Exercises on Fourier Series

Exercise Set 1

1. Find the Fourier series of the function f defined by

$$f(x) = \begin{cases} -1 & \text{if} \quad -\pi < x < 0, \\ 1 & \text{if} \quad 0 < x < \pi. \end{cases}$$

and f has period 2π. What does the Fourier series converge to at $x = 0$?

Answer: $f(x) \sim \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{(2n+1)}$. The series converges to 0. So, in order to make the Fourier series converge to $f(x)$ for all x we must define $f(0) = 0$.

2. What is the Fourier series of the function f of period 2π defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad -\pi < x < 0, \\ 3 & \text{if} \quad 0 < x < \pi. \end{cases}$$

What does the series converge to when $x = 0$?

Answer: $f(x) \sim 2 + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{(2n+1)}$. The series converges to 2, that is, the average value of f around 0, namely, $(1+3)/2 = 2$.

3. Let h be a given number in the interval $(0, \pi)$. Find the Fourier cosine series of the function

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < h, \\ 0 & \text{if} \quad h < x < \pi. \end{cases}$$

Answer: $f(x) \sim \frac{2h}{\pi} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin(nh)}{nh} \cos nx \right\}$.

4. Calculate the Fourier sine series of the function defined by $f(x) = x(\pi - x)$ on $(0, \pi)$. Use its Fourier representation to find the value of the infinite series
$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \dots.$$
Answer: $f(x) \sim \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3}$. Set $x = \frac{\pi}{2}$ and rearrange terms to get the value $\frac{\pi^3}{32}$.

5. Let h be a given number in the interval $(0, \frac{\pi}{2})$. Find the Fourier cosine series representation of
$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{2h - x}{2h} & \text{if } 0 < x < 2h, \\ 0 & \text{if } 2h < x < \pi. \end{cases}$$
where f is of period 2π.

Answer: $f(x) \sim \frac{2h}{\pi} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{\sin nh}{nh} \right)^2 \cos nx \right\}$

6. What is the Fourier sine series of $f(x) = \frac{\pi}{4} - \frac{x}{2}$, where $0 < x < \pi$.

Answer: $f(x) \sim \sum_{n=1}^{\infty} \frac{\sin 2nx}{2n}$.

7. What is the Fourier cosine series of $f(x) = \frac{\pi}{4} - \frac{x}{2}$, where $0 < x < \pi$.

Answer: $f(x) \sim \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2}$.

8. What is the Fourier sine series of $f(x) = x^2$ where $0 < x < \pi$.

Answer: $f(x) \sim \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \left\{ \frac{\pi^2}{n} + \frac{2}{n^3} \left((-1)^n - 1 \right) \right\} \sin nx$.

9. Find the Fourier series of $f(x) = |x|$ where $-L < x < L$.

Answer: $f(x) \sim \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos\left(\frac{(2n+1)\pi x}{L} \right)$.

10. Calculate the Fourier series of $f(x) = x^2$ where $0 < x < 2\pi$ and f has period 2π.

Answer: $f(x) \sim \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} - 4\pi \sum_{n=1}^{\infty} \frac{\sin nx}{n}$.

11. The function f is defined by $f(x) = e^x$ for $-L < x < L$. Find its Fourier series.

    Answer: $f(x) \sim \dfrac{e^L - e^{-L}}{2L} + L(e^L - e^{-L}) \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{L^2 + n^2\pi^2} \cos\left(\frac{n\pi x}{L}\right) +$
    $\pi(e^L - e^{-L}) \displaystyle\sum_{n=1}^{\infty} \frac{n\,(-1)^{n-1}}{L^2 + n^2\pi^2} \sin\left(\frac{n\pi x}{L}\right).$

12. Let a be a given integer. The function f is defined by $f(x) = \sin ax$ for $0 < x < \pi$. Find its Fourier cosine series.

    Answer:

$$\sin ax \sim \begin{cases} \dfrac{4a}{\pi} \displaystyle\sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{a^2 - (2n+1)^2} & \text{if } a \text{ is even,} \\[2ex] \dfrac{4a}{\pi} \left\{ \dfrac{1}{2a^2} + \displaystyle\sum_{n=1}^{\infty} \frac{\cos 2nx}{a^2 - 4n^2} \right\} & \text{if } a \text{ is odd.} \end{cases}$$

13. A function f has the property that $f(x+\pi) = -f(x)$ for all x. Show that all its even Fourier coefficients are zero (*i.e.*, $a_0 = a_2 = a_4 = a_6 = \ldots = 0$, $b_2 = b_4 = b_6 = \ldots = 0$).
    Hint: Show that f must be periodic with period 2π.

14. A function f satisfies the two conditions

$$f(-x) = f(x)$$

    and $f(x+\pi) = -f(x)$ for all x. Show that its Fourier coefficients satisfy $a_0 = a_2 = a_4 = a_6 = \ldots = 0$, $b_1 = b_2 = b_3 = b_4 = \ldots = 0$).

15. Let f be a function with the properties

$$f(-x) = -f(x)$$

    and $f(x+\pi) = -f(x)$ for all x. Show that its Fourier coefficients satisfy $a_0 = a_1 = a_2 = a_3 = \ldots = 0$, $b_2 = b_4 = b_6 = \ldots = 0$).

Suggested Homework Set 1 *Do problems 1, 3, 4, 5, 10, 13*