case 2: $y = -x \Rightarrow x^2 - x^2 + x^2 = 16$ or $x^2 = 16$.

$$x = \pm 4$$ But $y = -x \Rightarrow y = \pm 4$ too!

So we get four critical points:

$$x = \frac{4}{\sqrt{3}}, \quad y = \frac{4}{\sqrt{3}}$$

$$x = -\frac{4}{\sqrt{3}}, \quad y = -\frac{4}{\sqrt{3}}$$

$$x = 4, \quad y = -4$$

$$x = -4, \quad y = 4$$

<table>
<thead>
<tr>
<th>$x^2 + y^2$</th>
<th>$\sqrt{x^2 + y^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$32$</td>
<td>$\frac{\sqrt{32}}{3}$</td>
</tr>
<tr>
<td>$32$</td>
<td>$\frac{\sqrt{32}}{3}$</td>
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<td>$32$</td>
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<tr>
<td>$32$</td>
<td>$\frac{\sqrt{32}}{3}$</td>
</tr>
</tbody>
</table>

check critical points for max/min.

Note that maximum is at $(4, -4)$ and $(-4, 4)$ and the value at this point is $\sqrt{32}$

Note that minimum is at $(\pm \frac{4}{\sqrt{3}}, \pm \frac{4}{\sqrt{3}})$ and the value here is $\sqrt{\frac{32}{3}}$

1.5 Double and Iterated Integrals

Let domain $f = \mathcal{R}$ and write $z = f(x, y)$ continuous over a finite region $\mathcal{R}$ of the $x, y$ plane. We divide $\mathcal{R}$ into $n$ subregions area $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$ in any fashion what so ever.

Figure 4

Next we pick a point $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in each one of these subregions and form the sum:

$$\sum_{i=1}^{n} f(x_i, y_i) \Delta A_i = f(x_1, y_1) \Delta A_1 + \ldots + f(x_n, y_n) \Delta A_n$$

Think about $f(x_i, y_i) \Delta A_i$: This is the signed volume of a parallelepiped of base area $\Delta A_i$ and height $f(x_i, y_i)$. 

The ABC’s of Calculus
If this sum approaches a limit as \( n \to \infty \) and at the same time every subregion shrinks to a point \((\Delta A_i \to 0)\) the limit (if it is unique) is called the double integral of \( f \) over \( R \) and denoted by:

\[
\int \int_R f \, dA \quad \text{or} \quad \int \int_R f(x, y) \, dA
\]

by definition:

\[
= \lim_{n \to +\infty, \, \Delta A_i \to 0} \sum_{i=1}^n f(x_i, y_i) \Delta A_i
\]

Remark:

1) Let \( f(x, y) \geq 0 \) all \((x, y)\) in \( R \)

\[ \Rightarrow \int \int_R f \, dA = \text{volume under that point of } s \text{ whose projection is } R \]

2) If \( f(x, y) = 1 \) for \((x, y)\) in \( R \) then:

\[ \int \int_R f \, dA = \int \int_R dA = \text{volume under the part of the plane } z = 1 \text{ whose projection is in } R. \]

So we claim that \( \int \int_R dA = \text{area of } R \) (numerically)

The double integral (2D) has same properties as (1D) integral, namely:

1)

\[ \int \int_R (c \cdot f) \, dA = c \int \int_R f \, dA \quad \text{c is constant} \]

2)

\[ \int \int_R (f \pm g) \, dA = (\int \int_R f \, dA) \pm (\int \int_R g \, dA) \]

3) If \( R = R_1 \cup R_2 \cup \ldots \ldots \cup R_n \) where \( R_i \cap R_j = 0 \)

then

\[
\int \int_R f \, dA = \int \int_{R_1} f \, dA + \int \int_{R_2} f \, dA + \ldots + \int \int_{R_n} f \, dA
\]

In order to evaluate a double integral we need to set up an "iterated integral" in cartesian (rectangular) coordinates, such an integral takes the form:

\[
\int \int_{R_1} f(x, y) \, dx \, dy \quad \text{or} \quad \int \int_{R_2} f(x, y) \, dy \, dx
\]

where

\[ R_1 = \{(x, y) : f_1(y) \leq x \leq f_2(y), \quad a \leq y \leq b\} \]
and

\[ R_2 = \{ (x, y) : g_1(x) \leq y \leq g_2(x), \quad c \leq x \leq d \} \]

\( R_1 \& R_2 \) are simply descriptions of the given region \( \mathcal{R} \) in rectangular coordinates.

**Figure 5**

**Describing Regions in \( \mathcal{R}^2 \)**

**Example 15** Describe the circle of radius 4 centered at (0, 0) in two different ways = \( \{ x^2 + y^2 = 4 \} \)

1) Vertical slices:

**Figure 6**

\[ \{ (x, y) : -2 \leq x \leq 2, \quad -\sqrt{4-x^2} \leq y \leq +\sqrt{4-x^2} \} \]

2) Horizontal slices:

\[ \{ (x, y) : -\sqrt{4-y^2} \leq x \leq +\sqrt{4-y^2}, \quad -2 \leq y \leq 2 \} \]

**Example 16 Figure 7**

\[ \mathcal{R} = \{ (x, y) : 0 \leq y \leq (1-x), \quad 0 \leq x \leq 1 \} \]

**Example 17 Figure 8**

\[ \mathcal{R} = \{ (x, y) : x^2 \leq y \leq x, \quad 0 \leq x \leq 1 \} \]

**NOTE:** \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) are really the same region \( \mathcal{R} \) described in two different ways.

The iterated integral takes the form:

\[
\int \int_{\mathcal{R}_1} f(x, y) \, dxdy = \int \int_{\mathcal{R}} f \, dA \quad \text{(theorem......)}
\]

\[
\int \int f(x, y) \, dxdy \quad \text{means} \quad \int_a^b \int_{f_1(y)}^{f_2(y)} f(x, y) \, dxdy
\]
Functions and their properties

\[ a, b, f_1(y), f_2(y): \text{come from the description of } \mathcal{R}_1 \text{ by horizontal slices.} \]

\[
\int \int_{\mathcal{R}} f \, dA = \int_{a}^{b} \left( \int_{f_1(y)}^{f_2(y)} f(x, y) \, dx \right) \, dy
\]

Integrate this first, keep "y" fixed.

Then integrate the resulting function of y given inside the box with respect to "y" between a\&b.

The other technique for evaluation involves taking vertical slices.

\[
\int \int_{\mathcal{R}} f \, dA = \int_{c}^{d} \left( \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \right) \, dx
\]

Integrate this first, keep "x" fixed.

The double integral of (piece wise) continuous function \( f \) defined on \( \mathcal{R} \subset \mathbb{R}^n \) exists.

We say \( f \) is integrable over \( \mathcal{R} \) if \( \int \int_{\mathcal{R}} |f| \, dA \) is finite. If \( f \) is integrable over \( \mathcal{R} \Rightarrow \) Fubini’s theorem says that:

\[
\int \int_{\mathcal{R}_1} f(x, y) \, dx \, dy = \int \int_{\mathcal{R}_2} f(x, y) \, dy \, dx = \int \int_{\mathcal{R}} f \, dA
\]

Where \( \mathcal{R}_1, \mathcal{R}_2 \) are the two (different) representations of region \( \mathcal{R} \). We can reverse/interchange the order of integration.

**Example 18**

\[
\int_{0}^{1} \int_{1}^{2} \, dx \, dy = \int \int_{\mathcal{R}} f \, dA.
\]

\[
f_1^2 \, dx = (2 - 1) = 1 \quad \text{next} \quad \int_{0}^{1} (\int_{1}^{2} \, dx) \, dy = \int_{0}^{1} 1 \cdot dy = y \bigg|_{0}^{1} = 1
\]

But this iterated integral by definition:

\[
\int_{0}^{1} \int_{1}^{2} \, dx \, dy = \int \int_{\mathcal{R}} \, dA
\]

= area of the region \( \mathcal{R} \)

= area of square

**Example 19**

\[
I = \int_{0}^{1} \int_{x^2}^{x} xy^2 \, dy \, dx
\]
Figure 9

\[ f(x, y) = xy^2 \]

\[ \mathcal{R} = \{(x, y) : x^2 \leq y \leq x, \quad 0 \leq x \leq 1\} \]

\[
\begin{align*}
I &= \int_0^1 \left( \int_{x^2}^x y^2 \, dy \right) dx \\
  &= \int_0^1 \left( \frac{y^3}{3} \bigg|_{y=x^2}^{y=x} \right) dx \\
  &= \int_0^1 \left( \frac{x^3}{3} - \frac{x^6}{3} \right) dx \\
  &= \int_0^1 \left( \frac{x^4}{3} - \frac{x^7}{3} \right) dx \\
  &= \left[ \frac{x^5}{15} - \frac{x^8}{24} \right]_0^1 \\
  &= \frac{1}{15} - \frac{1}{24} \\
  &= \frac{1}{3} \left( \frac{1}{5} - \frac{1}{8} \right) \\
  &= \frac{1}{40}
\end{align*}
\]

Example 20 HORIZONTAL SLICES:

Redescribe:

\[ \mathcal{R} = \{(x, y) : 0 \leq y \leq 1, \ y \leq x \leq \sqrt{y}\} \]

Figure 10

\[
\begin{align*}
I &= \int_0^1 \left( \int_y^{\sqrt{y}} (xy^2) \, dx \right) dy \\
  &= \int_0^1 \left( y^2 \int_y^{\sqrt{y}} x \, dx \right) dy \\
  &= \int_0^1 \left( y^2 \int_y^{\sqrt{y}} \left( \frac{x^2}{2} \right)_{x=y} \right) dy \\
  &= \int_0^1 y^2 \left( \frac{y^2}{2} - \frac{y^2}{2} \right) dy \\
  &= \frac{1}{2} \int_0^1 \left( y^3 - y^4 \right) dy \\
  &= \frac{1}{2} \left[ \frac{y^4}{4} - \frac{y^5}{5} \right]_0^1 \\
  &= \frac{1}{40}
\end{align*}
\]
\[
= \frac{1}{2} \left( \frac{1}{4} - \frac{1}{5} \right) \\
= \frac{1}{2} \left( \frac{1}{20} \right) \\
= \frac{1}{20}
\]

**Example 21** Find the volume cut from \(9x^2 + 4y^2 + 36z = 36\) by plane \(z = 0\)

1) Ensure surface is above \(xy\)-plane.
2) Give an idea of shape of region.
3) Set up an integral for volume.
4) Describe the "projection" involved.
5) Use an appropriate coordinate system.
6) Use Fubini??
7) Write down iterated integrals
8) Evaluate

**SOLUTION**

1) Yes the surface lies above the \(xy\)-plane.
2) Done
3) \(\int \int_{\mathcal{R}} f \, dA = \text{volume under } s.\)

\[
\text{solve for } z \text{ then } 36z = 36 - 9x^2 - 4y^2 \Rightarrow z = 1 - \frac{x^2}{4} - \frac{y^2}{9}
\]

The projection of \(s\) on the \(xy\)-plane (ie \(z = 0\)) is given by the use of geometry \(z = 0\) \(1 \geq \frac{x^2}{4} + \frac{y^2}{9}\)

4) \(\text{volume} = \int \int_{\mathcal{R}} \left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right) \, dA\)

5) \(\mathcal{R}\) is an ellipse, use elliptical coordinates.

\[
x = ar \cos \Theta \\
y = br \sin \Theta \quad (\text{Jacobian } abr)
\]

**How to choose \(a,b?\)**

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow a = 2 \quad b = 3
\]
6) \( R = \{(x, y) : \frac{x^2}{4} + \frac{y^2}{9} \leq 1\} \) in rectangular coordinates.
\( R = \{(r, \Theta); 0 \leq \Theta < 2\pi, \ 0 \leq r \leq 1\} \) in elliptical coordinates.

\[
\text{Volume} = \int_0^{2\pi} \int_0^1 (1 - \frac{x^2}{4} - \frac{y^2}{9}) \, dr \, d\Theta
\]

\[
= \int_0^{2\pi} \int_0^1 (1 - r^2) \frac{a-b}{6r} \, dr \, d\Theta
\]

\[
= 6 \int_0^{2\pi} \left( \int_0^1 (r - r^3) \, dr \right) d\Theta
\]

\[
= 12\pi \int_0^1 (r - r^3) \, dr
\]

\[
= 12\pi \left[ \frac{r^2}{2} - \frac{r^4}{4} \right]_0^1
\]

\[
= 12\pi \left[ \frac{1}{2} - \frac{1}{4} \right]
\]

\[
= \frac{3\pi}{2}
\]

**Example 22** Find the volume common to the cylinders \( x^2 + y^2 = 16 \) and \( x^2 + z^2 = 16 \).

Idea: look at cylinder \( x^2 + y^2 = 16 \) and its projection onto the \( xy \)-plane namely:

\[
D = \{(x, y) : 0 < x^2 + y^2 \leq 16\} = \{(x, y) : -\sqrt{16 - x^2} \leq y \leq \sqrt{16 - x^2}, \ -4 \leq x \leq 4\} \text{ cartesian coordinates}
\]

The surface above \( D \) is given by \( x^2 + z^2 = 16 \) or \( z^2 = 16 - x^2 \) \( \Rightarrow z = \pm \sqrt{16 - x^2} \) but \( z \geq 0 \) \( \Rightarrow z = \sqrt{16 - x^2} \)

### 1.6 Double and Iterated Integrals

Let domain \( f = \mathcal{R} \) where \( f : \mathbb{R}^2 \rightarrow \mathcal{R} \) and let \( z = f(x, y) \) denote that part of the surface \( S \) which lies above the domain \( f = \mathcal{R} \).

**Figure 13**

We divide \( \mathcal{R} \) into \( m \) subregions of area \( \Delta A_1, \Delta A_2, \ldots, \Delta A_m \) in any fashion.

**Figure 14**
Next we pick a point \((x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)\) in each one of these subregions and form the sum:

\[
\sum_{i=1}^{m} f(x_i, y_i)\Delta A_i = f(x_1, y_1)\Delta A_1 + f(x_2, y_2)\delta A_2 + \ldots + f(x_m, y_m)\Delta A_m
\]

If this sum approaches a limit as the number \(m \to \infty\) and at the same time every subregion shrinks to a point (ie \(\Delta A_i \to 0\)), the limit (if it is unique) is called the double integral of \(f\) over \(R\) and is written as:

\[
\int \int_{R} f(x, y) dA = \lim_{m \to \infty, \Delta A_i \to 0} \sum_{i=1}^{m} f(x_i, y_i)\Delta A_i
\]

If \(f(x, y) \geq 0\) then:

\[
\int \int_{R} f dA = \text{Volume under surface } S \text{ lying directly above } R
\]

If \(f = 1\), then \(\int \int_{R} dA = \text{area of } R\)

**Figure 15**

The double integral (2 variables) has the following properties:

1) \[
\int \int_{R} (c \cdot f) dA = c \int \int_{R} f dA \quad \text{c is any constant}
\]

2) \[
\int \int_{R} (f \pm g) dA = (\int \int_{R} f dA) \pm (\int \int_{R} g dA)
\]

3) If

\[
R = R_1 \cup R_2 \cup \ldots \cup R_n \text{where } R_i \cap R_j = \emptyset \text{ (ie. these are pairwise disjoint)}
\]

then

\[
\int \int_{R} f dA = \int \int_{R_1} f dA + \int \int_{R_2} f dA + \ldots + \int \int_{R_n} f dA
= \sum_{i=1}^{m} \int_{R_i} f dA
\]
In order to evaluate a double integral we set up a so-called \textit{iterated integral}.

In cartesian coordinates, such an integral is of the form:
\[
\int \int_{\mathcal{R}_1} f(x,y) \, dx \, dy \quad \text{or} \quad \int \int_{\mathcal{R}_2} f(x,y) \, dy \, dx
\]

where
\[
\mathcal{R}_1 = \{ (x,y) : f_1(y) \leq x \leq f_2(y), \ a \leq y \leq b \}
\]
and
\[
\mathcal{R}_2 = \{ (x,y) : c \leq x \leq d, \ g_1(x) \leq y \leq g_2(x) \}
\]

Now each one of \( \mathcal{R}_1, \mathcal{R}_2 \) are a description of \( \mathcal{R} \) in cartesian coordinates:

\textbf{Figure 16}

So, \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) are really the same region described in two different ways.

The iterated integral then becomes:
\[
\int \int_{\mathcal{R}_1} f(x,y) \, dx \, dy = \int_a^b \left( \int_{f_1(y)}^{f_2(y)} f(x,y) \, dx \right) \, dy
\]

Limits come from the description of \( \mathcal{R} \) above

\[
= \int_a^b \left\{ \int_{f_1(y)}^{f_2(y)} f(x,y) \, dx \right\} \, dy
\]

integrate w.r.t. \( x \) and hold \( y \) constant

Then integrate w.r.t. \( y \) to get a number

So this is the inverse procedure to partial differentiation or we integrate with respect to \( x \) holding \( y \) constant in the \textit{inner-most} integral, and then integrate with respect to \( y \).

The double integral of a continuous function \( f \) defined on \( \mathcal{R} \subset \mathbb{R}^2 \) exists. We say \( f \) is integrable over \( \mathcal{R} \).

Indeed if \(|f(x,y)|\) is integrable over \( \mathcal{R} \), the iterated integrals of \( f \) over \( \mathcal{R} \) exist and are equal (Fubini), ie
\[
\int \int_{\mathcal{R}_1} f(x,y) \, dx \, dy = \int \int_{\mathcal{R}_2} f(x,y) \, dy \, dx
\]
or, we can reverse the order of integration.
Example 23
\[ \int_0^1 \int_1^2 dx \, dy = 1 \]

**Figure 17**

Here \( R = \{(x, y) : 1 \leq x \leq 2, 0 \leq y \leq 1\} \), so \( R \) is a square. The iterated integral represents the area of this square, \( R \).

Now
\[ \int_0^1 \int_1^2 dx \, dy = \int_0^1 (\int_1^2 dx) \, dy = \int_0^1 (2 - 1) \, dy = \int_0^1 \, dy = 1 \]

If we reverse the order (we need to change the limits!) of integration:
\[ \int_0^1 \int_1^2 dx \, dy = \int_1^2 \int_0^1 dy \, dx = \int_1^2 (\int_0^1 \, dy) \, dx = \int_1^2 (1 - 0) \, dx = 1 \]

Example 24

\[ I = \int_0^1 \int_{x^2}^x xy^2 \, dy \, dx = \frac{1}{40} \]

\[ I = \int_0^1 \left[ \frac{x^3}{3} - \frac{x^6}{3} \right] \, dx \]
\[ = \int_0^1 \left[ \frac{x^3}{3} \right]_{y=x^2}^{y=x} \, dx \]
\[ = \int_0^1 \left[ \frac{y^3}{3} - \frac{x^6}{3} \right] \, dx \]
\[ = \frac{1}{3} \int_0^1 (x^4 - x^2) \, dx \]
\[ = \frac{1}{3} \left( \frac{5}{5} - \frac{1}{8} \right) \]
\[ = \frac{1}{40} \]

Here \( R = \{(x, y) : 0 \leq x \leq 1, \ x^2 \leq y \leq x\} \) is shown below:

**Figure 18**

To reverse the order of integration we need to describe \( R \) in the "other" order, ie. by taking a horizontal fibre.

For a given \( y \) we need to find the coordinates of the "ends" of the fibre.

This gives: \( (\sqrt{y}, y) \) and \( (\sqrt{y} y^2 = \sqrt{y}, y) \)
Therefore:
\[ \mathcal{R} = \{(x, y) : y \leq x \text{leq}\sqrt{y}, \ 0 \leq y \leq 1\} \]

So
\[
I = \int_0^1 \int_y^\sqrt{y} xy^2 \, dx \, dy \\
= \int_0^1 y^2 \{ \int_y^\sqrt{y} x \, dx \} \, dy \\
= \int_0^1 y^2 \left( \frac{y}{2} - \frac{y^2}{2} \right) \, dy \\
= \frac{1}{2} \int_0^1 (y^3 - y^4) \, dy \\
= \frac{1}{2} \left( \frac{1}{4} - \frac{1}{5} \right) \\
= \frac{1}{40}
\]

Example 25
\[
\int_0^{2\pi} \int_0^{1-\cos \Theta} \rho^3 \cos^2 \Theta \, d\rho \, d\Theta = \frac{49\pi}{32}
\]
Here:
\[ \mathcal{R} = \{(\rho, \Theta) : 0 \leq \rho \leq 1 - \cos \Theta, \ 0 \leq \Theta < 2\pi\} \]
\( \mathcal{R} \) can also be described as:
\[ \mathcal{R} = \{(\rho, \Theta) : 0 \leq \Theta \leq \cos^{-1}(1 - \rho), \ 0 \leq \rho \leq 2\} \]

Figure 19

So:
\[
\int_0^2 \int_0^{\cos^{-1}(1-\rho)} \rho^3 \cos^2 \Theta \, d\Theta \, d\rho = \int_0^{2\pi} \int_0^{1-\cos \Theta} \rho^3 \cos^2 \Theta \, d\rho \, d\Theta \\
= \int_0^{2\pi} \cos^2 \Theta \, d\Theta \cdot \left[ \int_0^{1-\cos \Theta} \rho^3 \, d\rho \right] \\
= \int_0^{2\pi} \frac{(1 - \cos \Theta)^4}{4} \cos^2 \Theta \, d\Theta \\
= \frac{1}{4} \int_0^{2\pi} (1 - \cos \Theta)^4 \cos^2 \Theta \, d\Theta
\]

Example 26 Evaluate using iterated integrals (and interchange) \( f(x, y) = x \) over the region \( \mathcal{R} \) bounded by \( x^2 \) and \( x^3 \).

Note: \( x^3 < x^2 \) if \( 0 < x < 1 \) Describe \( \mathcal{R} \) in two different ways:
A horizontal fibre has the end-coordinates \((\sqrt{y}, y), (\sqrt{y}, y)\), therefore:

\[ R = \{ (x, y) : \sqrt{y} \leq x \leq \sqrt{y}, \quad 0 \leq y \leq 1 \} \]

Therefore:

\[
\int \int_{R} x \, dA = \int_{0}^{1} \int_{\sqrt{y}}^{y} x \, dx \, dy
\]

\[
= \int_{0}^{1} \frac{1}{2} (y^{3} - y) \, dy
\]

\[
= \frac{1}{2} \left[ \frac{3}{5} y^{5} - \frac{y^{2}}{2} \right]_{y=0}^{y=1}
\]

\[
= \frac{1}{20}
\]

Furthermore, a vertical fibre has end-coordinates \((x, x^{3})\) and \((x, x^{2})\), So:

\[ R = \{ (x, y) : 0 \leq x \leq 1, \quad x^{3} \leq y \leq x^{2} \} \]

Therefore

\[
\int \int_{R} f \, dA = \int \int_{R} x \, dA
\]

\[
= \int_{0}^{1} \int_{x^{3}}^{x^{2}} x \, dy \, dx
\]

\[
= \int_{0}^{1} x (\int_{x^{3}}^{x^{2}} dy) \, dx
\]

\[
= \int_{0}^{1} x (x^{2} - x^{3}) \, dx
\]

\[
= \int_{0}^{1} (x^{3} - x^{4}) \, dx
\]

\[
= \frac{1}{4} - \frac{1}{5}
\]

\[
= \frac{1}{20}
\]

\[
= \int_{0}^{1} \int_{\sqrt{y}}^{y} x \, dx \, dy \quad \text{as required by Fubini’s theorem}
\]

Example 27 Evaluate \(\int \int_{R} 1 \, dA\) where \(R\) is the first quadrant region bounded by \(2y = x^{2}\), \(y = 3x\) and \(x + y = 4\)

1) Sketch the region Find all points of intersection:

\[ y = 3x \quad \text{meets} \quad y = 4 - x \quad \text{at:}
\]

\[ x = 1 \quad (y = 3) \]

\[ y = \frac{x^{2}}{2} \quad \text{meets} \quad y = 3x \quad \text{at}
\]

\[ \frac{x^{2}}{2} = 3x \Rightarrow x = 0, \quad x = 6 \]
\[ y = \frac{x^2}{2} \] meets \( y = 4 - x \) when:
\[ x = -4 \quad \text{and} \quad x = 2 \]

2) Describe \( \mathcal{R} \) by vertical or horizontal fibres:

(a) **Vertical** Then \( \mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \) where:

\[ \mathcal{R}_1 = \{(x, y) : 1 \leq x \leq 2, \quad 4 - x \leq y \leq 3x\} \]

\[ \mathcal{R}_2 = \{(x, y) : 2 \leq x \leq 6, \quad \frac{x^2}{2} \leq y \leq 3x\} \]

**Figure 21**

So:

\[
\int_{\mathcal{R}} 1 \cdot dA = \int_{\mathcal{R}_1 \cup \mathcal{R}_2} 1 \cdot dA \\
= \int_{\mathcal{R}_1} dydx + \int_{\mathcal{R}_2} dydx \\
= \int_1^3 \int_{4-x}^{3x} dydx + \int_2^6 \int_{\frac{x^2}{2}}^{3x} dydx \\
= \int_1^3 (3x - (4 - x))dx + \int_2^6 (3x - \frac{x^2}{2})dx \\
= 4 \cdot \frac{(x - 1)^2}{2}_{x=1}^{x=2} + (3 \cdot \frac{x^3}{2} - \frac{x^3}{6})|^2_3 \\
= 2 + [3 \cdot 18 - 36 - \frac{1}{6}]
\]

\[ \text{Area of } \mathcal{R} = \boxed{\frac{46}{3}} \]

(b) **Horizontal** Here \( \mathcal{R} = \mathcal{R}_3 \cup \mathcal{R}_4 \) where:

\[ \mathcal{R}_4 = \{(x, y) : 3 \leq y \leq 18, \quad \frac{y}{3} \leq x \leq \sqrt{2y}\} \]

and

\[ \mathcal{R}_3 = \{(x, y) : 2 \leq y \leq 3, \quad 4 - y \leq x \leq \sqrt{2y}\} \]

**Figure 22**

So area of \( \mathcal{R} \):

\[
\int_{\mathcal{R}_3 \cup \mathcal{R}_4} dA = (\int_{\mathcal{R}_3} + \int_{\mathcal{R}_4})dA \\
= \int_2^3 \int_{\sqrt{2y}}^{\sqrt{2y}} dx dy + \int_3^{18} \int_{\sqrt{2y}}^{\sqrt{2y}} dx dy \\
\]

\[ \text{Area of } \mathcal{R} = \boxed{\int_2^3 (\sqrt{2y} - (4 - y))dy + \int_3^{18} (\sqrt{2y} - \frac{y}{3})dy} \]

\[ = \left(\sqrt{2} \cdot \frac{2}{3} y^{\frac{3}{2}} - 4y + \frac{y^2}{2}\right)|_{y=2}^{y=3} + \left(\sqrt{2} \cdot \frac{2}{3} y^{\frac{3}{2}} - \frac{y^2}{6}\right) \]

\[ = \]
\[
\begin{align*}
&= \left[ \frac{2\sqrt{2}}{3} \cdot 3^4 - \frac{9}{2} - \frac{2\sqrt{2}}{3} (\sqrt{2})^3 + 8 - 2 \right] \\
&+ \left[ \frac{2\sqrt{2}}{3} 18^2 - \frac{18^2}{6} - \frac{2\sqrt{2}}{3} 6^2 + 6 \right] \\
&= \frac{46}{3}
\end{align*}
\]

Applications:

1) Finding the area of planar regions \( \mathcal{R} \), (as we have).

2) Evaluating complicated integrals by interchanging order of integration.

3) Centroids of planar regions:
   
   If \( \mathcal{R} \) has area \( A = \int \int \mathcal{R} \, dA \) then the relations:
   
   \[
   \bar{x} = \frac{M_y}{A} = \frac{\int \int \mathcal{R} \, x \, dA}{A}, \quad \bar{y} = \frac{M_x}{A} = \frac{\int \int \mathcal{R} \, y \, dA}{A}
   \]
   
   define the centroid \( (\bar{x}, \bar{y}) \) of \( \mathcal{R} \)

4) finding the mass of planar regions having mass density, per unit area given by \( \delta(x, y) \). then:
   
   \[
   \text{mass} \, \mathcal{R} = \int \int \mathcal{R} \, \delta(x, y) \, dA
   \]

Example 28 POLAR:

\[
I = \int_0^\infty \int_0^\infty e^{-x^2-y^2} \, dx \, dy = \int_0^\infty (e^{-y^2} \int_0^\infty e^{-x^2} \, dx) \, dy
\]

Figure 23

\{ \mathcal{R} = (x, y) : 0 \leq x \leq \infty, \, 0 \leq y \leq \infty \} \\
\{ \mathcal{R} = (x, y) : 0 \leq y \leq \infty, \, 0 \leq x \leq \infty \}

when \( x^2 + y^2 \) appearing in integrand use "polar".

Why? Because:

\[
x^2 + y^2 = (r \cos \Theta)^2 + (r \sin \Theta)^2 \\
= r^2 (\cos^2 \Theta + \sin^2 \Theta) \\
= r^2
\]

Then \( \mathcal{R} \) can be described easily in polar coordinates:
\( \mathcal{R} = \{(r, \Theta) : 0 < r < \infty, 0 \leq \Theta \leq \frac{\pi}{2}\} \)

But \( dA \) polar = \( rdrd\Theta \) (= \( dx\,dy \))

So:

\[
\begin{align*}
\int \int_{\mathcal{R}} f dA &= \int \int_{\mathcal{R} \text{polar}} f(r \cos \Theta, r \sin \Theta) \, dA_{\text{polar}} \\
&= \int_0^\infty \int_0^{2\pi} e^{-r^2} r \, rdr \, d\Theta \\
&= \int_0^\infty \left( \int_0^{2\pi} e^{-r^2} \, d\Theta \right) \cdot \left( \int_0^\infty re^{-r^2} \, dr \right) \\
&= \pi \cdot \int_0^\infty re^{-r^2} \, dr \cdot \left( \int_0^\infty 1 \cdot d\Theta \right) \\
&= \pi \cdot \left[ \int_0^\infty re^{-r^2} \, dr \right] \\
&= \pi \cdot \left[ \left. -\frac{1}{2} e^{-r^2} \right|_0^\infty \right] \\
&= \pi \cdot \left[ 0 - \left( -\frac{1}{2} \right) \right] \\
&= \frac{\pi}{4}
\end{align*}
\]

**Example 29** Evaluate:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \, dy
\]

By changing to polar coordinates.

Here \( \mathcal{R} = \{(x, y) : -\infty < x < \infty, -\infty < y < +\infty\} \). In polar coordinates this becomes:

\( \mathcal{R}' = \{(r, \phi) : 0 \leq r < \infty, 0 \leq \phi < 2\pi\} \)

Furthermore, the elements of area:

\( dx \, dy = rdrd\phi \) earlier

Also \( x^2 + y^2 = r^2 \), thus:
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \, dy = \int_0^\infty \int_0^\infty e^{-r^2}r \, dr \, d\phi \\
= \int_0^{2\pi} \left( \int_0^\infty r e^{-r^2} \, dr \right) d\phi \\
= 2\pi \int_0^\infty r e^{-r^2} \, dr \\
= 2\pi \lim_{R \to \infty} \int_0^R r e^{-r^2} \, dr \\
= 2\pi \lim_{R \to \infty} \left( \frac{1}{2} \int_0^R (-2r) e^{-r^2} \, dr \right) \\
= 2\pi \cdot \left( \frac{1}{2} \lim_{R \to \infty} (e^{-r^2} \big|_{r=0}^{r=R}) \right) \\
= -\pi \cdot \lim_{R \to \infty} [e^{-R^2} - 1] \\
= \pi
\]

**REMARK** Now
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \, dy = \int_{-\infty}^{\infty} e^{-x^2} \, dx \cdot \int_{-\infty}^{\infty} e^{-y^2} \, dy \\
= (\int_{-\infty}^{\infty} e^{-x^2} \, dx)^2 \\
= \pi \text{ above}
\]

Therefore \( \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \)

Plane areas by double integration

Recall the area of a planar region \( \mathcal{R} \) is given by:
\[
\int \int_{\mathcal{R}} dA = \text{area of } \mathcal{R}
\]
where \( dA \) is the element of area in the particular coordinate system used.

**Example 30** Find the area of the region \( \mathcal{R} \) bounded by \( 3x + 4y = 24, x = 0, y = 0 \) (by double integration).

1) **sketch the region:** Find all points of intersection of the curves making up its boundary.

**Figure 24**

Here the points are: \((0, 0), (0, 6)\) and \((8, 0)\).
2) Choose a description of $\mathcal{R}$. Either by horizontal or vertical fibres, say, horizontal.

**Figure 25**

3) Find the coordinates of the end-points of the fibre. Do this by finding "inverse functions".

**Figure 26**

For fixed $y$, the set of points on a typical fibre is given by:

$$\{ (x, y) : 0 \leq x \leq 8 - \frac{4}{3}y \}$$

4) Give a description of $\mathcal{R}$. Thus:

$$\mathcal{R} = \{ (x, y) : 0 \leq x \leq 8 - \frac{4}{3}y, \ 0 \leq y \leq 6 \}$$

5) Set up an iterated integral for evaluating the double integral. Here:

$$\int_{\mathcal{R}} dA = \int_0^6 \int_0^{8-\frac{4}{3}y} dx dy \quad \text{(in rectangular coordinates)}$$

6) Evaluate the iterated integral to find the area

$$\text{Area of } \mathcal{R} = \int_0^6 (8 - \frac{4}{3}y) dy$$

$$= \left[ (8y - \frac{2}{3}y^2) \right]_{y=0}^{y=6}$$

$$= 48 - \frac{2}{3} \cdot 36 = 48 - 24$$

$$= 24$$

(As a check: Area of $\Delta = \frac{1}{2} \text{(base)} \times \text{(height)} = \frac{1}{2}(8) \times (6) = 24$)

Remark: Use of vertical fibres in step(3) gives the iterated integral:

$$\text{Area of } \mathcal{R} = \int_0^8 \int_0^{6-\frac{4}{3}x} dy dx \quad (= 24)$$

Example 31 Area within $\rho = 2(1 - \cos \Theta)$

A typical "fibre" here is $\Theta = \text{constant}$ which gives a ray from $O$ to the boundary $\rho = 2(1 - \cos \theta)$

**Figure 27**
So:

\[ R = \{(\rho, \Theta) : 0 \leq \rho \leq 2(1 - \cos \Theta \quad 0 \leq \Theta < 2\pi)\} \]

So the area:

\[
\int \int_{R} dA = \int_{0}^{2\pi} \int_{0}^{2-2\cos \Theta} \rho d\rho d\Theta
\]

\[ = \int_{0}^{2\pi} (1 - \cos \Theta)^2 d\Theta \]

\[ = 2 \int_{0}^{2\pi} (1 - 2 \cos \Theta + \cos^2 \Theta) d\Theta \]

\[ = 2 \cdot 2\pi + 2 \int_{0}^{2\pi} \cos^2 \Theta d\Theta \]

\[ = 4\pi + 2 \left( \frac{1}{2} \cdot 2\pi \right) \]

\[ = 6\pi \]

**Example 32** \( f(x, y) = x \) over \( R \) bounded by \( y = x^2 \) and \( y = x^3 \). Graph the region (if possible).

**NOTE:** \( x^3 < x^2 \) if \( x < 1 \).

setting \( x^3 = x^2 \) we get either \( x = 0 \) or \( x = 1 \) so points of intersection of these curves are \( (0, 0), (1, 1) \). Describe \( R \) (in some coordinate system), stick with cartesian coordinates, \( (x, y) \)...so:

\[ R = \{(x, y) : \sqrt[y]{y} \leq x \leq \sqrt[3]{y}, \quad 0 \leq y \leq 1\} \]

Therefore using a horizontal slice:

\[
\int \int_{R} f dA = \int \int_{R} x \, dxdy \rightarrow \text{notice the order} \]

\[ = \int_{0}^{1} \int_{\sqrt[3]{y}}^{\sqrt[y]{y}} x \, dxdy \]

\[ = \int_{0}^{1} \left[ \frac{x^2}{2} \right]_{x=\sqrt[y]{y}}^{x=\sqrt[3]{y}} dy \]

\[ = \int_{0}^{1} \left( \frac{y^{\frac{2}{3}}}{2} - \frac{y^{\frac{1}{3}}}{2} \right) dy \]

\[ = \frac{1}{2} \left[ \frac{y^{\frac{5}{6}}}{5} - \frac{y^{\frac{2}{3}}}{2} \right]_{0}^{1} \]

\[ = \frac{1}{2} \cdot \left( \frac{3}{5} - \frac{1}{2} \right) \]

\[ = \frac{1}{2} \cdot \left( \frac{6 - 5}{10} \right) \]

\[ = \frac{1}{20} \]
Vertical Slice:

\[ R = \{(x, y) : 0 \leq x \leq 1, \ x^3 \leq y \leq x^2\} \]

\[
\int \int_R \, dA = \int_0^1 x^{x^3} \, dy \, dx \quad \text{notice the order}
\]
\[
= \int_0^1 x(x^2) \, dx
\]
\[
= \int_0^1 x(x^2 - x^3) \, dx
\]
\[
= \int_0^1 (x^3 - x^4) \, dx
\]
\[
= \left[ \frac{x^4}{4} - \frac{x^5}{5} \right]_{n=0}^{n=1}
\]
\[
= \frac{1}{4} - \frac{1}{5}
\]
\[
= \frac{5 - 4}{20}
\]
\[
= \frac{1}{20}
\]

Example 33

\[
\int \int_R \frac{1}{2} \, dA
\]

\[ R \] is first quadrant region bounded by \( 2y = x^2, \ y = 3x \) and \( x + y = 4 \)

1) sketch & find all points of intersection of these curves:

- \( y = 3x \) meets \( y = 4 - x \) at \( x = 1 \) or \( y = 3 \)
- \( y = \frac{x^2}{2} \) meets \( y = 3x \) at \( x = 6 \) or \( y = 0 \) and \( y = 18 \)
- \( y = \frac{x^2}{2} \) meets \( y = 4 - x \) at \( x = -4 \) and \( x = 2 \) or \( y = 8 \) not in 1st quadrant, so ignore it
- \( y = 2 \).

2) Describe \( R \)

Either way we need to divide \( R \) into two pieces.

a) vertical slice.

\[ R = R_1 \cup R_2 \]

Where:

\[ R_1 = \{(x, y) : 1 \leq x \leq 2, \ 4 - x \leq y \leq 3x\} \]
\[ R_2 = \{(x, y) : 2 \leq x \leq 6, \ \frac{x^2}{2} \leq y \leq 3x\} \]
\[
\int \int_{R} 1 \cdot dA = \int \int_{R_1} 1 \cdot dA + \int \int_{R_2} 1 \cdot dA \\
= \int_{1}^{2} \int_{4-x}^{3x} dydx + \int_{2}^{6} \int_{x/2}^{3x} dydx \\
= \int_{1}^{2} (4x - 4)x dx + \int_{2}^{6} \left(3x - \frac{x^2}{2}\right) dx \\
= 4 \cdot \int_{1}^{2} (x - 1) dx + \left(\frac{3x^2}{2} - \frac{x^3}{6}\right) \bigg|_{x=2}^{x=6} \\
= 2 + 18 - 6 + \frac{4}{3} \\
= \frac{46}{3}
\]

Similarly we can show:

\[
R = R_3 \cup R_4
\]

where:

\[R_3 = \{(x, y) : (4 - y) \leq x \leq \sqrt{7}y, \ 2 \leq y \leq 3\}\]
\[R_4 = \{(x, y) : (\frac{4}{3}) \leq x \leq \sqrt{2}y, \ 3 \leq y \leq 18\}\]

**Example 34** Evaluate the integral \(\int_{0}^{1} \int_{\sqrt{x}}^{1} \sqrt{1 + y^3} dydx\) by interchanging the order of integration.

\[
\int \int_{R} f dA \quad \text{Here} \quad f(x, y) = \sqrt{1 + y^3}
\]

\[R = \{(x, y) : \sqrt{x} \leq y \leq 1, \ 0 \leq x \leq 1\}\]

\[R = \{(x, y) : 0 \leq x \leq y^2, \ 0 \leq y \leq 1\}\]

So by Fubini, using horizontal slices:

\[
\int_{0}^{1} \int_{\sqrt{x}}^{1} \sqrt{1 + y^3} dydx = \int_{0}^{1} \int_{0}^{y^2} \sqrt{1 + y^3} dx dy
\]

\[
1 + y^3 = u \\
3y^2 dy = du \\
y^2 dy = \frac{du}{3} \\
y = 0 \Rightarrow u = 1 \\
y = 1 \Rightarrow u = 2
\]
\[
\begin{align*}
&= \int_0^1 \sqrt{1 + y^3} \left( \int_0^y dx \right) dy \\
&= \int_0^1 y^2 \sqrt{1 + y^3} dy \\
&= \int_1^2 \frac{du}{3 \cdot \sqrt{u}} \\
&= \frac{1}{3} \int_1^2 u^\frac{2}{3} du \\
&= \frac{1}{3} \cdot \frac{u^\frac{5}{3}}{\frac{5}{3}} \\
&= \frac{2}{9} \cdot u^\frac{5}{3} \\
&= \frac{2}{9} \cdot (2\sqrt{2} - 1)
\end{align*}
\]

**Example 35**

\(x^2 + y^2 + 2x - s\sqrt{x^2 + y^2} = 0\)  equation in polar?

\[
x = r \cos \Theta \\
y = r \sin \Theta
\]  \(\Rightarrow \) \(x^2 + y^2 = r^2\)

\(r^2 + 2 \cdot r \cos \Theta - 2r = 0\)

*Divided by \(r\), \(r \neq 0\)*

\(r + 2 \cos \Theta - 2 = 0\)  \(r = 2 \cdot (1 - \cos \Theta)\)

*Area of \(R\) in polar: \(\int \int_R dA = \int \int_R r \, dr \, d\Theta\)*

*Describe \(R\) in polar:*

\[R = \{(r, \Theta) : 0 \leq r \leq 2 \cdot (1 - \cos \Theta), \ 0 \leq \Theta < 2\pi\}\]

\[
\text{Area} = \int_0^{2\pi} \int_0^{2 \cdot (1 - \cos \Theta)} r \, dr \, d\Theta \\
= \int_0^{2\pi} \left[ \frac{r^2}{2} \cdot (1 - \cos \Theta) \right]_0^{2 \cdot (1 - \cos \Theta)} d\Theta \\
= \int_0^{2\pi} \left( \frac{4 \cdot (1 - \cos \Theta)^2}{2} - 0 \right) d\Theta \\
= 2 \cdot \int_0^{2\pi} (1 - \cos \Theta)^2 d\Theta
\]
Example 36 Find the centroid of:

\[ 3x + 4y = 24, x = 0, y = 0 \]

Centroid coordinates are \((\bar{x}, \bar{y})\) where \(\bar{x} = \frac{M_y}{A}\), and \(\bar{y} = \frac{M_x}{A}\). \(A\) is area and \(M_y = \int \int_R x \, dA, M_x = \int \int_R y \, dA\).

Here \(A = 24\) (found earlier) whereas:

\[
M_y = \int \int_R x \, dA = \int_0^8 \int_0^{6-4x} x \, dy \, dx = \int_0^8 x(6 - \frac{3}{4}x) \, dx = 64
\]

and

\[
M_x = \int \int_R y \, dA = \int_0^6 \int_0^{8-4y} y \, dx \, dy = \int_0^6 y(8 - \frac{4}{3}y) \, dy = 48
\]

ie

\[
(\bar{x}, \bar{y}) = \left( \frac{M_y}{A}, \frac{M_x}{A} \right) = \left( \frac{64}{24}, \frac{48}{24} \right) = (\frac{8}{3}, 2)
\]

Note Either description of \(R\) in the expression for the iterated integral could be used.

Example 37 Find the centroid of the first quadrant area bounded by \(y^2 = 4x, x^2 = 5 - 2y, x = 0\).

Idea: As above

\[
(\bar{x}, \bar{y}) = \left( \frac{\int \int_R x \, dA}{\int \int_R dA}, \frac{\int \int_R y \, dA}{\int \int_R dA} \right)
\]

Figure 36

The points of intersection are: \((0, 0), (0, \frac{5}{2})\), and those for which \(x = \frac{y^2}{4}\) and \(x^2 = 5 - 2y \Rightarrow \frac{y^4}{16} + 2y - 5 = 0\) or \(y = 2 \Rightarrow x = 1\) This gives \(1, 2\).

Take a vertical fibre (horizontal ones lead to the sum of two or more complicated integrals).

Then:

\[
R = \{ (x, y) : 2\sqrt{x} \leq y \leq \frac{5 - x^2}{2}, \ 0 \leq x \leq 1 \}
\]

Figure 37
area, \( A = 7 \iint_R dA \)
\[
= \int_0^1 \int_{2\sqrt{x}}^{5-x^2} dy \, dx
= \int_0^1 \left( \frac{5}{2} - \frac{x^2}{2} - 2x^\frac{3}{2} \right) dx
= 1
\]

Also
\[
\iint_R x \, dA = \int_0^1 \int_{2\sqrt{x}}^{5-x^2} x \, dy \, dx
= \int_0^1 x \left( \frac{5}{2} - \frac{x^2}{2} - 2x^\frac{3}{2} \right) dx
= \int_0^1 \left( \frac{5}{2} - \frac{x^2}{2} - 2x^\frac{3}{2} \right) dx
= \frac{13}{4}
\]

And
\[
\iint_R y \, da = \int_0^1 \int_{2\sqrt{x}}^{5-x^2} y \, dy \, dx
= \int_0^1 \left( \frac{y^2}{2} \right)_{y=2\sqrt{x}}^{y=5-x^2} dx
= \frac{1}{2} \int_0^1 \left( \frac{5 - x^2}{2} \right)^2 - 4x \right) dx
= \frac{1}{2} \int_0^1 \left( \frac{25 - 10x^2 + x^4}{4} - 4x \right) dx
= \frac{1}{8} \int_0^1 \left( 25 - 16x - 10x^2 + x^4 \right) dx
= \frac{1}{8} \left( \frac{208}{15} \right)
= \frac{26}{15}
\]

Therefore the centroid is:
\[
(x, y) = \left( \frac{13}{40}, \frac{26}{15} \right)
\]
1.7 Volumes under a surface by double integration

We recall that if $z = f(x, y)$ and $f(x, y) \geq 0$ over its domain $D$, a subset of the $xy$-plane (or $z = 0$), the volume of that part of the surface $S$ which lies directly above the domain $D$ is given by:

$$\text{Volume under } S = \int \int_D z\,dA = \int \int_D f(x, y)dA$$

where $dA$ is the element of area on $D$

**Example 38** Find the volume cut from $9x^2 + 4y^2 + 36z = 36$ by the plane $z = 0$.

**IDEA**

1) **Find the projection of this surface on one of the coordinate axes.**

**Figure 38**

Here, say, $z = 0$. This gives the ellipse:

$$D = \{(x, y) : 9x^2 + 4y^2 = 36\}$$

2) **solve for $z$ in terms of $x$ and $y$**

Here:

$$z = 1 - \frac{x^2}{4} - \frac{y^2}{9} = f(x, y)$$

Note that for $z$ on $D$, $z = 0$ while for $z \text{ in } D$, $z \geq 0$

3) **So volume under $S$:**

$$\int \int_D \left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right)dA$$

4) **need to describe $D$: We need to introduce elliptical (plane) coordinates centered at $(0, 0)$**

$$x = ar\cos \Theta, \quad y = br\sin \Theta; \quad r^2 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$$

**Figure 39**

The Jacobian is: $abr$

Here $a = 2, b = 3$

Therefore $1 - \frac{x^2}{4} - \frac{y^2}{9} = 1 - r^2$
5) **Volume**

\[
\int \int_{D} \left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right) dA = \int_{0}^{2\pi} \int_{0}^{1} (1 - r^2) \left(1 \cdot b \cdot r\right) dr d\theta
\]

\[
= \int_{0}^{2\pi} \int_{0}^{1} (1 - r^2) \cdot 6r dr d\theta
\]

\[
= 12\pi \int_{0}^{1} (1 - r^2) r dr
\]

\[
= 12\pi \cdot \left(\frac{1}{2} - \frac{1}{4}\right)
\]

\[
= \frac{3\pi}{4}
\]

Remark: S can be "reconstructed" via its "projections" eg.

\[S = \{(x, y, z) : 9x^2 + 4y^2 + 36z = 36\}\]

Projection of S onto:

- **xy-plane** is: (set \(z = 0\)) \(\Rightarrow 4x^2 + 9y^2 = 36\)
- **xz-plane** is: (set \(y = 0\)) \(\Rightarrow 9x^2 + 36z = 36\)
- **yz-plane** is: (set \(x = 0\)) \(\Rightarrow 4y^2 + 36z = 36\)

**Figure 40**

The surface S is reconstructed (basically) by superimposing the three projections. the more projections, the better the schematic representation of S.

**Figure 41**

**Example 39** Find the volume in the first octant bounded by \(xy = 4z\), \(y = x\) and \(x = 4\).

Here \(z = \frac{xy}{4} = f(x, y) \geq 0\) for \((x, y)\) in D.

Volume = \(\int \int_{D} z dA\) because \(x > 0\) over D.

\[Vol = \int \int_{D} \left(\frac{xy}{4}\right) dA\]

Stick with cartesian coordinates, because \(z = \left(\frac{xy}{4}\right)\) has an "interesting form" and because D is a "triangle" so:

\[D = \{(x, y) : 0 \leq y \leq x, \ 0 \leq x \leq 4\}\]
or
\[ D = \{(x, y) : y \leq x \leq 4, \ 0 \leq y \leq 4\} \]

**Using Vertical Slices:**

\[
\text{vol} = \int_0^4 \int_0^x \frac{xy}{4} \, dy \, dx
\]
\[
= \frac{1}{4} \int_0^4 x \cdot \left( \int_0^x y \, dy \right) \, dx
\]
\[
= \frac{1}{4} \int_0^4 x \cdot \left[ \frac{x^2}{2} - 0 \right] \, dx
\]
\[
= \frac{1}{8} \int_0^4 x^3 \, dx
\]
\[
= \left[ \frac{x^4}{8} \right]_0^4
\]
\[
= \frac{1}{8} (4^3)
\]
\[
= \frac{1}{8}
\]

**Example 40**  
*\(D\) is found by projection of \(S\) on \(xy\)-plane, \(z = 0\).* 

The domain of \(f(x, y) = \frac{xy}{4}\) is clearly all \((x, y)\) on the \(xy\)-plane.

We need to restrict \(D\) to that part of \(xy\)-plane which is bounded by:

\(y = x\) (projection of plane \(y = x\) on \(z = 0\))
\(x = 4\) (projection of plane \(x = 4\) on \(z = 0\))
\(y = 0\) (part of projection of \(z = xy\) on \(z = 0\))

**Figure 45**

Thus volume under \(S\) is:
\[
\int \int_D \left( \frac{xy}{4} \right) \, dA
\]

**Figure 46**

Now:
\[ D = \{(x, y) : 0 \leq y \leq x, \ 0 \leq x \leq 4\} \]

So the volume under \(S\) is:
The ABC’s of Calculus

\[
\int_0^4 \int_0^x \frac{xy}{4} dy dx = \frac{1}{4} \int_0^4 \left\{ \int_0^x xy dy \right\} dx \\
= \ldots \ldots \\
= \frac{1}{8} \int_0^4 x^3 dx \\
= 8
\]

or, using horizontal slices,

\[
vol = \int_0^4 \int_y^4 \frac{xy}{4} dx dy \\
= \frac{1}{4} \int_y^4 y(\int_x^4 x dx) dy \\
= \frac{1}{8} \int_0^4 y(16 - y^2) dy \\
= \frac{1}{8} [128 - 64] \\
= \frac{64}{8} \\
= 8
\]

Example 41 Find the volume common to the cylinders \(x^2 + y^2 = 16\) and \(x^2 + z^2 = 16\).

\textbf{Idea:} We look at the cylinder \(x^2 + z^2 = 16\) and consider the projection of \(x^2 + y^2 = 16\) on the \(xy\)-plane or \(z = 0\).

\textbf{Figure 47}

the projection of \(x^2 + y^2 = 16\) on \(z = 0\) is the circle \(x^2 + y^2 = 16\). The projection of \(x^2 + z^2 = 16\) on \(z = 0\) is the region bounded by the parallel lines \(x = \pm 4\). So, the projection of the solid of intersection is simply:

\[
D = \{(x, y) : 0 \leq x^2 + y^2 \leq 16\} \\
= \{(x, y) : -\sqrt{16 - x^2} \leq y \leq \sqrt{16 - x^2}, \ -4 \leq x \leq +4\}
\]

Hence the volume of the solid:

\[
= 2 \int \int_D \frac{\sqrt{16 - x^2}}{z} dA
\]

\textbf{Figure 48}
because there is a symmetric part of the solid below $z = 0$, (as $z = \pm \sqrt{16 - x^2}$)

Thus:

$$2 \int \int_D \sqrt{16 - x^2} \, dA = 2 \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (\sqrt{16 - x^2}) \, dy \, dx$$

$$= 2 \int_{-4}^{4} 2(16 - x^2) \, dx$$

$$= 4 \int_{-4}^{4} (16 - x^2) \, dx$$

$$= 8 \int_{0}^{4} (16 - x^2) \, dx \quad \text{(because integrand is an even function)}$$

$$= 8 \left[ 16x - \frac{x^3}{3} \right]_0^4$$

$$= 8 \cdot \frac{128}{3}$$

$$= \frac{1024}{3}$$

**Example 42** Find the volume inside $\rho = 2$ and outside the cone $z^2 = \rho^2$. (cylindrical coordinates)

**Figure 49**

The projection of the solid on $z = 0$ is given by the inside of the circle $\rho = 2$.

There are four identical parts making up the required volume and so it suffices to find the volume of that part of the solid which lies above the semi-circle $D$ shown in the figure.

Next,

$$\text{volume of solid} = \int \int_D z \, dA$$

$$= 4 \int \int_D \rho \, dA \quad (z^2 = \rho^2 \Rightarrow z = \rho \text{ above } D)$$

Recall $z^2 = \rho^2 = x^2 + y^2$ (in cylindrical coordinates). since $D$ is on the $xy$-plane $z = 0$, it is best to pick a description of $D$ in polar coordinates. Therefore:

$$D = \{ (\rho, \Theta) : 0 \leq \rho \leq 2, \quad 0 \leq \Theta \leq \frac{\pi}{2} \}$$

Therefore:

$$\text{Volume} = 4 \int \int_D \rho (\rho \, d\rho \, d\Theta)$$
\[
= 4 \int_0^\pi \left( \int_0^2 \rho^2 \, d\rho \right) \, d\Theta
= 4 \int_0^\pi \frac{8}{3} \, d\Theta
= \frac{32\pi}{3}
\]

**Example 43** Find the volume common to \( r^2 + z^2 = a^2 \) and \( r = a \sin \Theta \) (cylindrical coordinates).

Note \( r^2 + z^2 = a^2 \) is a sphere centered at \((0, 0, 0)\) of radius \( a \) (since \( r^2 = x^2 + y^2 \)).

Furthermore, \( r = a \sin \Theta \) \( \iff \sqrt{x^2 + y^2} = a \frac{y}{\sqrt{x^2 + y^2}} \) (polar coordinates). Or,
\[
x^2 + y^2 = ay, \text{ But } x^2 + y^2 - ay = 0 \Rightarrow x^2 + (y - \frac{a}{2})^2 = \frac{a^2}{4} \text{ (completing the square)}.
\]

Therefore:
\[
r = a \sin \Theta \iff (x - 0)^2 + (y - \frac{a}{2})^2 = \frac{a^2}{4}
\]
And this is a cylinder centered at \((0, \frac{a}{2})\) of radius \( \frac{a}{2} \).

**Figure 50**
The resulting volume is shown in the diagram.

Therefore the volume of the solid:
\[
= 4 \int \int_D z \, dA
\]

where \( D = \{(r, \Theta) : 0 \leq r \leq a \sin \Theta, \quad 0 \leq \Theta \leq \frac{\pi}{2}\} \) centered at \((0, \frac{a}{2})\) in polar coordinates. Note that \( \theta \leq \frac{\pi}{2} \) and **not** \( \Theta \leq 2\pi \) (Why?? else \( r < 0 \)).

Or
\[
Volume = 4 \int_0^{\frac{\pi}{2}} \int_0^{a \sin \Theta} \sqrt{a^2 - r^2} \, r \, dr \, d\Theta
\]
(Why? because volume = \( 4 \times \) (volume of a half-circular region))

\[
= 4 \int_0^{\frac{\pi}{2}} \left\{ \int_0^{a \sin \Theta} r(a^2 - r^2)^{\frac{3}{2}} \, dr \right\} d\Theta
= \frac{4}{(-2)} \int_0^{\frac{\pi}{2}} \left\{ \int_0^{a \sin \Theta} (-2r)(a^2 - r^2)^{\frac{3}{2}} \, dr \right\} d\Theta
\]
(\( u = a^2 - r^2, \quad du = -2rdr \))
\[
= (-2) \int_0^{\frac{\pi}{2}} \left\{ \int_{a^2}^{u^{\frac{3}{2}}} u^2 \, du \right\} d\Theta
\]
\[ r = 0 \rightarrow u = a^2 \quad r = a \sin \Theta \rightarrow u = a^2 \cos^2 \Theta \]

\[
\begin{align*}
&= (-2) \int_0^{\pi/2} \left\{ \frac{2}{3} u^2 \right\} a^2 \cos^2 \Theta \, d\Theta \\
&= \frac{4(-1)}{3} \int_0^{\pi/2} \{a^3 \cos^3 \Theta - a^3\} \, d\Theta \\
&= \frac{-4a^3}{3} \int_0^{\pi/2} (1 - \sin^2 \Theta) \cos \Theta \, d\Theta \\
&= \frac{-4a^3}{3} + \frac{4a^3}{3} \int_0^{\pi/2} \sin^2 \Theta \cos \Theta \, d\Theta + \frac{2a^3\pi}{3}
\end{align*}
\]

\[
\begin{align*}
&= \frac{-4a^3}{3} + \frac{4a^3}{3} \cdot \left[ \frac{\sin^3 \Theta \sqrt{3}}{3} \right] + \frac{2a^3\pi}{3} \\
&= \frac{-4a^3}{3} + \frac{4a^3}{9} + \frac{2a^3\pi}{3} \\
&= \frac{2a^3}{9}(3\pi - 4)
\end{align*}
\]

**Example 44** Find the volume cut from paraboloid \(4x^2 + y^2 = 4z\) by the plane \(z - y = 2\).

Note that paraboloid is \(z = x^2 + \frac{y^2}{4}\) and plane \(z = y + 2\). Therefore the curve of intersection of the two surfaces is (equates \(z\)'s):

\[ y + 2 = x^2 + \frac{y^2}{4} \]

Collecting terms we obtain (after completing the square):

\[ 4x^2 + (y - 2)^2 - 12 = 0 \]

Or:

\[ \epsilon : \frac{x^2}{(\sqrt{3})^2} + \frac{(y - 2)^2}{(\sqrt{12})^2} = 1 \]

**Figure 51**

Which is an ellipse. So, the projection of this solid of intersection on the plane \(z = 0\) is given by This ellipse, whose center is at \((0, 2)\).

The paraboloid is basically constructed by its projections:

- \(xz\)-plane \((y = 0)\): \(z = x^2\)
- \(yz\)-plane \((x = 0)\): \(z = \frac{y^2}{4}\)

**Figure 52**

To compute the volume of this solid we compute the height of a typical vertical fibre which intersects the solid.
In our case (height of fibre) = \((y + 2) - (x^2 + \frac{y^2}{4})\) (= difference in z-coordinates)

So a volume element = \(((y + 2) - ((x^2 + \frac{y^2}{4}))dA\)

Where \(dA = \) element of area on the projection \(D\) of the solid on \(z = 0\). (or the interior of \(\epsilon\)). Therefore:

\[
\text{Volume} = \int \int_D \left( (y + 2) - ((x^2 + \frac{y^2}{4}) \right) dA
\]

We introduce elliptical coordinates centered at \((0, 2)\) (Why? because our \(D\) is centered there). 

\[\epsilon : \left(\frac{x}{\sqrt{3}}\right)^2 + \left(\frac{y - 2}{\sqrt{12}}\right)^2 = 1\]

So, let:

\[
x = \sqrt{3}r \cos \Theta
\]
\[
y = 2 + \sqrt{12}r \sin \Theta
\]

and the

\[\text{Jacobian} = \sqrt{3} \cdot \sqrt{12}r\]

**Figure 53**

(then \(\left(\frac{x}{\sqrt{3}}\right)^2 + \left(\frac{y - 2}{\sqrt{12}}\right)^2 = r^2\) the basic equation of our \(\epsilon\) occurs when \(r = 1\))

So in our coordinate system

\[D = \{(r, \Theta) : 0 \leq r \leq 1, \ 0 \leq \Theta \leq 2\pi\}\]

and thus:

\[
\text{Volume} = \int \int_D \left( (y+2) - (x^2 + \frac{y^2}{4}) \right) dA
= \int_0^{2\pi} \int_0^1 3(1-r^2) \cdot \sqrt{3\sqrt{12}} r dr d\Theta
\]

(area element in new coord. syst.

(Also \(y + 2 - x^2 - \frac{y^2}{4} = \ldots = 3(1-r^2)\) as defined above).

\[
\text{Volume} = \int_0^{2\pi} \int_0^1 3(1-r^2) \cdot 6r dr d\Theta \quad (\sqrt{3\sqrt{12}} = 6)
\]
\[
= 36\pi \int_0^1 (1-r^2) \cdot r dr
\]
\[
= 36\pi \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\bigg|_{r=0}^{r=1}
\]
\[
= 36\pi \left(\frac{1}{2} - \frac{1}{4}\right)
\]
\[
= 36\pi \cdot \frac{1}{4}
\]
\[
= \frac{9\pi}{2}
\]
Remark: for elliptical coordinates centered at \((x_0, y_0)\) based on the ellipse \((\frac{x-x_0}{a})^2 + (\frac{y-y_0}{b})^2 = 1 : \epsilon\)

**Figure 54**

we set

\[
\begin{align*}
  x &= x_0 + ar \cos \Theta \\
  y &= y_0 + br \sin \Theta \\
  \text{Jacobian} &= abr \quad (\text{as before}) \text{ etc.}
\end{align*}
\]