Computer Science CSCC51 | Spring 1999
Scarborough Campus | University of Toronto

Homework Assignment #4
Due: Wednesday April 7 1999, in lecture
Worth: 8%

1. (30 points) Consider the integral $I = \int_0^1 \frac{1}{1+x^2} dx$.

(a) Approximate I using the following quadrature rules:

(a.1) midpoint
(a.2) trapezoid
(a.3) Simpson's
(a.4) corrected trapezoid
(a.5) Gauss based on 2 nodes
(a.6) Gauss based on 4 nodes

Show your calculations.

(b) Find (by hand) the exact value of the integral I and compute the error (including its sign) for each of the quadrature rules in part a). Comment on the magnitude and the sign of the error of each rule (e.g., compare midpoint with trapezoid, Simpson's with corrected trapezoid, and the Gauss rules with Simpson's; compare with the bounds expected from theory).

Note: This question is to be done mainly by hand calculations. Doing hand calculations with Gauss rules' points and weights is quite error-prone. It is advisable to implement these rules in MATLAB, using the weights and points for the interval [-1, 1], as found in the file `Gauss` in the webpage of the course, and an appropriate transformation. Show the transformation you have used. Hand-in a hard-copy of your code, if you have programmed any rules.

2. (20 points)

(a) Convert the following system of differential equations into an Initial Value Problem (IVP) for a system of first order ODEs. Write the IVP in standard form. Indicate the vector of unknown functions and the initial conditions.

$$\begin{aligned} x'' &= (\sin(t) - 4)x' + \frac{x}{1+y^2} \\ y' &= -y\cos(t) - \frac{x'}{1+t^2} \\ x(0) &= 3,\ y(0) = 4,\ x'(0) = -1. \end{aligned}$$

Indicate the Jacobian of the system.

(b) Apply one step of forward Euler's method with step-size $h = 0.1$ to obtain an approximate value of the components of the unknown vector at time $t_1 = 0.1$.

This question is to be done by hand calculations.

3. (50 points) Consider a simple ecosystem consisting of rabbits (prey) and foxes (predators). Assume that rabbits have an infinite supply of grass (food) and foxes prey upon the rabbits for their food. A classical mathematical model for studying the population dynamics of this ecosystem is given by the following pair of nonlinear first order differential equations, due to Volterra and Lotka (early this century):

$$\begin{aligned} \frac{dr}{dt} &= b_r r - d_r r f \\ \frac{df}{dt} &= -d_f f + b_f r f, \end{aligned}$$

where $r(t)$ and $f(t)$ are the number of (alive) rabbits and foxes, respectively, at time t, and b_r, d_r, d_f and b_f are given positive constants representing the birth rate of rabbits, death rate of rabbits, death rate of foxes and birth rate of foxes, respectively. [Often, d_r and b_f are taken to be equal, in which case they represent the rate at which foxes and rabbits encounter each other.]

With initial conditions $r(0) = 33$ and $f(0) = 19$ (the numbers are normalized and we treat them as continuous variables), $b_r = 2$, $d_r = b_f = 0.01$, and $d_f = 1$, integrate the system of ODEs in the time interval $[0, 15]$, using MATLAB's function `ode45`.

We would like to compute the local maxima and minima of the computed $r(t)$ and $f(t)$. For this reason, consider the following technique. Let t_{i-1}, t_i and t_{i+1} be any three consecutive points in time, on which the computed solution is available. Let r_{i-1}, r_i and r_{i+1} be the respective computed solution for the rabbits. Based on the three data, $\{(t_{i-1}, r_{i-1}), (t_i, r_i), (t_{i+1}, r_{i+1})\}$, construct a quadratic interpolant $q(x)$, calculate its derivative $q'(x)$, and find its root, say ρ. If ρ is in $[t_{i-1}, t_{i+1}]$, then ρ is taken as a local maximum or minimum depending on the sign of $q''(\rho)$. The same technique can be applied to the computed $f(t)$.

Write a MATLAB function that calculates the local maxima and minima of the computed $r(t)$ and $f(t)$, using the above technique. The function should return the value of the points in time where a local maximum or minimum of r or f occurs as well as the value of the maximum or minimum.

Output the results in the following format:

Time			$\cdots$	
Rabbits max			$\cdots$	
Time			$\cdots$	
Rabbits min			$\cdots$	
Time			$\cdots$	
Foxes max			$\cdots$	
Time			$\cdots$	
Foxes max			$\cdots$	

Graph the computed $r(t)$ and $f(t)$ versus t in a single plot. Use a solid line for rabbits and a dashed one for foxes. On the same plot, use MATLAB to write a "+" for each local maximum of r or f and a "o" for each local minimum of r or f. [Use the MATLAB function `text` to write a symbol in an already displayed plot.]

In a separate plot, graph $f(t)$ versus $r(t)$. [Such a plot is often referred to as a phase plot.]

Comment on the plots and give a physical interpretation of what you observe in a few words.

Hand in a hard-copy of your code, the 2 plots and the output, together with your comments.

Note: An example of using MATLAB's function `ode45` is found in the file `ode.m` in the webpage of the course.