CSC 238F 1999, Assignment 4, St. George Campus, Day Section
Due: Nov. 24, 10:10 AM (beginning of tutorial).

On the cover page of your assignment, you must write **and sign** *the following statement: "I have read and understood the policy on collaboration in homework stated in the webpage of the course." Without such a signed statement your homework will not be marked.*

---

1. (10 marks) Consider the first-order language of arithmetic described on page 142 of the notes. Let $\mathcal{N}$ and $\mathcal{Z}$ be structures for this language, with domains $\mathbb{N}$ and $\mathbb{Z}$, respectively, and the standard meaning for the predicate symbols. More formally:

$$\begin{array}{ll} S^{\mathcal{N}} = \{(a,b,c) \in \mathbb{N}^3 : a+b=c\} & S^{\mathcal{Z}} = \{(a,b,c) \in \mathbb{Z}^3 : a+b=c\} \\ P^{\mathcal{N}} = \{(a,b,c) \in \mathbb{N}^3 : a\cdot b=c\} & P^{\mathcal{Z}} = \{(a,b,c) \in \mathbb{Z}^3 : a\cdot b=c\} \\ L^{\mathcal{N}} = \{(a,b) \in \mathbb{N}^2 : a<b\} & L^{\mathcal{Z}} = \{(a,b) \in \mathbb{Z}^2 : a<b\} \\ \approx^{\mathcal{N}} = \{(a,b) \in \mathbb{N}^2 : a=b\} & \approx^{\mathcal{Z}} = \{(a,b) \in \mathbb{Z}^2 : a=b\} \\ \mathit{0}^{\mathcal{N}} = 0 & \mathit{0}^{\mathcal{Z}} = 0 \\ \mathit{1}^{\mathcal{N}} = 1 & \mathit{1}^{\mathcal{Z}} = 1 \end{array}$$

For each of the sentences below, state whether it is true or false in each of $\mathcal{N}$ and $\mathcal{Z}$. Justify your answer by translating the formula into a statement (in precise English) about numbers, and then explain why that statement is true or false for natural numbers and for integers.

(a) $\exists x \, \forall y \, (L(x,y) \vee \approx(x,y))$

(b) $\exists x \, \forall y \, (L(y,x) \vee \approx(x,y))$

(c) $\forall x \, \exists y \, L(x,y)$

(d) $\forall x \, \exists y \, L(y,x)$

(e) $\forall x \, \forall y \, \Big(L(x,y) \rightarrow \forall u \, \forall v \, (P(x,x,u) \wedge P(y,y,v) \rightarrow L(u,v))\Big)$

2. (10 marks) Consider the same first-order language of arithmetic as in Question 1, enriched with a new constant symbol $\mathit{2}$ which is intended to represent the integer 2. In this question we will be working with this language in the structure $\mathcal{N}$ also defined in Question 1. The following formula $\mathit{Prime}(x)$ expresses the predicate "x is prime":

$$L(\mathit{1},x) \wedge \forall y \, \forall z \, \Big(P(y,z,x) \rightarrow (\approx(y,\mathit{1}) \vee \approx(z,\mathit{1}))\Big)$$

(a) ***Goldbach's conjecture*** asserts that every even integer greater than 2 can be expressed as the sum of two prime numbers. Nobody knows whether this is true or false. Write a formula to express Goldbach's conjecture. In your answer you may use the predicate $\mathit{Prime}(x)$ for which a formula was given above.

(b) ***Twin primes*** are prime numbers that are two apart; e.g., 3 and 5 are twin primes, as are 17 and 19. The ***twin-prime conjecture*** asserts that there are infinitely many twin primes. Nobody knows whether this is true or false. Write a formula to express the twin-prime conjecture. (**Hint:** You can say that there are infinitely many numbers with property P by saying that for each number there is a larger number with property P.)

(c) Write a formula to express Proposition 1.7 (page 27) in the notes.

3. (10 marks) For each of the following assertions, state whether it is true or false, and justify your answer. In (a)–(d) A and B are unary predicates in the first-order language.

(a) $\forall x\,(A(x) \rightarrow B(x))$ logically implies $\exists x\,(A(x) \wedge B(x))$.

(b) $\exists x\,(A(x) \rightarrow \neg B(x))$ is logically equivalent to $\neg\forall x\,(A(x) \wedge B(x))$.

(c) $\exists x\,A(x) \wedge \exists x\,\neg A(x)$ is logically equivalent to $\exists x\,(A(x) \wedge \neg A(x))$.

(d) $\exists x\,A(x) \vee \exists x\,\neg A(x)$ is logically equivalent to $\exists x\,(A(x) \vee \neg A(x))$.

(e) For any first-order formulas E and F such that x does not appear free in F, $\forall x\,E \leftrightarrow F$ is logically equivalent to $\forall x\,(E \leftrightarrow F)$.

4. (10 marks) Consider a relational database that describes an aspect of the activities in a university. Specifically, the database schema consists of the following relations:

- *Student*(s, n, a) — a tuple (s, n, a) belongs to this relation if the student whose SIN is s has name n and address a. We assume that the SIN uniquely identifies a student; however, there could be different students with the same name (or address).
- *Course*(c, n) — a tuple (c, n) belongs to this relation if there is a course whose code is c (e.g., CSCB38) and whose name is n (e.g., Discrete Mathematics).
- *Takes*(s, c, y, m) — a tuple (s, c, y, m) belongs to this relation if the student whose SIN is s took (or is now taking) course c in year y and received a mark of m. We assume that if a student is presently taking the course but has not completed it yet, the mark m has value I (for "incomplete").
- *Teaches*(n, c, y) — a tuple (n, c, y) belongs to this relation if the professor whose name is n taught (or is now teaching) course c in year y. We assume that each professor is uniquely identified by her/his name, and that each course is offered only once in each year.

Write formulas to express the following queries:

(a) Find the name and address of every student who took (or is taking) CSCB38 in 1999.

(b) Find the names of all courses taught by MacLean.

(c) Find the names of all students who received an A in CSCB70 when Panario taught the course.

(d) Find the names of all courses that were taught *only* by Rackoff.

(e) Find the names of all courses taught in 1999 by any professor who has never failed a student in any course that she/he ever taught. (A student fails a course, if her/his mark is F.)

5. (10 marks) Transform the following first-order formula into an equivalent PNF formula in which the quantifier-free part uses only the connectives $\vee$ and $\neg$. Demonstrate the steps through which you obtain your final formula.

$$\exists x\,\neg\exists y\,A(x,y) \;\rightarrow\; \exists y\,(B(x,y) \leftrightarrow \forall z\,C(z))$$