12. 0 for all $x \neq 0$, and the slope does not exist when $x = 0$.
13. The derivative does not exist since f is not continuous there.
14. The derivative does not exist because $f(x)$ is undefined for any x slightly less than -1 . However, its right-derivative at $x = -1$ is $+\infty$.
15. Yes. The absolute value can be removed so that $f(x) = x^2$. It turns out that $f'(0) = 0$.
16. $f'(1) = -\frac{1}{2}$.
17. $f'(1) = -2$.
18. a) $f'(1)$ does not exist since f is not continuous at $x = 1$. Alternately note that the left- and right-derivatives at $x = 1$ are unequal: $f'_+(1) = 1$, $f'_-(1) = \infty$.
b) No. In this case f is continuous at $x = 2$ but the one-sided derivatives are unequal: $f'_+(2) = -4$, $f'_-(2) = 1$.
c) Since $2 < \frac{5}{2} < 3$, we see that $f'(\frac{5}{2}) = -5$.

3.2 Exercise Set 11 (page 93)

1. $\frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$.
2. $-2t^{-3} = -\frac{2}{t^3}$.
3. 0.
4. $\frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$.
5. $\frac{t^{-4/5}}{5} = \frac{1}{5\sqrt[5]{t^4}}$.
6. 0.
7. $4t^3$.
8. $-3x^{-4} = -\frac{3}{x^4}$.
9. $-x^{-2} = -\frac{1}{x^2}$.
10. $\pi x^{\pi-1}$.
11. $2t$. Use the Power Difference Rules
12. $6x + 2$. Use the Power, Sum and Difference Rules
13. $1(t^2 + 4) + 2t(t - 1)$. Use the Product Rule
14. $f(x) = 3x^{5/2} + x^{1/2}$ so $f'(x) = \frac{15}{2}x^{3/2} + \frac{1}{2\sqrt{x}}$. Use the Power Rule

15. $\frac{(2x+1)(0.5)x^{-0.5} - 2x^{0.5}}{(2x+1)^2}$. Use the Quotient Rule
16. $\frac{(x+1) - (x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$. Use the Quotient Rule
17. $\frac{(x^2+x-1)(3x^2) - (x^3-1)(2x+1)}{(x^2+x-1)^2}$. Use the Quotient Rule
18. $\frac{(\sqrt{x}+3x^{3/4})((2/3)x^{-1/3}) - (x^{2/3})((1/2)x^{-1/2} + (9/4)x^{-1/4})}{(\sqrt{x}+3x^{3/4})^2}$. Use the Quotient Rule

3.3 Exercise Set 12 (page 105)

1. 0.
2. 3.
3. $\frac{2}{3}$.
4. $\frac{3}{2}\sqrt{x-4}$.
5. $-\frac{5}{2}x^{-7/2}$.
6. $\frac{1}{3}(2t+1)(t^2+t-2)^{-2/3}$.
7. $\frac{d^2f}{dx^2} = 6$.
8. $4x(x+1)^3 + (x+1)^4 = (x+1)^3(5x+1)$.
9. $-\frac{1}{2}$.
10. $(t+2)^2 + 2(t-1)(t+2) = 3t^2 + 6t$.
11. $32(\frac{4}{3}x^2 - x)(x-1)^{-1/3} = \frac{32}{3}(4x^2 - 3x)(x-1)^{-1/3}$.
12. $210(2x+3)^{104}$.
13. $\frac{1}{2}\frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$.
14. $3x^2 - 6x + 3$, or $3(x-1)^2$: Both are identical.
15. $-\frac{1}{x^2} + \frac{x}{\sqrt{x^2-1}}$.
16. $\frac{1}{4} \frac{1+3\sqrt{x}}{x\sqrt{x}(1+\sqrt{x})^3} = \frac{1+3\sqrt{x}}{4x^{3/2}(1+\sqrt{x})^3}$.
17. -10. Note that $f''(x) = 6x - 10$.
18. $3.077(x+0.5)^{-3.324}$.
19. Use the Chain Rule; For instance, let $\square = x^2$, from which we get $\frac{d}{dx}f(\square) = f'(\square)D\square$. Put x^2 in the Box, note that $D\square = 2x$ and simplify. You'll find $\frac{d}{dx}f(x^2) = 2x f'(x^2)$.
20. Use another form of the Chain Rule: Putting $u = g(x)$ and $w = \sqrt[3]{u} \equiv u^{1/3}$, we have $w = \sqrt[3]{g(x)}$ and

$$\frac{d}{dx} \sqrt[3]{g(x)} = \frac{dw}{dx} = \frac{dw}{du} \cdot \frac{du}{dx} = \frac{1}{3}u^{-2/3} \cdot g'(x) = \frac{g'(x)}{3\sqrt[3]{g(x)^2}}.$$
21. Let $y(x) = f(x^2)$. By the Chain Rule, we have $y'(x) = f'(x^2) \cdot 2x = 2xf'(x^2)$. Replacing x by x^2 in $f'(x) + f(x) = 0$, we have $f'(x^2) + f(x^2) = 0$, or $f'(x^2) = -f(x^2) = -y(x)$. So $y'(x) = 2xf(x^2)$ can be rewritten as $y'(x) = -2xy(x)$, that is, $y'(x) + 2xy(x) = 0$.
22. Use the Chain Rule once again on both sides of $f(F(x)) = x$. We find $f'(F(x))F'(x) = 1$, which gives $F'(x) = \frac{1}{f'(F(x))}$.
23. Use another form of the Chain Rule: $\frac{dy}{du} = \frac{dy}{dt} \cdot \frac{dt}{du} = 3t^2 \cdot \frac{1}{2\sqrt{u}}$. At $u = 9$ we have $t = \sqrt{9} + 6 = 9$ and

$$\frac{dy}{du} = 3 \cdot 9^2 \cdot \frac{1}{2\sqrt{9}} = \frac{81}{2}.$$
24. $y = 32(x-2) + 1$ (or $32x - y - 63 = 0$).
25. Just use the Chain Rule. You don't even have to know f , g explicitly, just their values: So, $y'(2) = f'(g(2)) \cdot g'(2) = f'(0) \cdot 1 = 1$.
26. $\left(1 - \frac{2}{(3t-2\sqrt{t})^2}\right) \left(3 - \frac{1}{\sqrt{t}}\right)$. Use the Chain Rule in the form: $\frac{dy}{dt} = \frac{dy}{dr} \frac{dr}{dt}$. But $\frac{dy}{dr} \frac{dr}{dt} = (1 - 2r^{-2})(3 - t^{-1/2})$. Now set $r = 3t - 2\sqrt{t}$.
27. $f'(9) = \frac{7}{24\sqrt{3}}$, since $f'(x) = \frac{1}{2} \frac{1+\frac{1}{2\sqrt{x}}}{\sqrt{x+\sqrt{x}}}$. On the other hand, since $\sqrt{t^2} = |t|$, we see that $\frac{df}{dt} = \frac{2t+1}{2\sqrt{t^2+t}}$, if $t \geq 0$ and $\frac{df}{dt} = \frac{2t-1}{2\sqrt{t^2-t}}$ if $t < 0$.
28. Let $y = |x| = \sqrt{x^2}$. Now, set $g(u) = \sqrt{u}$, $u = u(x) = x^2$. Then, $y = g(u(x))$. Using the Chain Rule we get $y'(x) = g'(u(x)) \cdot u'(x) = \frac{1}{2\sqrt{u}} \cdot (2x) = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}$, whenever $x \neq 0$.
29. By definition, $\lim_{h \rightarrow 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$. Look at the limit

$$\lim_{h \rightarrow 0} [f(x_0+h) - f(x_0)] = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} \cdot h = f'(x_0) \cdot 0 = 0.$$

We have shown that $\lim_{h \rightarrow 0} f(x_0+h) - f(x_0) = 0$, which forces

$$\lim_{h \rightarrow 0} f(x_0+h) = f(x_0).$$

This, however, is another way of writing

$$\lim_{x \rightarrow x_0} f(x) = f(x_0).$$

Hence f is continuous at x_0 (by an equivalent definition of continuity).

3.4 Exercise Set 13 (page 112)

1. -2 . Implicit differentiation gives $(2x + y) + y'(x)(x + 2y) = 0$. Now set $x = 1$, $y = 0$ and solve for $y'(1)$.
2. $\frac{dy}{dx} = \frac{3x^2 - 2y^2}{4xy - 4y^3}$. $\frac{dx}{dy} = \frac{4xy - 4y^3}{3x^2 - 2y^2}$.
3. $-\frac{1}{129}$. Implicit differentiation gives an expression of the form $\frac{1}{2}(x + y)^{-1/2}(1 + y') + xy' + y = 0$. Now solve for y' after setting $x = 16$ and $y = 0$.
4. $\frac{1}{2y}$. Implicit differentiation gives an expression of the form $1 - 2yy'(x) = 0$. Now solve for y' .
5. 0. Implicit differentiation gives an expression of the form $2x + 2yy'(x) = 0$. Now set $x = 0$, $y = 3$. You see that $y'(0) = 0$.
6. $y + 1 = \frac{1}{2}(x + 1)$. Note that $y'(x) = \frac{x}{2y}$.
7. $y - 1 = \frac{1}{3}(x - 1)$, or $x - 3y + 2 = 0$. Note that $y'(x) = \frac{2 - y}{x + 2y}$.
8. $y = \frac{5}{2}(x - 4)$, or $5x - 2y - 20 = 0$. Note that $y'(x) = \frac{x + 1}{2 - y}$.
9. $y = -(x - 1) - 1$, or $x + y = 0$. Note that $y'(x) = -\frac{y(2x + y)}{x(x + 2y)}$.

3.5 Exercise Set 14 (page 119)

1. $\frac{\cos 1}{2}$. The derivative is given by $\frac{\cos \sqrt{x}}{2\sqrt{x}}$.
2. $2 \sec(2x) \cdot \tan(2x) \cdot \sin x + \sec(2x) \cdot \cos x$.
3. 1. The derivative is given by $\cos^2 x - \sin^2 x$. Now evaluate this at $x = 0$.
4. $\frac{1}{1 - \sin x}$. The derivative is given by $\frac{1 + \sin x}{\cos^2 x}$. Now use an identity in the denominator and factor.
5. $\frac{1}{2}$. Note that $y'(t) = \frac{\cos t}{2\sqrt{1 + \sin t}}$. Now set $t = 0$.
6. $-2x \sin(x^2) \cos(\cos(x^2))$.
7. $2x \cos 3x - 3x^2 \sin 3x$.
8. $\frac{2}{3}x^{-1/3} \tan(x^{1/3}) + \frac{1}{3} \sec^2(x^{1/3})$.
9. $-(1 + \cos x) \csc^2(2 + x + \sin x)$. Don't forget the minus sign here!
10. $-3 \cot 3x \csc 3x$. The original function is the same as $\csc 3x$.
11. 1. In this case, the derivative is given by $\frac{-\sin x + x \cos x + \cos x}{\cos^2 x - 1}$. Remember that $\cos(\pi/2) = 0$, $\sin(\pi/2) = 1$.
12. $4x \cos(2x^2)$.
13. 1. In this case, the derivative is given by $2 \sin x \cos x$. When $x = \frac{\pi}{4}$ we know that $\cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$.
14. $-3 \csc^2(3x - 2)$.
15. $2 \csc x - (2x + 3) \csc x \cot x$.
16. $-(\sin x + x \cos x) \sin(x \sin x)$.
17. $\frac{1}{2\sqrt{x}} \sec \sqrt{x} + \frac{1}{2} \sec \sqrt{x} \cdot \tan \sqrt{x}$.
18. 0, except when $x^2 = 2 + 2n\pi$, where $n \geq 0$ is an integer. This is because $\csc \square \cdot \sin \square = 1$ for any symbol, $\square$, by definition, whenever the cosecant is defined.
19. $-\sin 2(x - 6) - 2 \csc 2x \cot 2x$. (Use the identity $2 \sin u \cos u = \sin 2u$ to simplify.)
20. $4 \sec^2 2x \tan 2x$. The given function is equal to $\sec^2(2x)$.
21. Notice that, for $x \neq 0$, $y(x) = \sin x / \tan x = \sin x \cdot \cot x = \cos x$. On the other hand, at $x = 0$, we have $y(0) = 1$, which coincides with the value of the cosine function at $x = 0$. Therefore, $y(x) = \cos x$ for all x . Now all three parts are clear.

3.6 Exercise Set 15 (page 127)

1. $y(x) = 3x - 2$ is continuous on $[0, 2]$ and $y(0) = -2 < 0$, $y(2) = 4 > 0$.
2. $y(x) = x^2 - 1$ is continuous, $y(-2) = 3 > 0$ and $y(0) = -1 < 0$.
3. $y(x) = 2x^2 - 3x - 2$ is continuous, $y(0) = -2 < 0$ and $y(3) = 7 > 0$.
4. $y(x) = \sin x + \cos x$ is continuous on $[0, \pi]$, $y(0) = 1 > 0$ and $y(\pi) = -1 < 0$.
5. $y(\pi) = -\pi < 0$. But $y(0) = 0$; so 0 is already a root. Try another point instead of 0, say $\frac{\pi}{2}$: $y(\frac{\pi}{2}) = \frac{\pi}{2} \cdot 0 + 1 = 1 > 0$. So there is a root in $[\frac{\pi}{2}, \pi]$ and hence in $[0, \pi]$ (besides the root 0.)
6. (This is hard.) In the proof we use several times the following basic fact in differential calculus: if the derivative of a function is identically zero, then this function must be a constant. Let's begin by applying this fact to the function y'' : its derivative $y''' = 0$ implies that y'' is a constant, say $y'' = a$. Let $u = y' - ax$. Then $u' = y'' - a = 0$ and hence u is a constant, say $u = b$, that is, $y' - ax = b$. Let $v = y - \frac{a}{2}x^2 - bx$. Then $v' = y' - \frac{a}{2} \cdot 2x - b = 0$ and hence v is a constant, say $v = c$. Thus $y - \frac{a}{2}x^2 - bx = c$, or $y = \frac{a}{2}x^2 + bx + c$. We can finish the proof by setting $A = \frac{a}{2}$, $B = b$ and $C = c$.

7. From the assumption that $\frac{dy}{dx} + y(x)^4 + 2 = 0$, we know that $\frac{dy}{dx}$ exists on (a, b) , and $y(x)$ is continuous on $[a, b]$. Assume the contrary that there are two zeros in $[a, b]$, say x_1, x_2 . Using the **Mean Value Theorem**, we see that there exists some c between x_1 and x_2 (a fortiori, between a and b , such that $\frac{dy}{dx}(c) = 0$. Thus $y(c)^4 + 2 = 0$. **Impossible!** So there cannot be two zeros for $y(x)$.
8. Consider the function $y(x) = x - \sin x$. By the Mean Value Theorem we see that, for each $x > 0$, there exists some c between 0 and x such that $y(x) - y(0) = y'(c)(x - 0)$, or $x - \sin x = y'(c)x$; (notice that $y(0) = 0$.) Now $y'(x) = 1 - \cos x$, which is always ≥ 0 . So, from $x > 0$ and $y'(c) \geq 0$ we see that $y'(c)x \geq 0$. Thus $x - \sin x \geq 0$, or $\sin x \leq x$.
9. Use Rolle's Theorem on $[0, \pi]$ applied to the function $f(x) = \sin x$. Since $f(0) = f(\pi) = 0$, we are guaranteed that there exists a point c inside the interval $(0, \pi)$ such that $f'(c) = \cos c = 0$. This point c is the root we seek.
10. Note that $(\sin x)' = \cos x \leq 1$. For any x in $[0, \frac{\pi}{2}]$, the function $\sin x$ satisfies all the conditions of the **Mean Value Theorem** on $[x, \frac{\pi}{2}]$. So, there exists c in $(x, \frac{\pi}{2})$ such that

$$\frac{\sin \frac{\pi}{2} - \sin x}{\frac{\pi}{2} - x} = \cos c \leq 1.$$

This statement is equivalent to the stated inequality, since $\sin(\pi/2) = 1$.

11. (a) $\frac{f(2)-f(0)}{2-0} = \frac{5-(-1)}{2} = 3$ and $f'(c) = 2c + 1 = 3$ give $c = 1$.
 (b) $\frac{g(1)-g(0)}{1-0} = \frac{4-3}{1} = 1$ and $g'(c) = 2c = 1$ give $c = \frac{1}{2}$.
12. Let $x(t)$ denote the distance travelled (in meters) by the electron in time t . We assume that $x(0) = 0$ and we are given that $x(0.3 \times 10^{-8}) = 1$. Now apply the MVT to the time interval $[0, 0.3 \times 10^{-8}]$. Then,

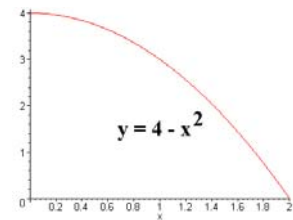
$$\frac{x(0.3 \times 10^{-8}) - x(0)}{0.3 \times 10^{-8} - 0} = x'(c),$$

for some time $t = c$ in between. But this means that the speed of the electron at this time $t = c$ is $\frac{1}{0.3 \times 10^{-8}} = 3.3 \times 10^8$ m/sec, which is greater than 2.19×10^8 m/sec, or the speed of light in that medium!

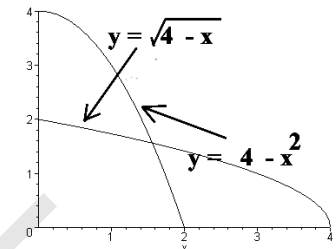
3.7 Exercise Set 16 (page 134)

You can use your Plotter program to sketch the graphs.

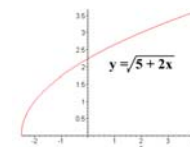
- $f(x) = 4 - x^2$, $0 \leq x \leq 2$. $f^{-1}(x) = \sqrt{4-x}$, $0 \leq x \leq 4$. See the margin.
- $g(x) = (x-1)^{-1}$, $1 < x < \infty$. $g^{-1}(x) = x^{-1} + 1$, $0 < x < \infty$.
- $f(x) = 2 - x^3$, $-\infty < x < \infty$. $f^{-1}(x) = \sqrt[3]{2-x}$, $-\infty < x < \infty$.
- $f(x) = \sqrt{5+2x}$, $-\frac{5}{2} \leq x < \infty$. $f^{-1}(x) = \frac{1}{2}(x^2 - 5)$, $0 \leq x < \infty$.
- $f(y) = (2+y)^{1/3}$, $-2 < y < \infty$. $f^{-1}(y) = y^3 - 2$, $0 < y < \infty$.
- (i) $F(0) = 2$, since $f(2) = 0$ forces $2 = F(f(2)) = F(0)$.
 (ii) $f(-1) = 6$, since $F(6) = -1$ means that $6 = f(F(6)) = f(-1)$.
 (iii) Indeed, if $f(x) = 0$ then $x = F(f(x)) = F(0) = 2$, and so this is the only possibility.
 (iv) $y = 8$, because $f(-2) = 8$ means (by definition) that $F(8) = -2$ so $y = 8$ is a solution. No, there are no other solutions since if we set $F(y) = -2$ then $y = f(F(y)) = f(-2) = 8$, so that $y = 8$ is the only such solution.
 (v) No. The reasoning is the same as the preceding exercise. Given that $f(-1) = 6$, the solution x of $f(x) = 6$ must satisfy $x = F(f(x)) = F(6) = -1$, by definition of the inverse function, F .
- We know that $F'(-1) = \frac{1}{f'(F(-1))} = \frac{1}{f'(-2.1)} = \frac{1}{4}$.
- $F(x) = x$, $\text{Dom}(f) = \text{Ran}(F) = (-\infty, +\infty) = \{x : -\infty < x < +\infty\}$, and $\text{Dom}(F) = \text{Ran}(f) = (-\infty, +\infty)$ too.
- $F(x) = \frac{1}{x}$, $\text{Dom}(f) = \text{Ran}(F) = \{x : x \neq 0\}$, and $\text{Dom}(F) = \text{Ran}(f) = \{x : x \neq 0\}$.
- $F(x) = \sqrt[3]{x}$, $\text{Dom}(f) = \text{Ran}(F) = \{x : -\infty < x < +\infty\} = \text{Dom}(F) = \text{Ran}(f)$.
- $F(t) = \frac{t-4}{7}$, $\text{Dom}(f) = \text{Ran}(F) = \{x : 0 \leq x \leq 1\}$ while $\text{Dom}(F) = \text{Ran}(f) = \{x : -4 \leq x \leq 11\}$.
- $G(x) = \frac{x^2-1}{2}$, $\text{Dom}(g) = \text{Ran}(G) = \{x : -\frac{1}{2} \leq x < +\infty\}$ while $\text{Dom}(G) = \text{Ran}(g) = \{x : 0 \leq x < \infty\}$.
- Note that g is one-to-one on this domain. Its inverse is given by $G(t)$ where $G(t) = \frac{\sqrt{1-t^2}}{2}$, $\text{Dom}(g) = \text{Ran}(G) = \{t : 0 \leq t \leq \frac{1}{2}\}$ while $\text{Dom}(G) = \text{Ran}(g) = \{t : 0 \leq t \leq 1\}$.
- This f is also one-to-one on its domain. Its inverse is given by $F(x)$ where $F(x) = \frac{3x-2}{2x+3}$, $\text{Dom}(f) = \text{Ran}(F) = \{x : x \neq -\frac{3}{2}\}$ while $\text{Dom}(F) = \text{Ran}(f) = \{x : x \neq -\frac{3}{2}\}$.
- This g is one-to-one if $y \geq -\frac{1}{2}$ and so it has an inverse, G . Its form is $G(y)$ where $G(y) = \frac{-1 + \sqrt{1+4y}}{2}$, $\text{Dom}(g) = \text{Ran}(G) = \{y : -\frac{1}{2} \leq y < +\infty\}$ while $\text{Dom}(G) = \text{Ran}(g) = \{y : -\frac{1}{4} \leq y < +\infty\}$.



Set 16, # 1:
 $f(x) = 4 - x^2$, x in $[0, 2]$.



Set 16, # 1:
 $f^{-1}(x) = \sqrt{4-x}$, x in $[0, 4]$.



3.8 Exercise Set 17 (page 139)

- $\sin(\arccos(0.5)) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$.
- $\cos(\arcsin(0)) = \cos 0 = 1$.
- $\sec(\sin^{-1}(\frac{1}{2})) = \sec(\frac{\pi}{6}) = \frac{2}{\sqrt{3}}$.
- $-\sqrt{5}$. (This is hard!) Let $\tan^{-1}(-\frac{1}{2}) = \alpha$. Then $-\frac{\pi}{2} < \alpha < 0$; (see the graph of the Arctangent function in this Section.) Also, $\tan \alpha = -\frac{1}{2}$. Thus

$$\sec^2 \alpha = 1 + \tan^2 \alpha = 1 + (-1/2)^2 = \frac{5}{4}.$$

But $-\frac{\pi}{2} < \alpha < 0$ implies that $\sec \alpha = 1/\cos \alpha > 0$. Therefore

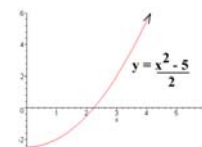
$$\csc \alpha = \frac{1}{\sin \alpha} = \frac{\cos \alpha}{\sin \alpha} \cdot \frac{1}{\cos \alpha} = \frac{1}{\tan \alpha} \cdot \sec \alpha = (-2) \frac{\sqrt{5}}{2} = -\sqrt{5}.$$

- $\sec(\sin^{-1}(\frac{\sqrt{3}}{2})) = \sec \frac{\pi}{3} = 2$.
- $\arcsin(\tan(-\frac{\pi}{4})) = \arcsin(-1) = -\frac{\pi}{2}$.

3.9 Exercise Set 18 (page 142)

- $\frac{d}{dx} \arcsin(x^2) = \frac{2x}{\sqrt{1-x^4}}$ which is 0 at $x = 0$.
- $2x \arccos x - \frac{x^2}{\sqrt{1-x^2}}$.

Set 16 # 4:
 $f(x) = \sqrt{5+2x}$,
 x in $[-5/2, \infty)$.



Set 16 # 4:
 $f^{-1}(x) = \frac{x^2-5}{2}$, x in $[0, \infty)$.

3. $\frac{1}{2(1+x)\sqrt{x}}$.
4. $-\frac{\sin x}{|\sin x|}$. Remember the identity?
5. $\frac{\sin x - \sin^{-1} x \cdot \cos x \sqrt{1-x^2}}{\sin^2 x \cdot \sqrt{1-x^2}}$.
6. $\frac{1}{2|x|\sqrt{(x^2-1)\sec^{-1} x}}$.
7. 2, because $\cos(2\text{Arccsin } x) \cdot \frac{2}{\sqrt{1-x^2}}$ which is 2 at $x = 0$.
8. $-\frac{16x}{\sqrt{1-16x^2}}$.
9. $-\frac{1}{(\text{Arctan } x)^2(1+x^2)}$.
10. $3x^2 \text{Arc sec}(x^3) + \frac{3|x|^3}{x\sqrt{x^6-1}}$.

3.10 Special Exercise Set (page 149)

1. $3200 \pi \approx 10,053.1 \text{ mm}^3/\text{min}$. First, convert all centimeters to millimeters. Then use the volume formulae $V = \frac{4}{3}\pi r^3$ and $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. Set $r = 20$ and $r' = 2$.
2. 0.5 m/sec . Let x be the length of one of its walls. Use the volume formulae $V = x^3$ and $\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$. Solve for dx/dt and set $x = 2$ and $V' = 6$.
3. $16.1 \text{ cm}^2/\text{sec}$. Let x be the length of one of its sides and y be the other. Use the area formula $A = xy$ and $x = \tau y$. Then $A = \tau y^2$ and $dA/dt = 2\tau y \, dy/dt$. Finally, set $y = 6.2$, $dy/dt = 2.1$.
4. 379.32 km/hr . Use Pythagoras with D , the distance between them, as the hypotenuse. Let x, y be the positions of the cars at a given time. Now use the fact $D^2 = x^2 + y^2$ from which we get

$$\frac{dD}{dt} = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{D}.$$

Set $x = 4$, $dx/dt = 281$ and $y = 6.7$, $dy/dt = 274$. From this derive that $D = 7.8$ and then the answer.

5. $\frac{50\pi}{67} \approx 2.34 \text{ cm/sec}$. Let $A = \pi r^2$ be the area and $C = 2\pi r$ its circumference. Relate A to C to find

$$A = \frac{C^2}{4\pi}.$$

Differentiate this formula and solve for dC/dt . Set $C = 67$ and $A' = 25$.

6. $549,988,154.1 \text{ km}^3/\text{hr}$. Let $V = \frac{4}{3}\pi a^2 c$. Differentiate this with respect to t and note that $dc/dt = 0$ by hypothesis. This gives

$$\frac{dV}{dt} = \frac{8\pi a}{3} \frac{da}{dt}.$$

Set $a = 50,500$ and $da/dt = 1300$.

7. Use the Cosine Law to find their mutual distance, D . Note that $\cos(2\pi/3) = -1/2$, so that $D = 1351.04 \text{ mi}$. Now, let A, B denote the distances of each one of the planes from the airport. Since the planes are approaching the airport it follows that $A' = dA/dt < 0$ and $B' = dB/dt < 0$. Since, by the Cosine Law, $D^2 = A^2 + B^2 - 2AB \cos(2\pi/3)$ it follows by implicit differentiation that

$$D' = A A' + B B' - (A' B + A B') \cos(2\pi/3).$$

Finally, set $A' = -790$, $B' = -770$, $A = 30$, $B = 46$, $D = 1351.04$ to get the answer.

8. 3.5 dollars/wk. Find dC/dt using implicit differentiation and set $x = 100$, $dx/dt = 10$.
9. **Project:** -254.78 km/sec . Let x be the length of one side. Show that the area of the triangle is given by $A = \frac{\sqrt{3}}{4} x^2$. Find A' and solve for dx/dt . Next, find x in terms of A and derive the formula

$$\frac{dx}{dt} = \frac{1}{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{A}} \frac{dA}{dt}.$$

Finally, set $A = 500,000$ and $A' = -237,100$, (note the minus sign).

3.11 Exercise Set 19 (page 160)

1. 0.73909
2. 1.5193. Newton's Method will require 4 iterations.
3. 1.259. This will require 3 iterations.
4. The "answer" may read $-0.20005 < \text{answer} < -0.19995$. This root is NOT equal to -0.2 !
5. $-1.57079 \approx -\frac{\pi}{2}$. But you only see this approximation after 7 iterations! In fact, note that $-\frac{\pi}{2}$ is the root in this interval as one can check directly.
6. 1.287 after 3 iterations if we start with $x_0 = 1.5$.
7. 2.05 (actually, 2.055255...)

3.12 Exercise Set 20 (page 172)

1. $-\frac{1}{2}$.
2. 1.
3. 0.
4. -2. Factor the numerator and simplify.
5. 2. Factor the numerator and simplify.
6. 2. Use L'Hospital's Rule.
7. 0. Use L'Hospital's Rule.
8. 0. Indeed, $\lim_{t \rightarrow 0} \frac{\sin^2 t - \sin(t^2)}{t^2} = \lim_{t \rightarrow 0} \left[\left(\frac{\sin t}{t} \right)^2 - \frac{\sin(t^2)}{t^2} \right] = 1 - 1 = 0$.
9. $\frac{1}{2}$. The quotient is continuous at $x = 1$. Use of L'Hospital's Rule will give nonsense here.
10. This limit does not exist. In fact, applying L'Hospital's rule to the one-sided limits at $x = 0$ shows that $\lim_{x \rightarrow 0^+} \frac{\arctan x}{x^2} = \lim_{x \rightarrow 0^+} \frac{(1+x^2)^{-1}}{2x} = +\infty$ and $\lim_{x \rightarrow 0^-} \frac{\arctan x}{x^2} = \lim_{x \rightarrow 0^-} \frac{(1+x^2)^{-1}}{2x} = -\infty$. So there is NO limit at $x = 0$.
11. 1. By L'Hospital's rule, $\lim_{x \rightarrow 0} \frac{(1-x^2)^{-1/2}}{(1+x^2)^{-1}} = 1$.
12. $\frac{6}{5}$. Multiply the expression by $1 = \frac{5x}{3x} \cdot \frac{3x}{5x}$, re-arrange terms and take the limit.
13. $\frac{1}{3}$. Use L'Hospital's Rule.
14. $\frac{1}{36}$. In this exercise we must apply L'Hospital's rule "three" times before we can see the answer.
15. 2. Indeed,

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{x \sin(\sin x)}{1 - \cos(\sin x)} &= \lim_{x \rightarrow 0} \frac{\sin(\sin x) + x \cos(\sin x) \cdot \cos x}{\sin(\sin x) \cdot \cos x} \\
 &= \lim_{x \rightarrow 0} \left(\frac{1}{\cos x} + \frac{x \cos(\sin x)}{\sin(\sin x)} \right) \\
 &= 1 + \lim_{x \rightarrow 0} \left(\frac{x}{\sin(\sin x)} \cdot \cos(\sin x) \right) \\
 &= 1 + \lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \cdot \frac{\sin x}{\sin(\sin x)} \cdot \cos(\sin x) \right) = 2.
 \end{aligned}$$

3.13 Chapter Exercises (page 173)

1. $27(x+1)^{26}$.
2. $-3 \cos^2 x \sin x$.
3. $\csc 2x - 2(x+1) \csc 2x \cot 2x$. Note that $\csc(2x) = \frac{1}{\sin 2x}$.
4. $2(x+5) \cos((x+5)^2)$. You can easily do this one using the "Box" form of the Chain Rule!
5. $\frac{\sin x - \cos x}{(\sin x + \cos x)^2}$.
6. $\frac{1}{\sqrt{2x-5}}$. Use the Generalized Power Rule.
7. $2 \cos 2x$.
8. $-4 \cos 4x \cdot \sin(\sin 4x)$.
9. $6 \tan 2x \cdot \sec^3 2x$. The two minus signs cancel out!
10. $2x \sec 2x + 2(x^2 + 1) \tan 2x \sec 2x$.
11. $-3 \csc 3x \cdot \cot 3x$.
12. $\sec 2x + 2(x+2) \tan 2x \sec 2x$.
13. $\frac{2x^2 + 6x - 2}{(2x+3)^2}$.
14. $3 \cos 3x \cdot (x^{1/5} + 1) + \frac{1}{5} x^{-4/5} \cdot \sin 3x$.
15. $(2x+6) \cos(x^2 + 6x - 2)$.
16. 2.8.
17. $\frac{2}{3 \sqrt[3]{2}} = \frac{\sqrt[3]{4}}{3}$.
18. $-\frac{\sqrt{3}}{3(2+\sqrt{3})}$. Be careful with the square root terms.
19. $210 \times 5^{104} = 42 \times 5^{105}$.
20. 0.
21. 4. The derivative is $4 \cos(\sin(4x)) \cos(4x)$.

22. 2.

23. 1. $f(x) = x + 2$ for $x > -2$. In this case, $x = -1 > -2$ so this is our f .

24. $\frac{1}{2\sqrt{2}}$.

25. $87, 318 \cdot (3x - 2)^{97}, \quad 87, 318.$

26. Putting $u = 3x^2$ and $y = f(u)$, we have

$$\frac{d}{dx} f(3x^2) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot 6x = 6x \cdot f'(3x^2).$$

27. $y - 1 = 24(x - 2)$, or $24x - y - 47 = 0$.

28. $\frac{81}{2} - \frac{\sin 9}{6}$. When $u = 9$ we have $t = 9$ also. We know that $\frac{dy}{du} = \frac{dy}{dt} \cdot \frac{dt}{du}$ or, $(3t^2 - \sin t) \cdot \frac{1}{2\sqrt{u}} = \frac{1}{6}(3 \cdot (81) - \sin 9)$.

29.

$$\begin{aligned} \frac{dy}{dt} &= \frac{dy}{dr} \cdot \frac{dr}{dt} = \left(\frac{1}{2} r^{-1/2} + 3r^{-2} \right) (3 - t^{-1/2}), \\ &= \left(\frac{1}{2} (3t - 2t^{1/2})^{-1/2} + 3(3t - 2t^{1/2})^{-2} \right) (3 - t^{1/2}). \end{aligned}$$

30. Notice that, for $x > 0$ we have $y(x) = x^2$ and hence y is differentiable at x with $y'(x) = 2x$. Similarly, for $x < 0$ we have $y(x) = -x^2$ and hence $y'(x) = -2x$. Finally, for $x = 0$ we have

$$\frac{y(h) - y(0)}{h} = \frac{h|h|}{h} = |h| \rightarrow 0 \text{ as } h \rightarrow 0$$

and hence y is also differentiable at $x = 0$ with $y'(0) = 0$. From the above argument we see that $y'(x) = 2|x|$ for all x . It is well-known that the absolute value function $|x|$ is not differentiable at $x = 0$. Therefore the derivative of y' at 0 does not exist. In other words, $y''(0)$ does not exist.

31. $-\frac{3}{2}$. The derivative is $3x^2 + 2xy' + 2y + 2yy' = 0$. Set $x = 1$, $y = 0$ and solve for y' .

32. $\frac{dy}{dx} = \frac{3x^2 - 2y^2}{4xy - 4y^3}$, $\frac{dx}{dy} = \frac{4xy - 4y^3}{3x^2 - 2y^2}$.

33. Implicit differentiation gives $\frac{1+y'}{2\sqrt{x+y}} + 2xy^2 + 2x^2yy' = 0$. So, at $(0, 16)$, we have $y' = -1$.

34. $\frac{dy}{dx} = \frac{3y^2 + y}{5y^4 - 6xy - x}$.

35. The tangent line to the curve at $(4, 0)$ is vertical. Here $2x + 2yy' = 0$ and we are dividing by 0 at $x = 4$.

36. $y + 1 = 2(x + 1)$, or $2x - y + 1 = 0$.

37. The vertical line through the origin: $x = 0$ (or the y -axis itself.) In this case, $(x + 2y)y' + (2 + y) = 0$. The derivative is undefined (or infinite) at $x = 0$.

38. $y = \frac{5}{2}(x - 4)$, or $5x - 2y - 20 = 0$.

39. $y = x$. At $(1, 1)$ we have $y' = 1$. So $y - 1 = 1(x - 1)$ and the result follows.

40. $y = x - \pi$. The derivative is $\cos x + y' \cos y - 6yy' = 0$. Set $x = \pi$, $y = 0$ and solve for y' .

41. $\frac{1}{2}$. Use L'Hospital's Rule.

42. 1

43. 0. Find a common denominator and use L'Hospital's Rule.

44. 0. Divide the numerator and denominator by x^2 and let $x \rightarrow -\infty$.

45. By L'Hospital's Rule, we have

$$\begin{aligned} \lim_{x \rightarrow -\infty} x^2 \left(\frac{\pi}{2} + \operatorname{Arctan} x \right) &= \lim_{x \rightarrow -\infty} \frac{\frac{\pi}{2} + \operatorname{Arctan} x}{x^{-2}} \\ &= \lim_{x \rightarrow -\infty} \frac{(1 + x^2)^{-1}}{-2x^{-3}} \\ &= \lim_{x \rightarrow -\infty} -\frac{x^3}{2(1 + x^2)} \\ &= +\infty. \end{aligned}$$