Computer Science CSCC51 | Spring 1999
Scarborough Campus | University of Toronto

Homework Assignment #3
Due: Monday March 22 1999, in lecture
Worth: 8%

1. (25 points)

(a) (5 points) Consider a continuous inner product in $[0, 1]$ using a weight function $w(x) = 1$. It is known that the polynomials $s_0(x)$, $s_1(x)$ and $s_2(x)$ of degree 0, 1 and 2, respectively, that are orthogonal to each other with respect to above (continuous) inner product and monic are $s_0(x) = 1$, $s_1(x) = x - \frac{1}{2}$ and $s_2(x) = x^2 - x + \frac{1}{6}$. Find the respective orthonormal polynomials $q_0(x)$, $q_1(x)$ and $q_2(x)$.

(b) (4 points) Find the polynomial $p_0(x)$ of degree 0, that is the constant least squares approximation to the function $f(x) = e^x$ with respect to the (continuous) inner product described in part (a). Give a numerical value to your answer. Explain how you constructed $p_0(x)$.

(c) (6 points) Find the polynomial $p_1(x)$ of degree 1, that is the linear least squares approximation to the function $f(x) = e^x$ with respect to the inner product described in part (a). Give numerical values to the coefficients of the polynomial $p_1(x)$. Explain how you constructed $p_1(x)$.

(d) (6 points) Consider a discrete inner product defined using n equidistant points in $[0, 1]$ and (discrete) weights $w_i = 1$, $i = 1, \dots, n$. Use MATLAB (i.e. the built-in functions `polyfit` and `polyval` to compute the polynomials $t_0(x)$ of degree 0, that are the constant least squares approximation to the function $f(x) = e^x$ with respect to the above (discrete) inner product, with $n = 1, \dots, 50$. Plot the value of the polynomials t_0 versus n. Use a solid line. In the same figure, plot the value of p_0 calculated in part (b) versus n (the same value for all n). Use a dotted line. Label your plot as "figure for question 1(d)". Use the built-in MATLAB functions `polyfit` and `polyval` to get the values of $t_0(x)$ and $t_1(x)$ on 50 equidistant evaluation points in $[0, 1]$.

(e) (4 points) In a few words, describe what you observe in the plot of part (d).

Note: This question is to be done by hand calculations with the exception of part (d).

2. (25 points)

(a) (3 points) Consider now a continuous inner product in $[0, 1]$ using a weight function $w(x) = e^{-x}$. Find the polynomial $q_{0w}(x)$ of degree 0, that is orthonormal with respect to the above (continuous) inner product. You may give the answer in terms of e. Explain how you constructed $q_{0w}(x)$.

(b) (5 points) Find the polynomial $q_{1w}(x)$ of degree 1, that is orthonormal (and orthogonal to $q_{0w}(x)$) with respect to the (continuous) inner product described in part (a). You may give the answer in terms of e. Explain how you constructed $q_{1w}(x)$.

(c) (3 points) Find the polynomial $p_{0w}(x)$ of degree 0, that is the constant least squares approximation to the function $f(x) = e^x$ with respect to the (continuous) inner product described in part (a). Give a numerical value to your answer. Explain how you constructed $p_{0w}(x)$.

(d) (5 points) Find the polynomial $p_{1w}(x)$ of degree 1, that is the linear least squares approximation to the function $f(x) = e^x$ with respect to the (continuous) inner product described in part (a). Give numerical values to the coefficients of the polynomial $p_{1w}(x)$. Explain how you constructed $p_{1w}(x)$.

(e) (2 points) In one figure, plot p_0 (from question 1), p_{0w} of part (c) and f in $[0, 1]$. Use a solid line for f, a dashed one for p_0 and a dotted one for p_{0w}. Label your plot as "figure for question 2(e)".

(f) (2 points) In one figure, plot p_1 (from question 1), p_{1w} of part (d) and f in $[0, 1]$. Use a solid line for f, a dashed one for p_1 and a dotted one for p_{1w}. Label your plot as "figure for question 2(f)".

(g) (5 points) In a few words, describe what you observe in the plots of parts (e) and (f).

Note: All but parts (e) and (f) are to be done by hand calculations. For parts (a) and (b) use the (first 2 steps of the) Gram-Schmidt algorithm.

3. (10 points) Consider interpolating the function $f(x) = \cosh(x) \equiv \frac{e^x + e^{-x}}{2}$ by a polynomial $p(x)$ of degree at most n, using $n + 1$ distinct data points, x_i, $i = 0, \ldots, n$, in $[-1, 1]$.

In terms of n, give a (good) upper bound for $\max_{-1 \le x \le 1} \frac{|f(x) - p(x)|}{|f(x)|}$ that holds when the x_i, $i = 0, \ldots, n$, are the roots of the Chebyshev polynomial of degree $n + 1$. Explain how you arrived at the bound.

Give a numerical value to each of your bound for $n = 4$ and for $n = 5$.

Note: This question is to be done by hand calculations.

4. (20 points) Consider approximating $I = \int_0^{\pi/2} \sin x dx$.

(a) (2 points) Compute the exact value of I.

(b) (3 points) Apply the trapezoidal rule to get an approximation T_1 to I.

(c) (4 points) Estimate a theoretical bound for the absolute value of the error $I - T_1$. Compute the actual error $I - T_1$, including its sign. Based on the formula for the trapezoidal rule error, discuss whether the sign of $I - T_1$ agrees with theory, and whether the bound for $|I - T_1|$ agrees with theory.

(d) (3 points) Apply the composite trapezoidal rule with 2 panels to get an approximation T_2 to I.

(e) (4 points) Estimate a theoretical bound for the absolute value of the error $I - T_2$. Compute the actual error $I - T_2$, including its sign. Based on the formula for the composite trapezoidal rule error, discuss whether the sign of $I - T_2$ agrees with theory, and whether the bound for $|I - T_2|$ agrees with theory.

(f) (4 points) Calculate $|I - T_1|/|I - T_2|$. Is this ratio close to what expected? Explain.

Note: This question is to be done by hand calculations.

5. (20 points)

(a) (4 points) Determine the constants w_1 and w_2 so that the rule of the form

$$\int_0^1 f(x)dx \approx w_1 f\left(\frac{1}{4}\right) + w_2 f\left(\frac{3}{4}\right)$$

is exact for all polynomials of degree ≤ 1.

(b) (4 points) Use transformation of variables in order to transform the rule constructed in part (a) into one which is appropriate for approximating $\int_a^b g(t)dt$ in a general interval (a, b). Show the transformation you have used.

(c) (4 points) What is the polynomial degree (degree of accuracy) of the rule constructed in part (b)? Explain.

(d) (4 points) Based on the rule constructed in part (b) obtain a composite (compound) quadrature rule for approximating $\int_a^b f(x)dx$ in a general interval (a, b) with n panels.

(e) (4 points) Consider now the trapezoidal rule

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\left(f(a) + f(b)\right).$$

Which rule has higher polynomial degree, the trapezoidal rule or the rule constructed in part (b)? Explain.

Which rule is preferable to use in a composite form and why?

Note: This question is to be done by hand calculations.