Construction of a clamped cubic spline interpolant

Problem:
Let the (distinct) knots x_i, $i = 0, \cdots, n$, be given. Without loss of generality, assume that the knots are in ascending order.
Let the data y_i, $i = 0, \cdots, n$, corresponding to the knots x_i, $i = 0, \cdots, n$, respectively, be given.
Let also y'_0 and y'_n corresponding to x_0 and x_n, respectively, be given.
Construct the clamped cubic spline with respect to the knots x_i, $i = 0, \cdots, n$, interpolating the data y_i, $i = 0, \cdots, n$ and y'_0 and y'_n.

A solution:
A convenient representation for a cubic piecewise polynomial $C(x)$ defined with respect to the knots x_i, $i = 0, \cdots, n$, is the following branch-function:

$$C(x) = \left\{ C_i(x) \quad \text{for} \quad x_i \le x \le x_{i+1} \right.$$

where the index i of the branches runs from 0 to $n-1$ and

$$C_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3$$

Note that

$$C'_i(x) = b_i + 2c_i(x-x_i) + 3d_i(x-x_i)^2$$

and

$$C''_i(x) = 2c_i + 6d_i(x-x_i)$$

Let the step-sizes be defined by $h_i = x_{i+1} - x_i$, $i = 0, \cdots, n-1$.

In order to ensure that $C(x)$ is a cubic spline, we impose continuity conditions on the interior knots for $C(x)$ and its first and second derivatives:

- Continuity conditions for $C(x)$:

$$C_{i+1}(x_{i+1}) = C_i(x_{i+1}) \ , \quad i = 0, \cdots, n-2$$

Thus

$$a_{i+1} = a_i + b_i h_i + c_i h_i^2 + d_i h_i^3 \ , \quad i = 0, \cdots, n-2 \qquad (1)$$

- Continuity conditions for $C'(x)$:

$$C'_{i+1}(x_{i+1}) = C'_i(x_{i+1}) \ , \quad i = 0, \cdots, n-2$$

Thus

$$b_{i+1} = b_i + 2c_i h_i + 3d_i h_i^2 \ , \quad i = 0, \cdots, n-2 \qquad (2)$$

- Continuity conditions for $C''(x)$:

$$C''_{i+1}(x_{i+1}) = C''_i(x_{i+1}) \ , \quad i = 0, \cdots, n-2$$

Thus

$$2c_{i+1} = 2c_i + 6d_i h_i \ , \quad i = 0, \cdots, n-2$$

or equivalently

$$d_i = \frac{c_{i+1} - c_i}{3h_i} \ , \quad i = 0, \cdots, n-2 \qquad (3)$$

For convenience, we also let $c_n = C''(x_n)/2$. This is equivalent to $c_n = C''_{n-1}(x_n)/2$ or $c_n = c_{n-1} + 3d_{n-1}h_{n-1}$ or $d_{n-1} = \dfrac{c_n - c_{n-1}}{3h_{n-1}}$. Thus (3) holds for $i = 0, \cdots, n-1$.

In order to ensure that $C(x)$ is interpolating the data y_i, $i = 0, \cdots, n$ at the knots x_i, $i = 0, \cdots, n$, we impose the interpolating conditions:

- Interpolating conditions:

$$C(x_i) = y_i \ , \quad i = 0, \cdots, n$$

or equivalently

$$C_i(x_i) = y_i \ , \quad i = 0, \cdots, n-1 \quad \text{and} \quad C_{n-1}(x_n) = y_n$$

or equivalently

$$a_i = y_i \ , \quad i = 0, \cdots, n-1 \quad \text{and} \qquad (4)$$
$$a_{n-1} + b_{n-1}h_{n-1} + c_{n-1}h_{n-1}^2 + d_{n-1}h_{n-1}^3 = y_n$$

In order to ensure that $C'(x)$ is interpolating the data y'_0, y'_n at the knots x_0, x_n, we impose the clamped boundary conditions:

- Clamped boundary conditions:

$$C'(x_0) = y'_0 \ , \quad C'(x_n) = y'_n$$

or equivalently

$$C'_0(x_0) = y'_0 \ , \quad C'_{n-1}(x_n) = y'_n$$

or equivalently

$$b_0 = y'_0 \ , \quad b_{n-1} + 2c_{n-1}h_{n-1} + 3d_{n-1}h_{n-1}^2 = y'_n \qquad (5)$$

After some manipulation, relations (1), (2), (3), (4) and (5) give rise to the following linear system:

$$\begin{bmatrix} 2h_0 & h_0 & & & & & \\ h_0 & 2(h_0+h_1) & h_1 & & & & \\ & h_1 & 2(h_1+h_2) & h_2 & & & \\ & & . & . & . & & \\ & & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ & & & & & h_{n-1} & 2h_{n-1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ . \\ c_{n-1} \\ c_n \end{bmatrix} =$$

$$= \begin{bmatrix} \dfrac{3}{h_0}(y_1-y_0) - 3y'_0 \\ \dfrac{3}{h_1}(y_2-y_1) - \dfrac{3}{h_0}(y_1-y_0) \\ \dfrac{3}{h_2}(y_3-y_2) - \dfrac{3}{h_1}(y_2-y_1) \\ . \\ \dfrac{3}{h_{n-1}}(y_n-y_{n-1}) - \dfrac{3}{h_{n-2}}(y_{n-1}-y_{n-2}) \\ -\dfrac{3}{h_{n-1}}(y_n-y_{n-1}) + 3y'_n \end{bmatrix} \qquad (6)$$

Note that the matrix of the linear system in (6) is of size $(n+1)\times(n+1)$ and is tridiagonal, symmetric and strictly diagonally dominant.
By solving the linear system (6), we compute c_i, $i = 0, \cdots, n$. Then
d_i, $i = 0, \cdots, n-1$, are recovered from (3) and
b_i, $i = 0, \cdots, n-1$, are recovered from (2) (with $b_0 = y'_0$), while
a_i, $i = 0, \cdots, n-1$, are already given by (4).