Substitution Tilings and Groupoids

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Introduction

We study tilings of $\mathbb{R}^2$ which are aperiodic, but not completely random.

Tiling $T \longrightarrow$ topological space Ω_T

Elements of Ω_T are tilings, and $\mathbb{R}^2$ acts by translating them.

We replace $(\Omega_T, \mathbb{R}^2)$ with a groupoid $\mathcal{R}_{punc}$ which captures all essential features of the action.

Tiling groupoid $\mathcal{R}_{punc} \longrightarrow$ groupoid C*-algebra $C^*(\mathcal{R}_{punc})$.

Properties of $C^*(\mathcal{R}_{punc}) \iff$ properties of T.

Tilings

Definition

A **tiling** T of $\mathbb{R}^2$ is a countable set $T = \{t_1, t_2, \dots\}$ of subsets of $\mathbb{R}^2$, called **tiles** such that

- Each tile is homeomorphic to the closed ball (they are usually polygons),
- $t_i \cap t_j$ has empty interior whenever $i \neq j$, and
- $\cup_{i=1}^{\infty} t_i = \mathbb{R}^2$.

- A **patch** is a finite subset of T. The **support** of a patch is the union of its tiles.
- If T is a tiling, $x \in \mathbb{R}^2$, $T + x$ is the tiling formed by translating every tile in T by x.
- T is **aperiodic** if $T + x \neq T$ for all $x \in \mathbb{R}^2 \setminus \{0\}$.

Substitution Rules

Frequently we have a finite number of "tile types".

$\mathcal{P} = \{p_1, p_2, \ldots, p_N\}$ is called a set of **prototiles** for T if $t \in T \implies t = p + x$ for some $p \in \mathcal{P}$ and $x \in \mathbb{R}^2$.

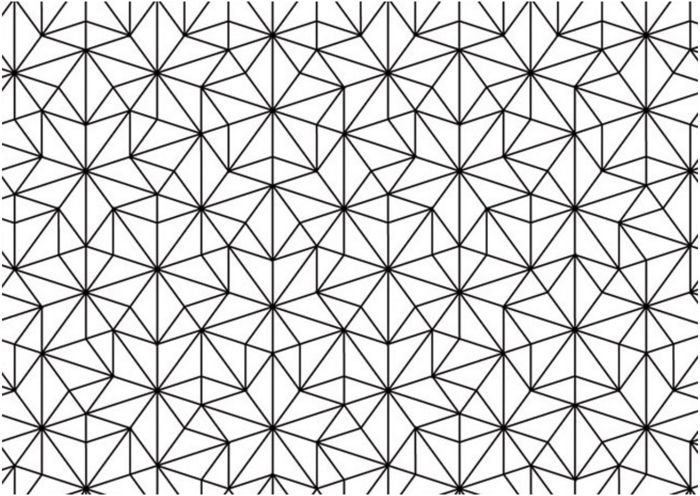
Definition

A **substitution rule** on a set of prototiles $\mathcal{P}$ consists of

- A scaling constant $\lambda > 1$
- A rule ω such that, for each $p \in \mathcal{P}$, $\omega(p)$ is a patch whose support is λp and whose tiles are translates of members of $\mathcal{P}$.

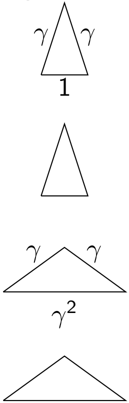
- ω can be applied to patches and tilings by applying it to each tile.
- ω can be iterated, since $\omega(p)$ is a patch.

Example: Penrose Tiling

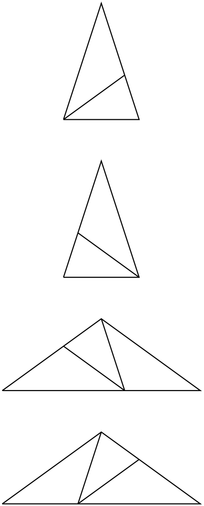


Example: Penrose Tiling

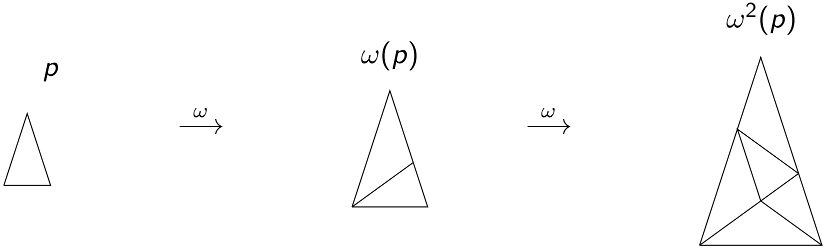
Prototiles
(+ rotates by $\frac{\pi}{5}$)
γ = golden ratio

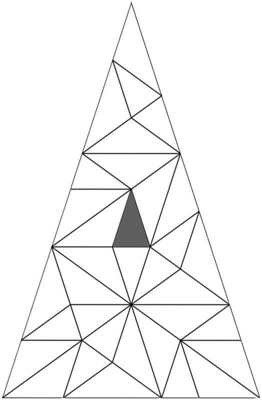


$\overset{\omega}{\overset{\lambda=\gamma}{\longrightarrow}}$



Example: Penrose Tiling



$\omega^4\Big($  $\Big) =$ 

$p \subset \omega^4(p) \subset \omega^8(p)$

$\omega^{4n}(p) \subset \omega^{4(n+1)}(p)$

Then

$$T = \bigcup_{n=1}^{\infty} \omega^{4n}(p)$$

is a tiling.

The Tiling Metric

The Tiling Metric

The distance between two tilings T and T' is less than ε if T and T' agree on a ball of radius $\frac{1}{\varepsilon}$ up to a small translation of at most ε. The distance $d(T, T')$ is then defined as the inf of all these ε (or 1 if no such ε exists).

There are essentially two ways that T and T' can be close:

1. $T' = T + x$ for some $|x| < \varepsilon$.
2. T' agrees with T exactly on a large ball around the origin, then disagrees elsewhere.

In most cases, 1 looks like a disc while 2 looks like a Cantor set (ie, totally disconnected, compact, no isolated points).

The Tiling Space

Definition

The **tiling space** associated with a tiling T, denoted Ω_T, is the completion of $T + \mathbb{R}^2 = \{T + x \mid x \in \mathbb{R}^2\}$ in the tiling metric. This is also called the **continuous hull** of T.

It's not obvious, but the elements of Ω_T are tilings.

Ω_T is the set of all tilings T' such that every patch in T' appears somewhere in T.

Properties of the tiling space

Definition

A tiling T is said to have **Finite Local Complexity** (FLC) if for every $r > 0$, the number of different patches (up to translation) of diameter r in T is finite .

If T has FLC, then Ω_T is compact.

Definition

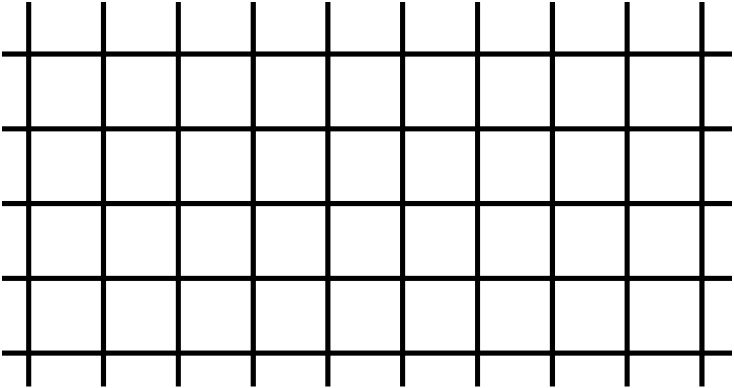
A substitution rule ω is said to be **primitive** if there exists some n such that such that $\omega^n(p_i)$ contains a copy of p_j for every $p_i, p_j \in \mathcal{P}$.

If T is formed by a primitive substitution rule, and $T' \in \Omega_T$, then $\Omega_{T'} = \Omega_T$.

T, T' both come from same primitive $\omega \implies \Omega_T = \Omega_{T'}$

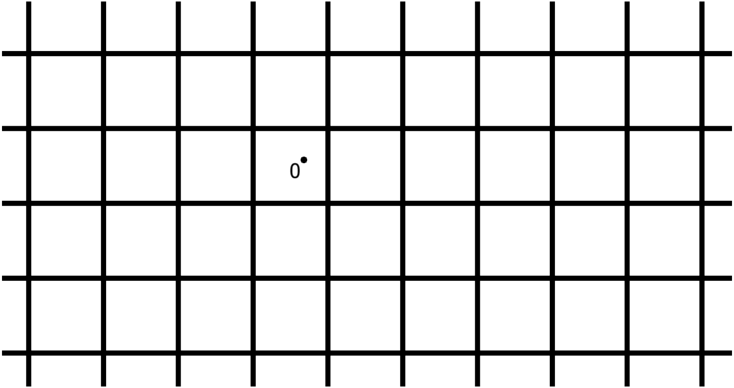
Replace $\Omega_T \to \Omega$.

Example: Grid



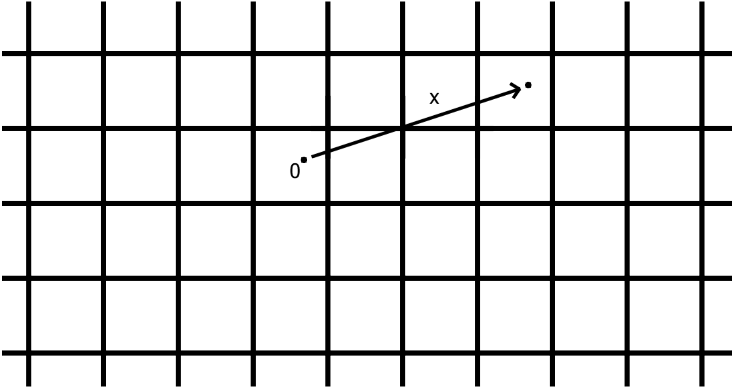
Infinite grid in $\mathbb{R}^2$

Example: Grid



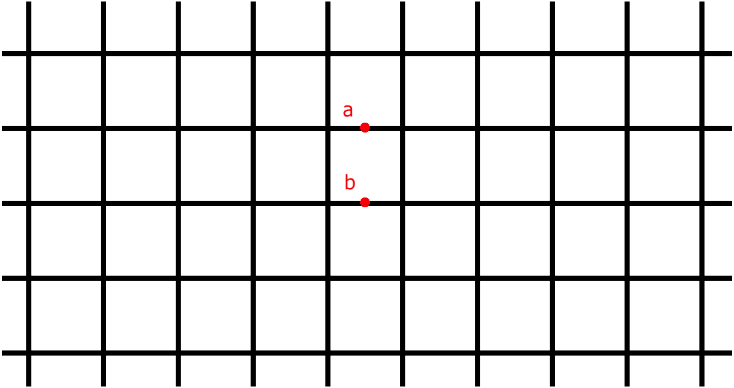
Placement of the origin in any square determines the tiling.

Example: Grid



Placement of the origin in any square determines the tiling. $T = T - x$

Example: Grid



a and b are the same in the tiling space $\implies \Omega_T \cong \mathbb{T}^2$

Example: Grid

The grid is periodic, and gives a somewhat boring tiling space.

Recall two ways that T and T' can be close:

1. $T' = T + x$ for some $|x| < \varepsilon$.
2. T' agrees with T exactly on a large ball around the origin, then disagrees elsewhere.

In the case of the grid, neighbourhoods consist of the first way only. The second way is much more interesting!

For this reason we assume finite local complexity, a primitive substitution rule, and that every tiling in Ω is aperiodic.

We produce a subspace of Ω to essentially make the first way vanish.

The discrete hull

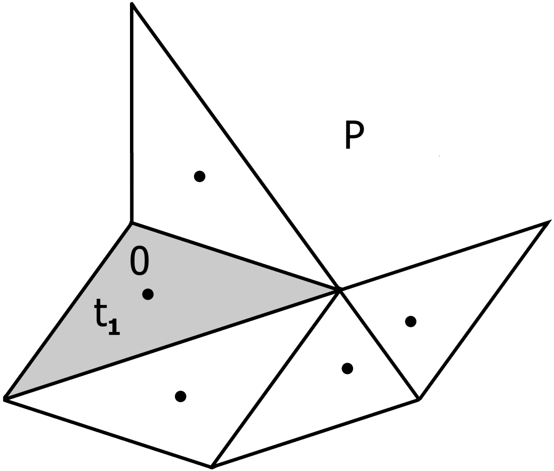
We replace each prototile $p \in \mathcal{P} \longrightarrow (p, x(p))$, where $x(p) \in$ the interior of p. The point $x(p)$ is called the **puncture** of p. If $t \in T$, then $t = p + y$ for some y and so we define $x(t) = x(p) + y$.

Define $\Omega_{punc} \subset \Omega$ as the set of all tilings $T \in \Omega$ such that the origin is on a puncture of a tile in T, ie, $x(t) = 0$ for some $t \in T$. Ω_{punc} is called the **discrete tiling space** or **discrete hull**.

Ω_{punc} is homeomorphic to a Cantor set (ie it is totally disconnected, compact, and has no isolated points).

Its topology is generated by clopen sets of the following form: if P is a patch and $t \in P$, then let

$$U(P, t) = \{T \in \Omega_{punc} \mid 0 \in t \in P \subset T\}$$



If T looks like this around the origin $0 \in \mathbb{R}^2$, then $T \in U(P, t_1)$.

The tiling groupoid

Let $\mathcal{R}_{punc} = \{(T, T+x) \mid T, T+x \in \Omega_{punc}\}$. Then $\mathcal{R}_{punc}$ is an equivalence relation.

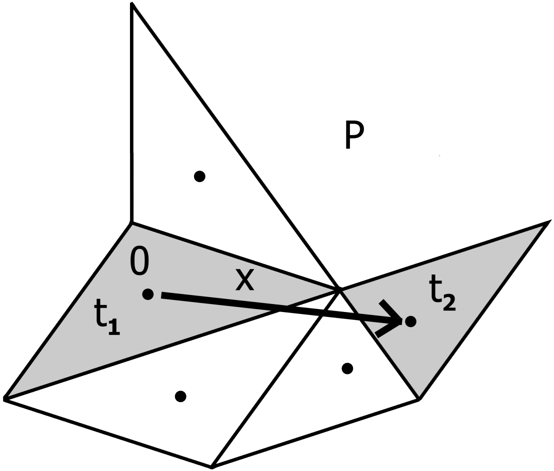
When given the topology inherited from $\Omega_{punc} \times \mathbb{R}^2$, $\mathcal{R}_{punc}$ is an étale equivalence relation.

The topology is generated by compact open sets of the following form: if P is a patch and $t_1, t_2 \in P$, then let

$$V(P, t_1, t_2) = \{(T, T-x) \mid T \in U(P, t_1), x(t_2) = x\}$$

When restricted to these neighbourhoods, the range and source maps are homeomorphisms, and

$$r(V(P, t_1, t_2)) = U(P, t_1), \quad s(V(P, t_1, t_2)) = U(P, t_2)$$



If $T \in U(P, t_1)$, then $(T, T - x) \in V(P, t_1, t_2)$.

The tiling groupoid

Properties of $\mathcal{R}_{punc}$:

- $r(T, T') = (T, T)$ and $s(T, T') = (T', T')$.
- The unit space $\mathcal{R}^0_{punc}$ is Ω_{punc} $\mathcal{R}_{punc}$ is r-**discrete** – Ω_{punc} is open in $\mathcal{R}_{punc}$.
- $\mathcal{R}_{punc}$ is **principal** (that is, $(r, s) : \mathcal{R}_{punc} \to \Omega_{punc} \times \Omega_{punc}$ is injective).
- $\mathcal{R}_{punc}$ is locally compact (indeed, there is a basis of its topology consisting of compact open sets).
- $\mathcal{R}_{punc}$ has a Haar system consisting of counting measures.

An AF-subgroupoid

If $p \in \mathcal{P}$, then $\omega^n(p)$ is a patch.

For each $n \in \mathbb{N}$, define

$$\mathcal{R}_n = \bigcup_{\substack{p \in \mathcal{P} \\ t_1, t_2 \in \omega^n(p)}} V(\omega^n(p), t_1, t_2)$$

$\mathcal{R}_n$ is a compact and open subequivalence relation of $\mathcal{R}_{punc}$.

$\mathcal{R}_n \subset \mathcal{R}_{n+1}$(it isn't obvious, but this depends on Ω not containing any periodic tilings).

Let $\mathcal{R}_{AF} = \cup_{n=1}^{\infty} \mathcal{R}_n$. $\mathcal{R}_{AF}$ is an increasing union of compact open groupoinds and is hence an AF-groupoid.

C*-algebras

Since $\mathcal{R}_{punc}$ is a principal étale groupoid, we can form its C*-algebra, $C^*(\mathcal{R}_{punc})$.

$C_c(\mathcal{R}_{punc})$ - the complex-valued compactly supported functions on $\mathcal{R}_{punc}$.

For $f, g \in C_c(\mathcal{R}_{punc})$, the convolution product and involution are

$$f * g(T, T') = \sum_{T'' \in [T]} f(T, T'')g(T'', T')$$

$$f^*(T, T') = \overline{f(T', T)}$$

$C_c(\mathcal{R}_{punc})$ is a $*$-algebra, and when completed in a suitable norm becomes $C^*(\mathcal{R}_{punc})$.

This algebra has a nice description in terms of patches and tiles.

C*-algebras

For an open neighbourhood $V(P, t_1, t_2)$, let $e(P, t_1, t_2)$ denote the characteristic function of this set.

Since these sets are clopen, these functions are continuous.

Let

$$\mathcal{E} = \mathrm{span}_{\mathbb{C}}\{e(P, t_1, t_2) \mid P \text{ is a patch in } T, t_1, t_2 \in P\}$$

This set becomes a dense $*-$subalgebra of $C^*(\mathcal{R}_{punc})$ with product and involution determined by the formulas

$$e(P, t_1, t_2)e(P', t_1', t_2') = \begin{cases} e(P \cup P', t_1, t_2') & \text{if } t_2 = t_1' \text{ and } P, P' \text{ agree} \\ 0 & \text{otherwise} \end{cases}$$

$$e(P, t_1, t_2)^* = e(P, t_2, t_1)$$

C^*-algebras

$e(P,t,t)e(P,t,t) = e(P,t,t) = e(P,t,t)^*$, so these elements are projections.

$$e(P,t_1,t_2)^*e(P,t_1,t_2) = e(P,t_2,t_1)e(P,t_1,t_2) = e(P,t_2,t_2)$$

$$e(P,t_1,t_2)e(P,t_1,t_2)^* = e(P,t_1,t_2)e(P,t_2,t_1) = e(P,t_1,t_1)$$

Thus each $e(P,t_1,t_2)$ is a partial isometry.

Finite dimensional subalgebras

Fix $n \in \mathbb{N}$ and $p \in \mathcal{P}$, and consider the patch $\omega^n(p)$.

For $t_1, t_2, t_1', t_2' \in \omega^n(p)$, we have

$$e(\omega^n(p), t_1, t_2)e(\omega^n(p), t_1', t_2') = \begin{cases} e(\omega^n(p), t_1, t_2') & \text{if } t_2 = t_1' \\ 0 & \text{otherwise} \end{cases}$$

Thus, the $e(\omega^n(p), t_1, t_2)$ act like matrix units, so if $k = |\omega^n(p)|$,

$$A_{n,p} = \mathrm{span}_{\mathbb{C}}\{e(\omega^n(p), t_1, t_2) \mid t_1, t_2 \in \omega^n(p)\} \cong \mathbb{M}_k$$

$$A_n = \bigoplus_{p \in \mathcal{P}} A_{n,p} = C^*(\mathcal{R}_n)$$

$$C^*(\mathcal{R}_{AF}) = \overline{\bigcup_n A_n}$$

K-theory

K_0 is a functor from C*-algebras to ordered abelian groups.

If C*-algebras are "non-commutative geometry", then K_0 is a non-commutative homology.

$$K_0(C^*(\mathcal{R}_{punc})) \cong \check{H}^0(\Omega) \oplus \check{H}^2(\Omega)$$