Computation of the Ruelle-Sullivan Map for Substitution Tilings

by

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B.Sc., University of Victoria, 2003

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

Abstract

We study the dynamics of tiling spaces through cohomology. An adaptation of the Čech-deRham theorem allows us to compute the Ruelle-Sullivan map for such spaces and consider its image together with cohomology as a more useful invariant than cohomology alone. Computation of the map is performed for the Penrose tiling and the Octagonal tiling.

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Chapter 1

Introduction

In the 60's and 70's, patterns were discovered in nature which were aperiodic, yet displayed some long-range order. These were usually crystaline in nature, but went against the laws of crystals as they were known at the time. These were studied with great interest, and eventually this phenomenon came to be called *aperiodic order*. Tilings are the major example of objects that can display aperiodic order. Much of the study of aperiodic order comes down to the study of certain tilings.

When one thinks of a tiling, what usually comes to mind is a collection of polygons fitting together to cover the plane. The mathematical definition of a tiling extends this to mean a collection of subsets of $\mathbb{R}^n$ homeomorphic to the closed unit ball in $\mathbb{R}^n$ whose interiors are pairwise disjoint and whose union is all of $\mathbb{R}^n$. From any tiling T we can form an $\mathbb{R}^n$ action on a topological space Ω_T; in this way we study aperiodic order though dynamics. As has been done in [AP] and elsewhere, one way of studying the order in these systems is though the cohomology. This provides some important invariants, but fails to show the whole picture. In [KP] the authors provide a way of obtaining a map on cohomology which can distinguish between two different $\mathbb{R}^n$-actions with the same cohomology. The goal of this thesis then is to compute this, the *Ruelle-Sullivan* map, for tiling systems.

Shortly after Chapter 2 begins we give the definition of a tiling given above with the hypothesis that all the tiles are translates of some member of a finite set $\{p_1, p_2, \ldots, p_N\}$; the p_i's are called *prototiles*. We also make some fairly standard constructions, including the notion of translating a tiling - the translate of a tiling is just the tiling obtained by translating each of the component tiles. The first non-intuitive construction is the definition of a metric on a collection of tilings - this metric basically states that two tilings are close if they agree up to a small translation on a large ball around the origin. We then form the tiling space Ω_T by taking T, taking all translates of it by vectors in $\mathbb{R}^n$ and completing this collection in the metric; the elements of this completion are shown to be tilings themselves. The space Ω_T is shown to be compact if we make a hypothesis on T called *finite local complexity* - that there are only a finite number of different looking patches in T of any given radius, up to translation. We then assume further that we have a substitution rule on our prototiles - we have a constant $\lambda > 1$ and, for each prototile, a rule for subdividing it into pieces, each of which is another prototile, scaled down by a factor of λ^{-1}. This idea extends to patches of tiles and whole tilings, so we can construct the dynamical system with the space Ω_T and the substitution map ω. Assumptions are made on the substitution so that ω is a homeomorphism and thus (Ω_T, ω) is a topological dynamical system.

We then construct a cell complex Γ from Ω_T as in [AP]. Basically, the n-cells of Γ are the prototiles with their faces identified if they are adjacent anywhere in any tiling in Ω_T. A map γ based on the substitution is defined on Γ and is shown to be onto, so we construct the inverse limit $\varprojlim_{\gamma} \Gamma$ and show it to be topologically conjugate to (Ω_T, ω) under the assumption that the substitution *forces its border* - this is explained in the section.

After defining cellular, Čech and dynamical cohomology, we begin to connect them for the case $n = 2$ in Chapter 4. To map cellular cocycles to Čech cocycles, an open cover $\mathcal{U}$ is constructed where each open set corresponds to a vertex pattern in T. We then have a map that takes a vertex pattern to the 0-cell at its center, and so this induces a map on the cellular 0-cochains to the Čech 0-cochains. We define a similar map on 1-cochains and extend this to a map on cohomology.

Next comes an adaptation of the Čech-deRham theorem [BT] to connect the Čech cohomology to the dynamical cohomology, mapping Čech cocycles to smooth functions on our space. We then define the Ruelle-Sullivan map which takes such functions and integrates them over an invariant probability measure on Ω_T. In the case of $n = 2$, these integrate to vectors in $\mathbb{R}^{n*}$. The philosophy, as suggested in [KP], is that the long range aperiodic order of an aperiodic tiling is given by the its first cohomology group together with the image of the Ruelle-Sullivan map.

To demonstrate this, in Chapter 5 we compute this map for two very different tilings which have isomorphic first cohomology groups. The first is the octagonal tiling, consisting of two labeled 1, 1, $\sqrt{2}$ triangles and a rhomb, along with all their rotations though $\frac{n\pi}{4}$. We find that the first cohomology group to be $\mathbb{Z}^5$, and we can find a generating set with one of the generators mapping to 0. Three of th remaining generators map to rotates of each other by multiples of $\frac{\pi}{4}$ while the fifth points in a direction between the first two, that is, at a multiple of $\frac{\pi}{8}$. This highlights the symmetry of the tiling through rotations of $\frac{\pi}{4}$.

The second is the famous kite-and-dart tiling of Penrose. To allow for a substitution rule, the kites and darts have been split into triangles, so that we have 40 prototiles - two differently shaped triangles each given two different labels, and all their rotations though $\frac{n\pi}{5}$. We compute the first cohomology group to be isomorphic to $\mathbb{Z}^5$ and the image of one of its generators under the Ruelle-Sullivan map to be 0. The image of the others are vectors in $\mathbb{R}^{n*}$ which are rotations of each other through different multiples of $\frac{\pi}{5}$. Here we see how our map captures some of the rotational symmetry in this aperiodic tiling.

Chapter 2

Tilings and Tiling Spaces

2.1 Cell Complexes

First let us solidify some terminology. Hereafter let

$$E^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$$
$$U^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$$
$$S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$$

ie, E^n is the closed unit ball in $\mathbb{R}^n$, U^n is its interior and S^{n-1} is its boundary.

A CW-Complex is, roughly speaking, a space built up by the successive adjoining of cells of dimension $0, 1, 2, \dots,$ etc. To be more precise:

Definition 2.1.1 *A CW-Complex on a Hausdorff space X is defined by the prescription of an ascending sequence of closed subspaces*

$$X^0 \subset X^1 \subset X^2 \subset \dots$$

which satisfy the following conditions:

(1) *$X^0 \subset X$ has the discrete topology.*

(2) *For* $n > 0$, X^n *is obtained from* X^{n-1} *by adjoining a collection* $\{e^n_\lambda\}_{\lambda\in\Lambda_n}$ *of disjoint sets homeomorphic to* U^n *(called* n***-cells****) such that for each* $\lambda \in \Lambda_n$ *there exists a continuous map*

$$f_\lambda : E^n \longrightarrow \overline{e}^n_\lambda$$

such that f_λ *maps* U^n *homeomorphically to* e^n_λ *and* $f_\lambda(S^{n-1}) \subset X^{n-1}$.

(3) X *is the union of the* X^i *for* $i \geq 0$.

(4) *The space* X *and the subspaces* X^i *all have the weak topology: A subset* A *is closed if and only if* $A \cap \overline{e}^n$ *is closed for all* n*-cells,* e^n, $n = 0, 1, 2 \ldots$.

A CW-complex is also called a *cell complex*. We denote K_n to be the set $\{e^n_\lambda\}_{\lambda\in\Lambda_n}$ of n-cells adjoined to the complex at stage n. Also, $\overline{e}^n_\lambda$ will denote the closure of an n-cell while $\dot{e}^n_\lambda$ will denote $\overline{e}^n_\lambda - e^n_\lambda$ and will be called the *boundary* of e^n_λ.[1] We say a CW-Complex is *regular* if it is a CW-Complex and we can choose each of our f_λ maps in part (2) of the definition to be homeomorphisms. If $K_n \neq \emptyset$ but $K_i = \emptyset$ for all $i > n$, then we say that the CW-Complex is n-dimensional. Also we say that, for two cells e^{n-1}_μ and e^n_λ, that e^{n-1}_μ is a *face* of e^n_λ if $e^{n-1}_\mu \subset \overline{e}^n_\lambda$

Example - If $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, then define

$$\begin{aligned}
K_0 &= \{(0, 0, -1), (0, 0, 1)\} \\
K_1 &= \Big\{\{(\sin \pi t, 0, \cos \pi t) \mid r \in (0, 1)\},\ \{(-\sin \pi t, 0, \cos \pi t) \mid r \in (0, 1)\}\Big\} \\
K_2 &= \Big\{\{(x, y, z) \in S^2 \mid y > 0\},\ \{(x, y, z) \in S^2 \mid y < 0\}\Big\}
\end{aligned}$$

[1] Note that this may not correspond to the topological definition of boundary.

S^2 is the boundary of the unit sphere, so we chose vertices to be at points of intersection of S^2 with the z-axis, edges to be lines connecting the two vertices down opposite sides and our 2-cells to be the two determined half-shells. This defines a regular CW-complex on S^2, as the edges are both homeomorphic to $(0, 1)$ with the homeomorphisms extending to their closures (ie, the two edges do not start and end at the same vertex). Also, the elements of K_2 are both homeomorphic to the unit ball in $\mathbb{R}^2$, with the homeomorphisms extending to the boundaries which themselves are homeomorphic to S^1.

2.2 Tilings

Consider $\mathbb{R}^n$, usual n-dimensional Euclidean space. If A is a subset of $\mathbb{R}^n$, we may translate it by a vector $x \in \mathbb{R}^n$,

$$A + x = \{a + x \mid a \in A\}$$

We shall begin with a finite set $\{p_1, p_2, \ldots p_N\}$ of subsets of $\mathbb{R}^n$ homeomorphic to the closed unit ball, which we call *prototiles*. These prototiles may carry labels to distinguish them, ie, two prototiles may have the same shape but have different labels. We then say that a *tile* is any subset of $\mathbb{R}^n$ which is a translate of one of the p_i. Then we define *partial tiling* and *tiling* as follows:

Definition 2.2.1 *A* ***partial tiling*** *is a collection* $\{t_j\}_{j\in J}$ *of subsets of* $\mathbb{R}^n$ *which are translates of prototiles with pairwise disjoint interiors. The* ***support*** *of a partial tiling is defined to be the union of its tiles; this is denoted supp*$(\cdot)$*. A* ***tiling*** *is a partial tiling whose support is* $\mathbb{R}^n$*. If* $T = \{t_j\}_{j\in J}$ *is a tiling, a* ***patch*** *in* T *is a subset of* T *with bounded support.*

When $n = 1$, a tiling can be thought of as a bi-infinite sequence of a finite number of symbols, and when $n = 2$, it is what one normally thinks of as a tiling; that is, shapes fitting together to cover the plane. If $T = \{t_j\}_{j\in J}$ is a tiling we can, for $x \in \mathbb{R}^n$, define the translation of T by x by $T + x = \{t_j + x\}_{j\in J}$.

We also think of a tiling T as a multi-valued function: for $u \in \mathbb{R}^n$ and $U \subseteq \mathbb{R}^n$, let

$$T(u) = \{t \in T \mid u \in t\}$$
$$T(U) = \bigcup_{u\in U} T(u)$$

Tilings T and T' are said to *agree* on U if $T(U) = T'(U)$.

Definition 2.2.2 *A tiling is said to be periodic if there exists a non-zero $x \in \mathbb{R}^n$ such that $T = T + x$. A tiling for which no such x exists is called aperiodic.*

Periodic tilings are generally not very interesting, so we usually want our tiling to be aperiodic. Unless stated otherwise, **tilings from here forward are assumed to be aperiodic**.

2.3 The Tiling Space Ω_T

If $\mathcal{T}$ is a collection of tilings, then we can define a metric on $\mathcal{T}$. If $T, T' \in \mathcal{T}$ with $T = \{t_j\}_{j\in J}$ and $T' = \{t'_i\}_{i\in I}$, then define

$$\begin{aligned} d(T, T') = \inf\{1, \epsilon \mid \exists\, x, x' \in \mathbb{R}^n \ni |x|, |x'| < \epsilon, \\ (T - x)(B_{1/\epsilon}(0)) = (T' - x')(B_{1/\epsilon}(0))\} \end{aligned}$$

This may look complicated, but it is really quite simple: two tilings are close if they agree up to a small translation of a large ball about the origin. To prove that this is a metric, we shall need a lemma.

Lemma 2.3.1 *If a and b are positive numbers such that $a + b \leq 1$, then*

$$\frac{1}{a+b} \leq \frac{1-ab}{a}.$$

Proof. Notice that since a, b, and $a + b \leq 1$, we have that $1 - a(a + b) \geq 0$, and so

$$\begin{aligned} & 0 \leq 1 - a(a+b) \\ \Rightarrow\ & 0 \leq b - a^2b - ab^2 \\ \Rightarrow\ & a \leq a - a^2b - ab^2 + b \\ \Rightarrow\ & a \leq (a+b)(1-ab). \end{aligned}$$

This implies the result. ■

Proposition 2.3.1 *d satisfies the conditions of a metric.*

Proof. That d is symmetric is clear, as the definition is symmetric in T and T'. $d(T, T) = 0$ for all T because we can always find arbitrarily large balls (and hence arbitrarily small ϵ) around the origin where T matches up with itself. Conversely, if $d(T, T') = 0$, then we must be able to find arbitrary large balls around the origin where T and T' agree up to arbitrarily small translation – this can only be true if $T = T'$. This d is also always non-negative as it is the inf of a set of positive numbers.

Now, let R, S and T be tilings. We need to show that

$$d(T,S) \leq d(T,R) + d(R,S).$$

If $d(T,R) + d(R,S) \geq 1$ then the equality holds because this $d(\cdot,\cdot) \leq 1$ always. So assume $d(T,R) + d(R,S) < 1$ and pick $\epsilon > 0$ small enough so that $d(T,R) + d(R,S) + \epsilon < 1$. Find x_{TR} and x'_{TR} with

$$|x_{TR}|\,,|x'_{TR}| < d(T,R) + \frac{\epsilon}{2}$$

such that

$$(T - x_{TR})\Big(B_{\frac{1}{d(T,R)+\frac{\epsilon}{2}}}(0)\Big) = (R - x'_{TR})\Big(B_{\frac{1}{d(T,R)+\frac{\epsilon}{2}}}(0)\Big).$$

Likewise, find x_{RS} and x'_{RS} with

$$|x_{RS}|\,,|x'_{RS}| < d(R,S) + \frac{\epsilon}{2}$$

such that

$$(R - x_{RS})\Big(B_{\frac{1}{d(R,S)+\frac{\epsilon}{2}}}(0)\Big) = (S - x'_{RS})\Big(B_{\frac{1}{d(R,S)+\frac{\epsilon}{2}}}(0)\Big).$$

Since $T - x_{TR}$ agrees with $R - x'_{TR}$ on $\Big(B_{\frac{1}{d(T,R)+\frac{\epsilon}{2}}}(0)\Big)$, then we must have that $T - x_{TR} - x'_{SR}$ agrees with $R - x'_{TR} - x'_{SR}$ on $\Big(B_{\frac{1}{d(T,R)+\frac{\epsilon}{2}}}(-x'_{SR})\Big)$. In a similar way we see that $S - x_{SR} - x'_{TR}$ agrees with $R - x'_{SR} - x'_{TR}$ on $\Big(B_{\frac{1}{d(S,R)+\frac{\epsilon}{2}}}(-x'_{TR})\Big)$. This means that $S - x_{SR} - x'_{TR}$ agrees with $T - x_{TR} - x'_{SR}$ wherever these two balls overlap. This overlap includes the origin because $|-x'_{TR}|\,,|-x'_{SR}| < 1$ and the radii $\frac{1}{d(S,R)+\frac{\epsilon}{2}}$ and $\frac{1}{d(T,R)+\frac{\epsilon}{2}}$ are both greater than 1. If r_1 and r_2 denote the largest balls around the origin which are contained in $\Big(B_{\frac{1}{d(T,R)+\frac{\epsilon}{2}}}(-x'_{SR})\Big)$ and $\Big(B_{\frac{1}{d(S,R)+\frac{\epsilon}{2}}}(-x'_{TR})\Big)$ respectively, then

$$r_1 = \frac{1}{d(T,R) + \frac{\epsilon}{2}} - |-x'_{SR}|$$

$$r_2 = \frac{1}{d(S,R)+\frac{\epsilon}{2}} - |-x'_{TR}|.$$

Now

$$\begin{aligned} r_1 &= \frac{1}{d(T,R)+\frac{\epsilon}{2}} - |-x'_{SR}| \\ &\geq \frac{1}{d(T,R)+\frac{\epsilon}{2}} - (d(S,R)+\frac{\epsilon}{2}), \\ &= \frac{1-(d(T,R)+\frac{\epsilon}{2})(d(S,R)+\frac{\epsilon}{2})}{d(T,R)+\frac{\epsilon}{2}}. \end{aligned}$$

The above lemma implies that $r_1 \geq \frac{1}{d(T,R)+\frac{\epsilon}{2}+d(S,R)+\frac{\epsilon}{2}}$, and a symmetric argument shows the same inequality for r_2. Thus $T-x_{TR}-x'_{SR}$ and $S-x_{SR}-x'_{TR}$ agree on $B_{\frac{1}{d(T,R)+d(S,R)+\epsilon}}(0)$. Since we have that $|x_{TR}+x'_{SR}|, |S-x_{SR}-x'_{TR}| \leq d(T,R)+d(S,R)+\epsilon$ by the usual triangle inequality, we have that $d(T,S) \leq d(T,R)+d(R,S)+\epsilon$ which proves the result. ■

Thus any collection of tilings can be made into a metric space. One way to produce our collection $\mathcal{T}$ of tilings is to start with a specific tiling T and let $\mathcal{T}$ be the set of all translates of T, ie, $\mathcal{T} = T + \mathbb{R}^n$.

Definition 2.3.1 *Let T be a tiling. Then we define Ω_T to be the metric space obtained by completing $T+\mathbb{R}^n$ in the above metric.*

Strictly speaking, Ω_T is a space of Cauchy sequences, but we can visualize its elements as tilings. For example, consider the tiling of $\mathbb{R}^2$ consisting of unit squares matching up edge-to-edge and vertex-to-vertex, (a checkerboard pattern) with the vertices on the $\mathbb{Z}^2$ lattice, except that imagine that the 4 squares centered at the origin are replaced with a single 2×2 square. The normal checkerboard tiling (call it C) can be identified with the Cauchy sequence $\{T+(n,0)\}_{n=0}^{\infty}$. This line of thinking leads easily to the identification $\Omega_T \cong (T+\mathbb{R}^2) \cup (C+\mathbb{R}^2)$.

When we take a tiling T and form the metric space Ω_T, we might like to know whether Ω_T possesses any nice topological properties. It is well known and shown in [AP] that Ω_T is compact if T satisfies the following condition.

Definition 2.3.2 *A tiling T is said to have **Finite Local Complexity** if, for every $R > 0$ there are only finitely many patches (up to translation) in T whose radii of their supports are less than R.*

2.4 Substitution and the Anderson-Putnam Complex

One of the difficulties encountered in the study of tilings is producing interesting examples. One method for doing this is called the *substitution method.* The substitution method starts with our usual set $\{p_1, p_2, \ldots p_N\}$ of a finite number of prototiles along with a rule for splitting each prototile into tiles which are smaller copies of the p_i's along with an inflation constant $\lambda > 0$ which inflates the smaller copies to be the same size as the originals. The simplest example is to take a square with side length 1 and split it into 4 squares of side length one-half. If we then multiply this by $\lambda = 2$, we end up with 4 copies of our original square. In general, the result of the procedure on p_i is denoted $\omega(p_i)$ and it is a partial tiling with support λp_i. This rule can be extended to the translates of the $p_i's$ by defining $\omega(p_i + x) = \omega(p_i) + \lambda x$. Thus, we can easily define ω of a partial tiling - simply divide all the tiles in the patch up according to the rule and inflate everything. This clearly results in a new partial tiling whose support is λ times the support of the old. This idea can be easily seen to extend to tilings, ie, $\omega(T)$ is the tiling obtained by dividing each tile in T according to the rule and then inflating everything - the result of this is also a tiling.

From here onward we shall be dealing with a tiling T, its tiling space Ω_T, and a substitution rule ω. We also make some assumptions for our substitution rule ω. The first is that ω maps Ω_T to itself and that ω is one-to-one; this is what's known as **recognizability**. It is a well known fact (proved in [S]) that if ω is one-to-one, then Ω_T contains no periodic tilings.

The second assumption is that the substitution is **primitive**, that is, there is an $M \in \mathbb{Z}^+$ such that $\omega^M(p_i)$ contains a translate of p_j for all $i, j = 1, 2, \ldots, N$.

The third assumption is that $\omega : \Omega_T \to \Omega_T$ is onto. These assumptions lead to the following fact from [AP].

Theorem 2.4.1 *Under the hypotheses above, (Ω_T, ω) is a topological dynamical system, that is, $\omega : \Omega_T \to \Omega_T$ is bijective and bicontinuous.*

From now on assume that all prototiles are polygons and in the substitution they meet vertex to vertex and edge to edge. We are now then ready produce what is known as the *Anderson-Putnam* complex [AP] for Ω_T. This is started by constructing a Hausdorff space Γ_0 which is the quotient of the disjoint union of the prototiles obtained by gluing the prototiles together in all ways in which the substitution rule allows them to be adjacent. The inflation map ω induces a continuous surjection γ_0 on Γ_0, and with respect to which we take the inverse limit to obtain a new space Ω_0. We begin by defining Γ_k for $k = 0, 1$. If t is a tile in a tiling T, we define $T^{(0)}(t) = \{t\}$ and $T^{(1)}(t) = T(t)$, that is, $T^{(k)}(t)$ is the set of tiles in T that are within k tiles of t. Consider the space $\Omega_T \times \mathbb{R}^n$ with the product topology. Let$\sim_k$ be the smallest equivalence relation on $\Omega_T \times \mathbb{R}^n$ that takes $(T_1, u_1), (T_2, u_2)$ to be equivalent whenever $T_1^{(k)}(t_1) - u_1 = T_2^{(k)}(t_2) - u_2$ for some tiles $t_i \in T_i$ such that $u_i \in t_i$. Now define $\Gamma_k = \Omega_T \times \mathbb{R}^n / \sim_k$ with the quotient topology.

For a simple example, consider again the tiling of the plane by unit squares matching edge-to-edge. There is one prototile, and the identification described above leads to identifyting the top edge with the bottom edge and the left edge with the right edge. In this example, we see that Γ_0 is isomorphic to $\mathbb{T}$, the 2-torus.

For the examples investigated in this thesis, the prototiles are going to be 2-cells, so let us see what we have constructed in this case. A point in $\Omega_T \times \mathbb{R}^2$ is a tiling T together with a vector u in $\mathbb{R}^2$, so if we think of T covering $\mathbb{R}^2$, we can think of (T, u) as $u \in supp(T)$. If u lies in the interior of a tile t, and t is the translate of a prototile p_i, then the equivalence class $(T, u)_0$ is the set of all (T_1, u_1) such that the point u_1 on the tiling T_1 is in the interior of a tile t_1 – which is also a translate of p_i – and lies at exactly the same place u lies on t. Thus, for each prototile p_i we can define $P_i = \{(T, u)_0 \mid T(u)$ is a translate of $p_i\}$. The following will be stated without proof.

Claim 2.4.1 *The P_i are 2-cells in a cell complex for Γ_0.*

To get the rest of this cell complex, imagine drawing all the prototiles, each with the p_i label; these are the P_i. Next, label the edges in the natural way: start with any edge of any 2-cell, give it a label, then give the same label to any edge on the other 2-cells that may be adjacent to it in any tiling in Ω_T; do this again for all edges labeled so far until no new labelings can occur. Repeat this for the other edges, and then for the vertices. This defines a CW-complex on Γ_0.

To get a similar construction for Γ_1, one simply has to start with more 2-cells in the complex. For each prototile, there will be several different 2-cells, each with a different label corresponding to different possible patterns of tiles around it (a tile with such a label will hereafter refered to as a *collared tile*).

Theorem 2.4.2 *If T has finite local complexity, Γ_k is a compact Hausdorff space.*

Proof Because T has finite local complexity, we can find an $r > 0$ such that every possible pairwise adjacency of the prototiles is represented in the partial tiling $T\Big(B_r(0)\Big)$. If (T_1, u_1) is an element of $\Omega_T \times \mathbb{R}^n$ it happens that either u_1 lies on in the interior of a tile in T_1 or on the edge between two tiles. In either case, we can find u in $B_r(0)$ such that $(T, u) \sim (T_1, u_1)$ (in the first case because all prototiles are represented in $T\Big(B_r(0)\Big)$, in the second case because of our pick of r). $\{T\} \times \overline{B_r(0)}$ is compact, and so Γ_k is the image of a compact set under the quotient map $\pi_\sim : \Omega_T \times \mathbb{R}^n \to \Gamma_k$, and hence is compact (the quotient map is always continuous with respect to the quotient topology). A cell complex is always Hausdorff (see [Ma]), so we are done. ■

2.5 Ω_T as an Inverse Limit

We aim to show that Ω_T is isomorphic to a space of sequences in elements of Γ_k called an *inverse limit*; spaces similar to the solenoids discussed in [BS]. To construct this, we first need a surjection on Γ_k.

Theorem 2.5.1 *The inflation map ω induces a continuous surjection*

$$\gamma_k : \Gamma_k \longrightarrow \Gamma_k \ ; \ \gamma_k((T, u)_k) = (\omega(T), \lambda u)_k.$$

Proof Let T_1 and T_2 be in Ω_T, and assume that $(T_1, u_1) \sim (T_2, u_2)$. Thus $T_1^{(k)}(t_1) - u_1 = T_2^{(k)}(t_2) - u_2$ for some $u_1 \in t_1 \in T_1$ and $u_2 \in t_2 \in T_2$, and so we must have that $t_1 - u_1 = t_2 - u_2$. Thus we can choose tiles t_1' and t_2' with $\lambda u_1 \in t_1' \in \omega(\{t_1\})$ and $\lambda u_2 \in t_2' \in \omega(\{t_2\})$ such that $\omega(T_1)^{(k)}(t_1') - \lambda u_1 = \omega(T_2)^{(k)}(t_2') - \lambda u_2$. Thus, $(\omega(T_1), \lambda u_1) \sim (\omega(T_2), \lambda u_2)$ and so γ_k is well-defined. The map on $\Omega_T \times \mathbb{R}^n$ that sends (S, u) to $(\omega(S), \lambda u)$ is co-ordinatewise continuous and hence continuous, so when we pass to the quotient we see that γ_k must be continuous. We have that ω is invertible on Ω_T, so $(\omega^{-1}(S), \lambda^{-1}u)_k$ maps to (S, u). Thus, γ_k is onto. ■

We now construct the inverse limit space of Γ_k with respect to γ_k. Define

$$\Omega_k = \varprojlim_{\gamma_k} \Gamma_k = \{\{x_i\}_{i=1}^{\infty} \mid x_i \in \Gamma_k, \gamma_k(x_i) = x_{i-1}\}$$

This is a topological space with the relative topology from the product topology (ie, $\Omega_k \subset \prod_{i=1}^{\infty} \Gamma_k$). Thus, a basis for the topology is the collection of sets of the form $B_{U,n}^{\Omega_k} = \{x \in \Omega_k \mid x_i \in \gamma_k^{n-i}(U);\ i = 1, 2, \ldots, n\}$, where $U \subset \Gamma_k$ is open and $n \in \mathbb{N}$. We can use the inflation map to define a right shift $\omega_k : \Omega_k \to \Omega_k$ by $\omega_k(x)_i = \gamma_k(x_i)$. We can see that ω_k is invertible with inverse $\omega_k^{-1}(x)_i = x_{i+1}$.

Before the last theorem of this chapter, we need a standard dynamical definition and a definition of a condition due to Kellendonk.

Definition 2.5.1 *Two topological dynamical systems (X, f) and (Y, g) are said to be* ***topologically semi-conjugate*** *if there exists a continuous surjection $\pi : X \to Y$ such that $\pi \circ f = g \circ \pi$. The systems are said to be* ***topologically conjugate*** *if, in addition, π is injective.*

Definition 2.5.2 *The substitution tiling space (Ω_T, ω) is said to* ***force its border*** *if there exists a fixed positive integer N such that for any tile t and tilings T_1 and T_2 in Ω_T containing t, we have that $\omega^N(T_1)\Big(\omega^N(\{t\})\Big) = \omega^N(T_2)\Big(\omega^N(\{t\})\Big)$.*

This says that there is a number of iterations N of the inflation after which the tiles surrounding the image of a tile in two different tilings must be the same. This is always satisfied if the tiles we are dealing with are collared tiles (the tiles surrounding collared tiles are known after each iteration of the substitution).

Theorem 2.5.2 *Let T be a substitution tiling under a substitution rule ω which has recognizability, is primitive, and is an onto map from Ω_T to itself. Then $\omega_k : \Omega_k \to \Omega_k$ is a homeomorphism, and thus (Ω_k, ω_k) is a topological dynamical system. The dynamical systems (Ω_T, ω) and (Ω_1, ω_1) are topologically conjugate. Furthermore, if T forces its border, then (Ω_T, ω) and (Ω_0, ω_0) are topologically conjugate.*

Proof We begin by showing that (Ω_T, ω) is conjugate to (Ω_1, ω_1), and then show that if the substitution forces its border that (Ω_1, ω_1) is conjugate to (Ω_0, ω_0). We must therefore find a homeomorphism between the two spaces that conjugates the actions. What is given below is a sketch of the construction and is also given in [AP].

For any $T' \in \Omega_T$, define $\pi : \Omega_T \to \Omega_1$ by $\pi(T') = \{x_i\}_{i=0}^{\infty}$ where $x_i = (\omega^{-1}(T'), 0)_1$. We have that $\gamma_1(x_i) = x_{i-1}$, so π is well-defined. Let $\{x_i\}_{i=0}^{\infty}$ be any element of Ω_1; we wish to find a tiling T' that maps to it under π. Since we must have that $x_0 = (T', 0)_0$, x_0 specifies the tile t_0 in T' which contains the origin. In the same sense, x_1 must specify the tile in $\omega^{-1}(T)$ that contains the origin - t_1 say. Thus T' contains the partial tiling $\omega(t_1)$. If we continue in this way, we obtain a nested sequence of partial tilings. To have that the limit of these is a tiling (our T'), we recall that in Ω_1 we are dealing with "collared tiles", so the tiles around any given patch are determined.

To see that π is one-to-one, suppose we have that $\pi(T_1) = \pi(T_2)$ for some $T_1, T_2 \in \Omega_T$. Define

$$r = \inf\left\{dist\Big(t, \partial(\cup T'(t))\Big) | T' \in \Omega_T, t \in T\right\}$$

where $dist(U, V)$ is defined as $\inf\{\|u - v\| \;\; |u \in U, v \in V\}$ for any sets $U, V \subset \mathbb{R}^n$. Finite Local Complexity of T implies that this is an inf over a finite set of positive numbers, and is thus positive. Suppose $v \in \mathbb{R}^n$; we show that T_1 and T_2 must agree on a ball around the origin containing v.

Let $n \in \mathbb{Z}^+$ such that $r\lambda^n > \|v\|$. Now $\pi(T_1) = \pi(T_2)$ as sequences in Ω_1, so $\pi(T_1)_n = \pi(T_2)_n$. We can see by finite induction that this reduces to saying that, for some tiles t_1 and t_2 containing the origin, we have $\omega^{-n}(T_1)^{(1)}(t_1) = \omega^{-n}(T_2)^{(1)}(t_2)$. Since we are in Ω_1, t_1 and t_2 are collared tiles. Thus, $\omega^{-n}(T_1)$ and $\omega^{-n}(T_2)$ agree at least on $B_r(0)$, and hence T_1 and T_2 agree on $B_{r\lambda^n}(0)$. Since $v \in B_{r\lambda^n}(0)$, we must have that T_1 and T_2 agree everywhere. Thus π is one-to-one.

To show that π is onto, suppose we have $x = \{(T_i, u_i)_1\}_{1=0}^{\infty} \in \Omega_1$. Define

$$T' = \bigcup_{i=1}^{\infty} \omega^i\Big(\bigcap_{u_i \in t \in T_i} T_i^{(1)}(t) - u_i\Big).$$

It needs to be verified that this is a partial tiling, and, using the r defined two paragraphs above, that it is in fact a tiling. Then it is clear that $T' \in \Omega_T$

and $\pi(T') = x$.

Bicontinuity is proven using standard methods, and it is easy to check that $\pi \circ \omega = \omega_1 \circ \pi$ to entwine the dynamics. Thus (Ω, ω) is topologically conjugate to (Ω_1, ω_1).

Now we assume that the substitution rule forces its border, and prove that (Ω_1, ω_1) is topologically conjugate to (Ω_0, ω_0). If $(T_1, u_1) \sim_1 (T_2, u_2)$ then trivially $(T_1, u_1) \sim_0 (T_2, u_2)$, so the natural map

$$f : \Omega_1 \to \Omega_0$$

$$f\left((T', u)_1\right) = (T', u)_0$$

is well-defined for all $T' \in \Omega_T$ and $u \in \mathbb{R}^n$. We clearly have that f is onto, but it is not in general one-to-one. Now

$$\begin{aligned} \gamma_0 \circ f\left((T', u)_1\right) &= \gamma_0 (T', u)_0 \\ &= (\omega(T'), \lambda u)_0 \\ &= f((\omega(T'), \lambda u)_1 \\ &= f \circ \gamma_1((T', u)_1). \end{aligned}$$

Thus we can define a well-defined map $F : \Omega_1 \to \Omega_0$ by $F(x)_i = x_i$ for $i \in \mathbb{N}^+$. We claim F is a homeomorphism that conjugates ω_1 and ω_0.

To show that F is injective, we need that the substitution forces its border. Suppose we have that $F\left(\{x_i\}_{i=0}^{\infty}\right) = F\left(\{x_i\}_{i=0}^{\infty}\right)$ with $x_i = (T_i, u_i)_1$ and $y_i = (T_i', u_i')_1$. We see that if $v_1 \in s_1 \in S_1$ and $v_2 \in s_2 \in S_2$ for S_1 and $S_2 \in \Omega_T$, and $(S_1, v_1)_0 = (S_2, v_2)_0$, then $S_1^{(0)}(s_1) - v_1 = S_2^{(0)}(s_2) - v_2$ and the forcing the border condition implies that $(\omega^N(S_1), \lambda^N v_1)_1 = (\omega^N(S_2), \lambda^N v_2)_1$. By our hypothesis, $F(\{x_i\}_{i=0}^{\infty}) = F(\{x_i\}_{i=0}^{\infty})$ and in particular $(T_{j+N}, u_{j+N})_0 = (T'_{j+N}, u'_{j+N})_0$, by a finite induction we have that $(\omega^N(T_{j+N}), \lambda^N u_{j+N})_1 = (\omega^N(T'_{j+N}), \lambda^N u'_{j+N})_1$. We know that $T_{j+N} = \omega^{-N}(T_j)$ and that $u_{j+N} = \lambda^{-N} u_j$, so $(T_j, u_j)_1 = (T'_j, u'_j)$ and F is one-to-one.

The compactness of Ω_T implies that F is onto; the following argument is due to Kellendonk. Say we are given $\{(T_i, u_i)\}_{i=0}^{\infty} \in \Omega_0$. Because Ω_T is compact, the sequence $\{\omega^n(T_n - u_n)\}_{n=0}^{\infty}$ has a convergent subsequence $\{\omega^{n_k}(T_{n_k} - u_{n_k})\}_{k=0}^{\infty}$ that converges to some tiling $T' \in \Omega_T$. Now, $\{(\omega^{-i}(T), 0)_1\}_{i=0}^{\infty}$ is in Ω_1, and we claim that it maps to $\{(T_i, u_i)\}_{i=0}^{\infty}$ under F. Since $(T_0, u_0)_0 = (\omega^n(T_n), \lambda^n u_n)_0$ and $\omega^n(T_n - u_n) = \omega^n(T_n) - \lambda^n u_n$, we have that $(\omega^n(T_n - u_n), 0)_0 = (T_0, u_0)_0$ for all n. Because T has Finite Local Complexity, this means that between all the tilings $(\omega^n(T_n - u_n)$, only finitely many different tiles contain the origin.

Pick an $i \geq 0$. Our subsequence being convergent means that given any $\epsilon > 0$, we can find a large enough k so that $\omega^{n_k}(T_{n_k} - u_{n_k})$ agrees with T' on $B_{\frac{1}{\epsilon}}(0)$. Let R be a real number greater than the diameter of each prototile, and chose k so that $n_k \geq i$ and $\omega^{n_k}(T_{n_k} - u_{n_k})$ agrees with T' on $B_{\lambda^i R}(0)$. Then $(\omega^{-i}(T'), 0)_0 = (\omega^{n_k - i}(T_{n_k} - u_{n_k}), 0)_0 = \gamma_0^{n_k - i}(T_{n_k}, u_{n_k})_0 = (T_i, u_i)_0$, which is what we were claiming.

We have easily that F is bicontinuous, and a simple calculation similar to the one above shows that it conjugates the actions of ω_0 and ω_1. Thus (Ω_0, ω_0) is topologically conjugate to (Ω_1, ω_1) when the substitution forces its border, and since topological conjugacy is an equivalence relation, (Ω_0, ω_0) is conjugate to (Ω_T, ω). ■

Conjugate dynamical systems have the same dynamics and can be thought of as, in a sense, the same system. The elements of Ω_k can be seen as tilings in the following way. If $\{x_i\}_{i=0}^{\infty} \in \Omega_k$, then $x_0 = (T_1, 0)_k$ for some tiling T_1. Furthermore, x_0 is equivalent to all $(S, 0)_k$ such that S has the same tile t_0 around the origin as T_1 – so we can see that the first co-ordinate specifies the tile at the origin. Then by extending this idea to $x_1 = (T_1, 0)_k$, we see that x_1 specifies a patch $\omega(t_1)$ around the origin, with the tile $t_0 \subset \omega(t_1)$. In the case $k = 1$ (or k=0 if T forces its border), we see that the limit of this process does indeed specify a tiling in Ω_T.

Chapter 3

Cohomology

Cohomology theories are ways to obtain important invariants of a space. The three we will talk about here concern topological spaces.

3.1 Cohomology in General

In general, when we talk about cohomology we mean the following. Let $X = \{X_1, X_2, \dots\}$ be a sequence of spaces. Define

$$C^i(X, G) = \{f : X_i \to G\}.$$

For some abelian group G. We call $C^i(X, G)$ the group of *i-cochains*. Suppose we have a sequence of maps δ_i such that

$$0 \to C^0(X, G) \xrightarrow{\delta_0} C^1(X, G) \xrightarrow{\delta_1} \cdots C^i(X, G) \xrightarrow{\delta_i} \cdots$$

such that $\delta_{i+1} \circ \delta_i = 0$ for all i. Then $\ker \delta_i$ is called the group of *i-cocycles* while the $\operatorname{Im} \delta_{i-1}$ is called the group of *i-coboundaries*. The *i-th cohomology* *group of X with coefficients in G* is then defined to be

$$H^i(X, G) = \ker \delta_i / \operatorname{Im} \delta_{i-1}.$$

Elements of $H^i(X, G)$ are still refered to as i-cocycles or merely cocycles.

With this, we can define three types of cohomology relevant to tilings.

3.2 Orientation, Incidence Number, and Cellular Cohomology

In a cell complex, *orientations* are needed on all the cells to define cellular cohomology. If we restrict our attention to 1, 2, and 3 cells for the moment, it's easy to guess what the orientation of cells would look like: a left or right arrow on a 1-cell, a clockwise or counter-clockwise curl on a 2-cell, or a left or right handed corkscrew in a 3-cell. In cohomology (and homology) theory, we need a way of expressing whether the orientations of cells and their faces "match up" - this is done with incidence numbers.

In Figure 3.1 the arrows indicate the orientations given to σ and the edges e, f and g. If we go around the cell according to the orientation of σ, then we see that the orientations of e and f match up to that of σ, while that of g does not. In this situation, we would like to define incidence numbers of the pairs (σ, e) and (σ, f) to be $+1$ and the incidence number of (σ, g) to be -1. If e^n_λ and e^{n-1}_μ are n and $n-1$ cells respectively, then we denote their incidence number by $[e^n_\lambda : e^{n-1}_\mu]$. Note that $[e^n_\lambda : e^{n-1}_\mu]$ is defined for arbitrary n and $n-1$ cells, but is zero if $e^{n-1}_\mu \nsubseteq \overline{e}^n_\lambda$.

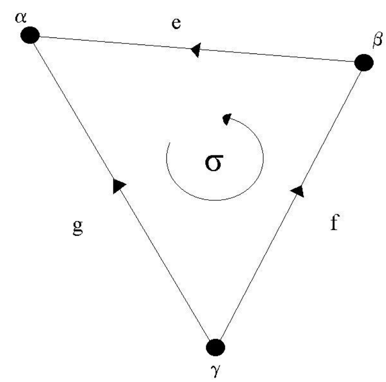

Figure 3.1: A typical 2-cell

Example: Looking at Figure 3.1, we have already established that:

$[\sigma : e] = 1$

$[\sigma : f] = 1$

$[\sigma : g] = -1$

In addition to these, we must relate edges to vertices. If π is an edge and x is a vertex of the edge, then we define $[\pi : x]$ to be ± 1; 1 when the edge points toward the vertex and –1 when it points away. Thus,

$[f : \beta] = 1$

$[f : \gamma] = -1$

$[e : \alpha] = 1$

$[e : \beta] = -1$

$[g : \alpha] = 1$

$[g : \gamma] = -1$

$[f : \alpha] = 0,$ *etc.*

It is, in fact, possible to define orientation rigorously to apply to arbitrary dimensions. For the purposes of the spaces presented here, the highest dimension we will have to deal with is 2, and so our discussion of orientation and incidence will end with the above paragraph.[1]

We are now ready to define the Cellular Cohomology of a cell complex K. For each i, let

$$F(K_i, G) = \{f \mid f \ : \ K_i \to G\}$$

where G is an abelian group (in this paper, G will always be either $\mathbb{R}$ or $\mathbb{Z}$ - for this reason we will often refer to group elements as "numbers"). If K is n-dimensional, let $F(K_i, G)$ be the zero group for all $i > n$. Define

$$\partial_i : F(K_i, G) \longrightarrow F(K_{i+1}, G)$$

$$\partial_i \varphi(e_\lambda^{i+1}) = \sum_{\mu \in \Lambda_i} [e_\lambda^{i+1} : e_\mu^i] \varphi(e_\mu^i); \quad \varphi \in F(K_i, G)$$

A minor problem that arises from this definition is the sum - it may not be finite. There are a couple ways ways to fix this - the first being to define $F(K_i, G)$ to be the finitely supported functions on K_i. This works fine, although it is rarely necessary. The other is to impose a mild restriction on our CW-complex stating that, for any n-cell e_λ^n, we have that $e_\mu^{n-1} \subset \overline{e}_\lambda^n$ for only finitely many μ. This would make $[e_\lambda^{i+1} : e_\mu^i] = 0$ for all but finitely many μ.

[1] In short, the orientation of a cell e_λ^n is derived from a group called the *nth relative homology group of* $\overline{e}_\lambda^n$ *with respect to* $\partial(e_\lambda^n)$, denoted $H_n(\overline{e}_\lambda^n, \partial(e_\lambda^n))$. This group is always infinite cyclic, and the orientation of e_λ^n is defined to be the choice of its generator. A full treatment of this object is given in [Ma].

Example: Looking again at our typical 2-cell, Figure 3.1, we see that if $\psi \in F(K_0, G)$, we have

$$\partial_0 \psi(f) = \psi(\beta) - \psi(\gamma)$$

and, if $\varphi \in F(K_1, G)$, we have

$$\partial_1 \varphi(\sigma) = \varphi(f) + \varphi(e) - \varphi(g)$$

We can see that if $\varphi \in F(K_1, G)$, we can think of ∂_1 acting on it by taking it to the function that takes a 2-cell and produces a number by adding up the values of φ on the edges multiplied by the respective incidence numbers.

With these sets and maps, we get a chain complex.

$$0 \longrightarrow F(K_0, G) \xrightarrow{\partial_0} F(K_1, G) \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{i-1}} F(K_i, G) \xrightarrow{\partial_i} F(K_{i+1}, G) \xrightarrow{\partial_{i+1}} \cdots$$

If we view each $F(K_i, G)$ as a group with the usual addition, then the ∂'s are all homomorphisms. In addition, the ∂'s are related in a very simple way.

Claim 3.2.1 $\partial_{i+1} \circ \partial_i = 0 \; \forall \; i.$

Proof Without a definition of orientation and incidence numbers for higher dimension at our disposal, proving this for $i > 2$ is impossible. We will prove this for the case $i = 0$; the others are done in a similar way with when equipped with proper definition of orientation for higher dimensions. Let $f \in F(K_0, G)$. We need to show that for all 2-cells e^2_λ, we have

$$\partial_1 \circ \partial_0 (e^2_\lambda) = 0$$

Suppose that e^2_λ is surrounded by edges $e_1, e_2, \ldots, e_n$ and let $t(e)$ and $i(e)$ denote the terminus and initial point of an edge e, respectively. Then

$$\begin{aligned}\partial_1 \circ \partial_0(e^2_\lambda) &= \sum_{k=1}^{n} \partial_0 f(e_k) \\ &= \sum_{k=1}^{n} \big[f\big(t(e_k)\big) - f\big(i(e_k)\big)\big]\end{aligned}$$

But $t(e_k) = i(e_k + 1)$, and $t(e_n) = i(e_1)$, so the sum collapses,

$$\begin{aligned}\partial_0 \circ \partial_1(e^2_\lambda) = &-f\big(i(e_1)\big) + \big[f\big(t(e_1)\big) - f\big(i(e_2)\big)\big] + \big[f\big(t(e_2)\big) - f\big(i(e_3)\big)\big] + \\ &\cdots + \big[f\big(t(e_n - 1)\big) - f\big(i(e_n)\big)\big] + f\big(t(e_n)\big) \\ = &\ 0 \quad \blacksquare\end{aligned} \tag{3.1}$$

Now we can form the cohomology.

Definition 3.2.1 *Let $K = \cup_{i=1}^{\infty} K_i$ be a CW-Complex. Define the* ***ith Cellular Cohomology Group*** *of K, denoted $H^i(K, G)$, to be*

$$H^i(K) := \ker \partial_i / \operatorname{Im} \partial_{i-1}$$

Note that this is well defined, as $\ker \partial_i$ and $\operatorname{Im} \partial_{i-1}$ are both groups (the ∂'s are homomorphisms) and $\operatorname{Im} \partial_{i-1} \subset \ker \partial_i$ by Claim 1.2.

3.3 Čech Cohomology

Using the notation from [BT], let X be a topological space, and let $\mathfrak{U} = \{U_a\}_{a \in J}$ be an open cover for X, where J is a countable linearly ordered index set. For $a < b < c$, denote the pairwise intersections $U_a \cap U_b$ by U_{ab}, triple intersections $U_a \cap U_b \cap U_c$ by U_{abc} etc. Let $\mathfrak{U}^{(n)}$ denote the set of n-fold intersections of elements of $\mathfrak{U}$ (0-fold intersections are just the sets themselves, 1-fold intersections are intersections of the form $U_a \cap U_b$ with $a < b$ etc). Let

$$F(\mathfrak{U}^{(n)}, G) \quad n \in \mathbb{N}$$

denote the group of functions on the set of n-fold intersections of elements of $\mathfrak{U}$ taking values in the abelian group G. By the 0-fold intersections we mean just the sets themselves. Define boundary maps $\check{\partial}_i$ by

$$\check{\partial}_i : F(\mathfrak{U}^{(i)}, G) \longrightarrow F(\mathfrak{U}^{(i+1)}, G)$$

$$(\check{\partial}_i f)(U_{a_1 a_2 \ldots a_{i+1}}) = \sum_{k=1}^{i+1} (-1)^{k+1} f(U_{a_1 a_2 \ldots a_{k-1} a_{k+1} \ldots a_{i+1}})$$

Then, as before, $\check{\partial}_{i+1}\check{\partial}_i = 0$ and we can form the cohomology of the complex

$$\cdots \xrightarrow{\check{\partial}_{i-1}} F(\mathfrak{U}^{(i)}, G) \xrightarrow{\check{\partial}_i} F(\mathfrak{U}^{(i+1)}, G) \xrightarrow{\check{\partial}_{i+1}} \cdots$$

We denote these groups

$$\check{H}^i(\mathfrak{U}, G) = \ker \check{\partial}_i / \operatorname{Im} \check{\partial}_{i-1}$$

and call these the *Čech Cohomology of the cover* $\mathfrak{U}$. A priori, these groups depend on the cover $\mathfrak{U}$. In this regard, we are rescued by a definition and a theorem from [BT].

Definition 3.3.1 *A* ***good cover*** *for a topological space X is an open cover of X for which each finite intersection is contractible.*

Theorem 3.3.1 *If $\mathfrak{U}$ and $\mathfrak{V}$ are good covers for a space X, then $\check{H}^i(\mathfrak{U}, G) \cong \check{H}^i(\mathfrak{V}, G)$ for all i.*

To define the Čech cohomology of a space, we should want a definition which is independent of the cover, good or otherwise. To do this, we follow the lead of [BT] and define the following.

Definition 3.3.2 *Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ and $\mathfrak{V} = \{V_\beta\}_{\beta \in J}$ be open covers of a space X. Then we say $\mathfrak{V}$ is a* ***refinement*** *of $\mathfrak{U}$, written $\mathfrak{U} < \mathfrak{V}$ if there is a map $\phi : J \to I$ such that $V_\beta \subset U_{\phi(\beta)}$.*

If $\mathfrak{U} < \mathfrak{V}$, then the map ϕ induces a map on Cohomology, via

$$\Phi : H^k(\mathfrak{U}, G) \longrightarrow H^k(\mathfrak{V}, G)$$

$$\Phi f(V_{a_1 a_2 \ldots a_k}) = f(U_{\phi(a_1)\phi(a_2)\ldots\phi(a_k)}).$$

It is possible that the indices in the last term are not in the correct order. To deal with this, we adopt the convention to take $f(U_{\sigma(a_1)\sigma(a_2)\ldots\sigma(a_k)}) = \mathrm{sgn}(\sigma) f(U_{a_1 a_2 \ldots a_k})$ for any permutation σ.

Definition 3.3.3 (BT) *A* ***direct system of groups*** *is a collection $\{G_i\}_{i \in I}$ of groups indexed by a directed set I such that for any pair $a < b$ there is a group homomorphism $f^a_b : G_a \to G_b$ satisfying*

$$f^a_a = identity$$

$$f^a_c = f^b_c \circ f^a_b, \ \forall \ a < b < c$$

Whenever we have a direct system of groups, we can form is direct limit.

Definition 3.3.4 *Let $\coprod G_i$ denote the disjoint union of the direct system of groups $\{G_i\}_{i \in I}$. Introduce an equivalence relation on $\coprod G_i$ by saying that*

$g_a \in G_a$ is equivalent to $g_b \in G_b$ if for some upper bound c of a and b we have $f_c^a(g_a) = f_c^b(g_b)$ in G_c. The ***direct limit*** *of the system, denoted by $\lim_{i\in I} G_i$, is the quotient of $\coprod G_i$ by this equivalence relation.*

Thus, two elements in $\coprod G_i$ are equivalent if they are "eventually equal". The direct limit is a group under the operation $[g_a] + [g_b] = [f_c^a(g_a) + f_c^b(g_b)]$, where c is an upper bound for a and b and the brackets indicate equivalence classes.

From all this we can see that for each k, $\{H^k(\mathfrak{U}, G)\}_{\mathfrak{U}}$ is a direct system of groups.

Definition 3.3.5 *The Čech Cohomology of a space X is defined as the direct limit*

$$H^k(X, G) = \lim_{\mathfrak{U}} H^k(\mathfrak{U}, G)$$

where the limit is over a directed set of refinements.

3.4 Dynamical Cohomology

Now, take (X, φ) to be a topological $\mathbb{R}^n$-dynamical system, ie, let X be a compact metric space and φ be a continuous $\mathbb{R}^n$ action on X. This just means that for each $v \in \mathbb{R}^n$, $\varphi_v : X \to X$ is a homeomorphism of X and the map sending (x, v) to $v \mapsto \varphi_v(x)$ is jointly continuous; we also have $\varphi_v \circ \varphi_w = \varphi_{v+w}$ for all $v, w \in \mathbb{R}^n$. Let $C(X)$ denote the algebra of continuous $\mathbb{R}$-valued functions on X. We call $f \in C(X)$ *continuously differentible* if

$$\frac{\partial f}{\partial v}(x) = \lim_{t\to 0} \frac{f(\varphi_{tv}(x)) - f(x)}{t}$$

exists and is back in $C(X)$ for all $x \in X$ and $v \in \mathbb{R}^d$. We say f is *smooth* if it is infinitely continuously differentiable, and let $C^\infty(X)$ denote the set of such functions. We can also take the same approach for a finite dimensional vector space; we let $C(X, W)$ denote the continuous W-valued functions on X. The definition of $C^\infty(X, W)$ extends naturally.

Let $\{x_1, x_2, \ldots x_n\}$ denote the standard basis for $\mathbb{R}^n$, and let $\mathbb{R}^{n*}$ denote the dual space of the real vector space $\mathbb{R}^n$. Then we can always find a basis for $\{dx_1, dx_2, \ldots, dx_n\}$ of $\mathbb{R}^{n*}$ such that $\langle x_i, dx_j \rangle = \delta_{ij}$ where δ_{ij} is the Kronecker delta symbol ($\delta_{ij} = 1$ if $i = j$, but $= 0$ otherwise). This is called the *dual basis for $\mathbb{R}^{n*}$ with respect to $\{x_1, x_2, \ldots x_n\}$*. Then we make the following definition (see [BT]).

Definition 3.4.1 *The **graded exterior algebra** of $\mathbb{R}^{n*}$ is the algebra over $\mathbb{R}$ generated by $dx_1, dx_2, \ldots, dx_n$ with the relations*

$$(dx_i)^2 = 0 \ \forall \ i$$

$$dx_i dx_j = -dx_j dx_i$$

We denote this algebra by $\Lambda\mathbb{R}^{n*}$.

Thus $\Lambda\mathbb{R}^{n*}$, when viewed as a real vector space, has basis (for $1 \leq i < j < k \leq n$)

$$1, dx_i, dx_i dx_j, dx_i dx_j dx_k, \ldots, dx_1 dx_2 \cdots dx_n$$

We also let $\Lambda^k\mathbb{R}^{n*}$ denote the subspace of $\Lambda\mathbb{R}^{n*}$ spanned by elements of the form $dx_{i_1} dx_{i_2} \cdots dx_{i_k}$ for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

Consider $C^\infty(X, \Lambda\mathbb{R}^{n*})$. Define

$$d : C^\infty(X, \mathbb{R}) \longrightarrow C^\infty(X, \mathbb{R}^{n*})$$

$$\langle v, df(x) \rangle = \frac{\partial f}{\partial v}(x)$$

for all $v \in \mathbb{R}^n$. This extends to a differential

$$d : C^\infty(X, \Lambda^k \mathbb{R}^{n*}) \longrightarrow C^\infty(X, \Lambda^{k+1} \mathbb{R}^{n*})$$

in the following way: every element of $C^\infty(X, \Lambda^k \mathbb{R}^{n*})$ may be written in the form

$$\sum_I f_I dx_{i_1} dx_{i_2} \cdots dx_{i_k}$$

where $I = \{i_1, i_2, \ldots i_k\} \subset \{1, 2, \ldots n\}$ and $f_I \in C^\infty(X, \mathbb{R})$. Furthermore, we have the following relations;

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i, \qquad \frac{\partial f}{\partial x_i} = \lim_{t \to 0} \frac{f(\varphi_{tx_i}(x)) - f(x)}{t}.$$

Proposition 3.4.1 *Let d_i denote the map $d_i = d : C^\infty(X, \Lambda^i \mathbb{R}^{n*}) \to C^\infty(X, \Lambda^{i+1} \mathbb{R}^{n*})$. Then $d_{i+1} \circ d_i = 0 \ \forall i$.*

Proof See [BT].

Thus, the chain

$$\cdots \xrightarrow{d} C^\infty(X, \Lambda^{k-1} \mathbb{R}^{n*}) \xrightarrow{d} C^\infty(X, \Lambda^k \mathbb{R}^{n*}) \xrightarrow{d} C^\infty(X, \Lambda^{k+1} \mathbb{R}^{n*}) \xrightarrow{d} \ldots$$

has the property that the image of the d map is contained in the kernel of the previous d map. Thus, we can form the following.

Definition 3.4.2 *We define the* ***dynamical cohomology*** *of (X, φ) to be the groups*

$$H^i(\mathbb{R}^n, C^\infty(X, \mathbb{R})) = \ker d_i / \operatorname{Im} d_{i-1}$$

Chapter 4

Connecting Cohomology Theories

4.1 Mapping Cellular Cohomology of Γ_0 to the Čech Cohomology of Ω_T when $n = 2$

To construct the Cech Cohomology of our tiling space Ω_T, we must first produce an open cover.

Let $p_1, p_2, \ldots, p_N$ be our N distinct prototiles in tiling T which generates our tiling space Ω_T. Assuming that each of the p_i is star-shaped (that is, for each i there exists a point $x_i \in p_i$ such that for any other point $y \in p_i$, the line from x_i to y is contained in p_i) pick a point in the interior of each tile about which it is star-shaped. Next, pick a point in the interior of each edge in the edge set. Because the tiles are star-shaped, these can be connected to the points in the interior of the tile on each tile by a straight line (see Figure

4.1). This splits the tile into regions, each containing exactly one vertex. For each vertex v around p_i we denote its corresponding region by r_v. Let

$$O(p_i, v) = \big(\bigcup_{x \in r_v} B_\epsilon(x)\big) \cap p_i$$

for some sufficiently small ϵ. If t is a tile, then $t = p_i + x$ for some i and x,

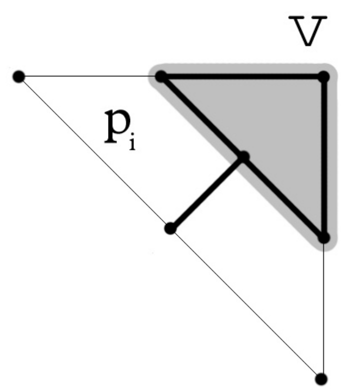

Figure 4.1: $O(p_i, V)$

so define $O(t, v) = O(p_i, v - x) + x$. Now define, for any tiling T and vertex v in any tile of T,

$$O(v) = \bigcup_{v \in t} O(t, v).$$

This is an open set in $supp(T) = \mathbb{R}^2$. Now, for all vertices v in T, look at the sets $T(v) - v$; we call such a set a *vertex pattern*. Because of the Finite Local Complexity property, for each v, $T(v) - v = T(v_i) - v_i$ for some finite set of vertices $v_1, v_2, \ldots, v_L$ for which $T(v_i) - v_i \neq T(v_j) - v_j$ if $i \neq j$, and so

$$\bigcup_{v \in T}(T(v) - v) = \bigcup_{i}^{L}(T(v_i) - v_i).$$

For each $i = 1, \ldots, L$, define

$$\begin{aligned} V_i &= \{T' : T'(0) = T(v_i) - v_i\} - (O(v_i) - v_i) \\ &= \{T' + x : T'(0) = T(v_i) - v_i; x \in O(v_i) - v_i\} \end{aligned}$$

The set V_i consists of all tilings which are translates of tilings which have the pattern $T(v_i) - v_i$ at the origin translated by vectors which keep the origin in r_{v_i}. Now we can define

$$\mathcal{V} = \{V_i : 1 \leq i \leq L\}$$

$\mathcal{V}$ is our open cover.

As stated earlier, we let Γ_k denote the Anderson-Putnam complex of the tiling space Ω_T, and we can also let Γ_{ki} denote the i-cells of said complex. We thus get the usual cellular chain complex

$$0 \longrightarrow F(\Gamma_{k0}, \mathbb{Z}) \xrightarrow{\partial} F(\Gamma_{k1}, \mathbb{Z}) \xrightarrow{\partial} F(\Gamma_{k2}, \mathbb{Z}) \longrightarrow 0$$

where $F(\Gamma_{k*}, \mathbb{Z})$ denotes the set of integer-valued functions on Γ_{k*}. Refering to Figure 1, the first boundary map is defined as

$$\begin{aligned} \partial : F(\Gamma_{ki}, \mathbb{Z}) &\longrightarrow F(\Gamma_{ki+1}, \mathbb{Z}) \\ \partial f(e^i) &= \sum_{e^{i-1}} [e^i : e^{i-1}] f(e^{i-1}), \quad f \in F(\Gamma_{k0}, \mathbb{Z}) \end{aligned}$$

where the sum is over all $i-1$ cells.

We now wish to produce a map from the cellular cohomology $H^1(\Gamma_k)$ to the Cech cohomology $\check{H}^1(\Omega)$. This is accomplished by producing 2 maps,

$$\alpha_0 : F(\Gamma_{k0}, \mathbb{Z}) \longrightarrow F(\mathcal{V}^{(0)}, \mathbb{Z})$$

$$\alpha_1 : F(\Gamma_{k1}, \mathbb{Z}) \longrightarrow F(\mathcal{V}^{(1)}, \mathbb{Z})$$

such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & F(\Gamma_{k0}, \mathbb{Z}) \arrow[r, "\partial"] \arrow[d, "\alpha_0"'] & F(\Gamma_{k1}, \mathbb{Z}) \arrow[r, "\partial"] \arrow[d, "\alpha_1"'] & F(\Gamma_{k2}, \mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[r] & F(\mathcal{V}^{(0)}, \mathbb{Z}) \arrow[r, "\check{\partial}"] & F(\mathcal{V}^{(1)}, \mathbb{Z}) \arrow[r, "\check{\partial}"] & F(\mathcal{V}^{(2)}, \mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

where the $\check{\partial}$'s in the bottom row indicate the usual Cech chain complex. In addition to wanting this diagram to commute, we also wish to have the kernel of the ∂ map from $F(\Gamma_{k1}, \mathbb{Z})$ to $F(\Gamma_{k2}, \mathbb{Z})$ to map into the kernel of the $\check{\partial}$ map from $F(\mathcal{V}^{(1)}, \mathbb{Z})$ to $F(\mathcal{V}^{(2)}, \mathbb{Z})$. This will ensure that the α_*'s can be translated to a map α between $H^1(\Gamma_k)$ and $\check{H}^1(\Omega)$.

We will first define α_0. It's easy to see that we have a map from $\mathcal{V}$ to Γ_{k0} – for every vertex pattern look at the vertex at the center of it, and then map it to that vertex in Γ_{k0}. We can then take α_0 to be the dual of this map.

Next, we need to decide what α_1 is. Since $n = 2$, we can assign to the 2-cells arbitrary orientation, so we pick them all to have the same orientation, say clockwise. Suppose we take $f \in F(\Gamma_{k1}, \mathbb{Z})$. Then we want $\alpha_1 f$ to be defined on two-fold intersections of our vertex patterns. If a two-fold intersection of vertex patterns is non-empty, then the vertices at the middle of them must lie on the outside of a common tile t in both vertex patterns. So we define

$$\begin{aligned}\alpha_1 f \; &: \; \mathcal{V} \longrightarrow \mathbb{Z} \\ \alpha_1 f(V_{ab}) &= \sum_{a\to b}[t:e]f(e)\end{aligned}$$

where the sum is over the edges starting from the vertex at the middle of pattern a to the vertex at the middle of pattern b according to the orientation of the cell. If these two vertices lie on the outside of two different tiles t_1 and t_2, ie, they lie at the beginning and end of an edge which connects t_1 and t_2, then this is not well defined. Since we said earlier that all the tiles must have the same orientation, then if e is the edge in question, $[t_1 : e]$ must be either plus or minus 1, with $[t_2 : e] = -[t_1 : e]$ ($[t : e]$ denotes the incidence number of t with respect to edge e). To make our map well-defined, we chose to sum around the tile which has positive orientation number with respect to e.

Claim 4.1.1 *The six-term diagram above commutes. In addition, $\alpha_1(\ker\partial) \subset \ker\check{\partial}$.*

Proof: Say we have $f \in F(\Gamma_{k1}, \mathbb{Z})$. Then

$$\begin{aligned}\alpha_1(\partial f)(V_{ab}) &= \sum_{a\to b}[t:e]\partial f(e) \\ &= \sum_{a\to b}[t:e]\big(f(t(e)) - f(i(e))\big) \\ &= f(v_b) - f(v_a)\end{aligned}$$

where $t(e)$ denotes the terminal point of an edge e and $i(e)$ denotes the initial point of e. The last step follows by collapsing the sum. Notice that if v_a and v_b were on the same edge that we end up with the same answer, as we pick the 2-cell with which the edge has positive orientation. On the other hand, we have

$$\begin{aligned}\check{\partial}(\alpha_0 f)(V_{ab}) &= \alpha_0 f(V_b) - \alpha_0 f(V_a) \\ &= f(v_b) - f(v_b)\end{aligned}$$

Now to show that $\alpha_1(\ker\partial) \subset \ker\check{\partial}$. Take $f \in \ker\partial$. Then, if $\sigma \in \Gamma_{k2}$,

$$(\partial f)(\sigma) = \sum_i [\sigma : e_i] f(e_i)$$

where the sum is over all edges e_i around the tile σ. f is in the kernel, so the sum must be zero. Thus f must sum to zero around all tiles σ. Thus, if V_a and V_b share an edge, summing around either tile that the edge is a part of will give values negative to each other, so that multiplying by the incidence number makes them equal. In other words, if V_a and V_b share and edge e , with the edge adjacent to two tiles t_1 and t_2, then

$$\sum_{a\to b}[t_1 : e] f(e) = \sum_{a\to b}[t_2 : e] f(e)$$

where it is understood that one of the sums is a single term. Now, if V_a, V_b and V_c are vertex patterns with non-empty intersection,

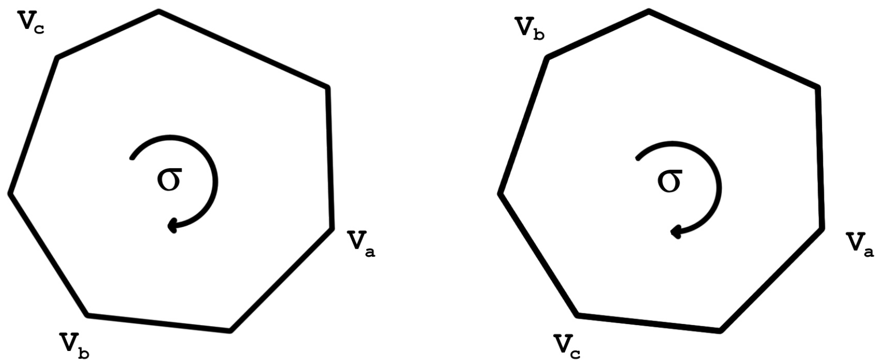

Figure 4.2: Two Possible Arrangements of 3 Vertex Patterns with Non-empty Intersection

$$
\begin{aligned}
\check{\partial}(\alpha_1 f)(V_{abc}) &= \alpha_1 f(V_{bc}) - \alpha_1 f(V_{ac}) + \alpha_1 f(V_{ab}) \\
&= \sum_{a\to b}[\sigma : e]f(e) - \sum_{a\to c}[\sigma : e]f(e) + \sum_{b\to c}[\sigma : e]f(e) \\
&= \sum_{a\to b}[\sigma : e]f(e) + \sum_{c\to a}[\sigma : e]f(e) + \sum_{b\to c}[\sigma : e]f(e)
\end{aligned}
$$

The figures above show the 2 ways in which 3 vertex patterns could be arranged around a tile, with $a < b < c$. In the first case, we see that we sum around once, as we sum from a to b, then from b to c, then from c to a. In the second, we sum around twice. In either case, our value is an integer multiplied by $\sum_i [\sigma : e_i] f(e_i)$, and this is zero because $f \in \ker \partial$. Thus $\alpha_1(\ker \partial) \subset \ker \check{\partial}$. This proves the claim. ∎

4.2 Mapping Čech Cohomology to Dynamical Cohomology to $\Lambda\mathbb{R}^{n*}$

We now describe the mapping Čech cohomology to dynamical cohomology and from dynamical cohomology to $\Lambda\mathbb{R}^{n*}$ in the case of a general topological dynamical system – the following section mentions nothing about tilings. Let (X, φ) be a topological $\mathbb{R}^n$-Dynamical System. Then let μ be an invariant probability measure on X (these always exist by [Gl]). Recall that we can form the dynamical cohomology groups

$$H^i(\mathbb{R}^n, C^\infty(X, \mathbb{R})) = \ker d_i / \operatorname{Im} d_{i-1}.$$

where $d_i : C^\infty(X, \Lambda^i\mathbb{R}^{n*}) \to C^\infty(X, \Lambda^{i+1}\mathbb{R}^{n*})$. From this we can define the following map.

Definition 4.2.1 *The Ruelle-Sullivan current C_μ associated with μ is the linear map*

$$\langle C_\mu, \cdot \rangle : C^\infty(X, \Lambda^k\mathbb{R}^{n*}) \longrightarrow \Lambda^k\mathbb{R}^{n*}$$

defined by

$$\langle C_\mu, f \rangle = \int_X f(x)d\mu(x), \quad f \in C^\infty(X, \Lambda^k\mathbb{R}^{n*})$$

Lemma 4.2.1 *Let μ be an invariant probability measure for the action φ. Let f be any φ-smooth function in $C^\infty(X, \Lambda^k\mathbb{R}^{n*})$ for some k. Then*

$$\langle C_\mu, df \rangle = 0$$

Proof It suffices to show that

$$\int_X \frac{\partial f}{\partial x_i}(x)d\mu(x) = 0$$

for $i = 1, 2, \ldots, n$. Since any invariant measure is the weak-* limit of a convex combination of ergodic measures, we can assume μ is ergodic (see []). By the Birkhoff ergodic theorem, for almost all x in X we have

$$\int_X \frac{\partial f}{\partial x_i}(x)d\mu(x) = \lim_{R\to\infty} \frac{1}{(2R)^n} \int_{[-R,R]^n} \frac{\partial f}{\partial x_i}(\varphi_u(x))d\lambda(u)$$

where λ indicates Lebesgue measure. We have

$$\begin{aligned}\left|\int_{[-R,R]^n} \frac{\partial f}{\partial x_i}(\varphi_u(x))d\lambda(u)\right| &= \left|\int_{[-R,R]^{n-1}} (f|_{u_i=R} - f|_{u_i=-R}du_1 \cdots \hat{du_i} \cdots du_n\right| \\ &\leq 2\left\|f\right\|_\infty (2R)^{n-1}\end{aligned}$$

where $\hat{du_i}$ indicates that du_i is omitted. This means that

$$\left|\int_X \frac{\partial f}{\partial x_i}(x)d\mu(x)\right| \leq \left|\lim_{R\to\infty} \frac{\|f\|_\infty}{R}\right| = 0.$$

This proves the result. ■

Since C_μ is zero on the image of the d maps, it extends to a map on cohomology:

$$\tilde{\tau}_{\varphi,\mu} : H^i(\mathbb{R}^n, C^\infty(X,\mathbb{R})) \longrightarrow \Lambda\mathbb{R}^{n*}$$

Thus, we have found a map from the dynamical cohomology to $\Lambda\mathbb{R}^{n*}$ for any (X, φ). The next step is then to find homomorphism from Čech cohomology to dynamical cohomology, and to compose these into a map from the Čech cohomology to $\Lambda\mathbb{R}^{n*}$, which is called the Ruelle-Sullivan map.

We make the following definition:

Definition 4.2.2 *Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be a finite open cover of X. A **partition of unity subordiante to** $\mathfrak{U}$ is a set of positive-valued functions $\{\rho_i\}$ on X such that $\sum_i \rho_i(x) = 1$ for all $x \in X$ and the support of ρ_i is contained in U_i.*

As before, for $i_0 < i_1 < \cdots < i_j$, we let $U_{i_0 i_1 \cdots i_j} = U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_j}$. For any open set U, we note that even if U is not invariant under φ, we have that, for any continuous function f on U and $v \in \mathbb{R}^n$, the expression $\frac{\partial f}{\partial v}$ still makes sense as a function on U (assuming the limit exists). Thus, we can define $C^\infty(U, \Lambda\mathbb{R}^{n*})$ as the smooth functions on U with values in $\Lambda\mathbb{R}^{n*}$.

Those familiar with the Čech-deRham theorem for manifolds and its proof will see how the following is an adaptation of the argument there to our case where X may not be a manifold. It involves constructing a double complex with appropriate maps to show that we can map the cohomologies to each other.

So, we define the double complex

$$K^{j,k}(\mathfrak{U}) = \bigoplus_{i_0 < i_1 < \cdots < i_j} C^\infty(U_{i_0 i_1 \cdots i_j}, \Lambda^k)$$

where we always sum only over $i_0 < i_1 < \cdots < i_j$ with nonempty $U_{i_0 i_1 \cdots i_j}$. If $f \in K^{j,k}(\mathfrak{U})$, then we denote the $i_0 i_1 \cdots i_j$ component by $f_{i_0 i_1 \cdots i_j}$. Also, for notational purposes, we want to set $f_{i_0 i_1 \cdots i_j} = 0$ if $U_{i_0 i_1 \cdots i_j} = \emptyset$ and $f_{\sigma(i_0)\sigma(i_1)\cdots\sigma(i_j)}$ =sgn$(\sigma) f_{i_0 i_1 \cdots i_j}$, for any permutation σ.

We have a diagram

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\check{\partial}\Big\uparrow & & \check{\partial}\Big\uparrow & & \check{\partial}\Big\uparrow & & \check{\partial}\Big\uparrow & & \\
K^{2,0}(\mathfrak{U}) & \xrightarrow{d} & K^{2,1}(\mathfrak{U}) & \xrightarrow{d} & K^{2,2}(\mathfrak{U}) & \xrightarrow{d} & K^{2,3}(\mathfrak{U}) & \xrightarrow{d} & \cdots \\
\check{\partial}\Big\uparrow & & \check{\partial}\Big\uparrow & & \check{\partial}\Big\uparrow & & \check{\partial}\Big\uparrow & & \\
K^{1,0}(\mathfrak{U}) & \xrightarrow{d} & K^{1,1}(\mathfrak{U}) & \xrightarrow{d} & K^{1,2}(\mathfrak{U}) & \xrightarrow{d} & K^{1,3}(\mathfrak{U}) & \xrightarrow{d} & \cdots \\
\check{\partial}\Big\uparrow & & \check{\partial}\Big\uparrow & & \check{\partial}\Big\uparrow & & \check{\partial}\Big\uparrow & & \\
K^{0,0}(\mathfrak{U}) & \xrightarrow{d} & K^{0,1}(\mathfrak{U}) & \xrightarrow{d} & K^{0,2}(\mathfrak{U}) & \xrightarrow{d} & K^{0,3}(\mathfrak{U}) & \xrightarrow{d} & \cdots
\end{array}$$

where

$$\check{\partial} : K^{j,k}(\mathfrak{U}) \longrightarrow K^{j+1,k}(\mathfrak{U})$$

$$(\check{\partial} f)_{i_0 i_1 \cdots i_j+1} = \sum_{l=0}^{j+1} (-1)^l f_{i_0 \cdots \hat{i}_l \cdots i_{j+1}}$$

with $\hat{i}_l$ indicating that i_l is omitted in the index.

Lemma 4.2.2 *The two differentials, $\check{\partial}$ and d, commute.*

Proof. Let $f \in K^{j,k}(\mathfrak{U})$. Then for any $i_0 < i_1 < \cdots < i_{j+1}$

$$\begin{aligned}
\check{\partial}(df)_{i_0 i_1 \cdots i_j+1} &= \sum_{l=0}^{j+1} (-1)^l df_{i_0 \cdots \hat{i}_l \cdots i_{j+1}} \\
&= \sum_{l=0}^{j+1} (-1)^l \Big(\sum_{i=1}^{n} \frac{\partial f_{i_0 \cdots \hat{i}_l \cdots i_{j+1}}}{\partial e_i} de_i \Big).
\end{aligned} \tag{4.1}$$

While we also have

$$\begin{aligned} d(\check{\partial} f)_{i_0 i_1 \cdots i_j+1} &= \sum_{i=1}^{n} \frac{\partial(\check{\partial} f_{i_0 i_1 \cdots i_j+1})}{\partial e_i} de_i \\ &= \sum_{i=1}^{n} \frac{\partial}{\partial e_i}\Big(\sum_{l=0}^{j+1}(-1)^l f_{i_0 \cdots \hat{i}_l \cdots i_{j+1}}\Big) de_i. \end{aligned} \tag{4.2}$$

The above definition of partial differentiation is linear, so we see by exchanging the order of summation that the two are equal. ■

We now want to define a new map on our complex using the partition of unity found above, $\{\rho_i\}_i$. For $j > 0$, we define

$$h : K^{j,k}(\mathfrak{U}) \longrightarrow K^{j-1,k}(\mathfrak{U})$$

$$(hf)_{i_0 \cdots i_{j-1}} = \sum_i \rho_i f_{i i_0 \cdots i_{j-1}}$$

Lemma 4.2.3 $h\check{\partial} + \check{\partial} h = 1$

Proof Let $f \in K^{j,k}(\mathfrak{U})$ for our cover $\mathfrak{U}$. Then, for $i_0 < i_1 < \cdots < i_j$

$$
\begin{aligned}
(h\check{\partial}f)_{i_0 i_1\cdots i_j} &= \sum_i \rho_i (\check{\partial}f)_{i i_0 i_1 \cdots i_j} \\
&= \sum_i \rho_i \Big(\sum_{l=0}^{j+1} (-1)^l f_{i i_0 \cdots \hat{i_l} \cdots i_j} \Big) \\
&= \sum_i \rho_i \Big(f_{i_0 \cdots i_j} - \sum_{l=0}^{j} (-1)^l f_{i i_0 \cdots \hat{i_l} \cdots i_j} \Big) \\
(\check{\partial}hf)_{i_0 i_1 \cdots i_j} &= \sum_{l=0}^{j} (-1)^l (hf)_{i_0 \cdots \hat{i_l} \cdots i_j} \\
&= \sum_{l=0}^{j} (-1)^l \Big(\sum_i \rho_i f_{i i_0 \cdots \hat{i_l} \cdots i_j} \Big).
\end{aligned} \tag{4.3}
$$

Adding the two sees $(\check{\partial}hf)_{i_0 i_1 \cdots i_j}$ cancel with the similar term in the first equation. Thus

$$
\big(h\check{\partial} + \check{\partial}h\big)(f)_{i_0 i_1 \cdots i_j} = \sum_i \rho_i f_{i_0 i_1 \cdots i_j} = f_{i_0 i_1 \cdots i_j} \Big(\sum_i \rho_i \Big) = f_{i_0 i_1 \cdots i_j},
$$

that is to say, $\big(h\check{\partial} + \check{\partial}h\big)(f) = f$. ■

Lemma 4.2.4 *The restriction map r defined by*

$$
r : C^\infty(X, \Lambda^k \mathbb{R}^{n*}) \longrightarrow K^{0,k}(\mathfrak{U})
$$

$$
r(f)_i = f|_{U_i},
$$

commutes with the differential d.

Proof. Take $f \in C^\infty(X, \Lambda^k\mathbb{R}^{n*})$. Then, for any i, we have

$$\begin{aligned} d(rf)_i &= \sum_{i=1}^{n} \frac{\partial r f_i}{\partial e_i} de_i \\ &= \sum_{i=1}^{n} \frac{\partial f|_{U_i}}{\partial e_i} de_i \\ r(df)_i &= \Big(\sum_{i=1}^{n} \frac{\partial f}{\partial e_i} de_i \Big)|_{U_i}. \end{aligned} \tag{4.4}$$

Since the restriction of a sum is the sum of the restrictions, and partial differentiation is linear, these are equal. ■

If $f \in K^{0,k}(\mathfrak{U})$, f has a unique pre-image under if r when $f_i = f_j$ on U_{ij} whenever U_{ij} is non-empty. Every f in $\ker \check{\partial} \subset K^{0,k}(\mathfrak{U})$ satisfies this, so we can define r^{-1} on the $\ker \check{\partial}$'s by

$$r^{-1} f(x) = f_i(x)$$

where U_i is any member of the cover that contains x. It is easy to see that the map r is bijective onto $\ker \check{\partial}$ and that r^{-1} its inverse.

Corollary 4.2.1 *For any k the following sequence is exact;*

$$0 \longrightarrow C^\infty(X, \Lambda^k\mathbb{R}^{n*}) \xrightarrow{r} K^{0,k}(\mathfrak{U}) \xrightarrow{\check{\partial}} \cdots \xrightarrow{\check{\partial}} K^{j,k}(\mathfrak{U}) \xrightarrow{\check{\partial}} K^{j+1,k}(\mathfrak{U}) \xrightarrow{\check{\partial}} \cdots$$

and thus has trivial cohomology.

Proof First we need to show that $\check{\partial} \circ r = 0$. Take $f \in C^\infty(X, \Lambda^k\mathbb{R}^{n*})$. Then

$$\begin{aligned} (\check{\partial} \circ r)(f)_{ab} &= (rf)_b - (rf)_a \\ &= f|_{U_b} - f|_{U_a} \end{aligned} \tag{4.5}$$

Since $(\check{\partial} \circ r)(f)_{ab}$ is a function on U_{ab}, this difference is 0. Conversely, if $\check{\partial} g = 0$ for $g \in K^{0,k}(\mathfrak{U})$, then we can act on g with r^{-1}, so $g = r \circ r^{-1}$, so $g \in \operatorname{Im} r$.

Now, take $f \in K^{j,k}(\mathfrak{U})$ with $\check{\partial} f = 0$. Then, because $\big(h\check{\partial} + \check{\partial} h\big)(f) = f$, we have that $f = \big(\check{\partial} h\big)(f)$, ie, $f \in \operatorname{Im} \check{\partial}$. On the other hand, we know that $\check{\partial}^2 = 0$ because it is the Čech differential, so we have that $\ker \check{\partial}_j = \operatorname{Im} \check{\partial}_{j-1}$ for $j \geq 1$. Thus the sequence is exact. ■

As before for $i_0 < i_1 < \cdots < i_j$ we let $\check{C}(U_{i_0 i_1 \cdots i_j}, \mathbb{R})$ and $\check{C}(U_{i_0 i_1 \cdots i_j}, \mathbb{Z})$ denote the locally constant functions from $U_{i_0 i_1 \cdots i_j}$ with values in $\mathbb{R}$ and $\mathbb{Z}$, respectively. Since these functions are locally constant, they are obviously smooth. We now define

$$\check{C}^j(\mathfrak{U}, \mathbb{R}) = \bigoplus_{i_0 < i_1 < \cdots < i_j} \check{C}(U_{i_0 i_1 \cdots i_j}, \mathbb{R})$$

$$\check{C}^j(\mathfrak{U}, \mathbb{Z}) = \bigoplus_{i_0 < i_1 < \cdots < i_j} \check{C}(U_{i_0 i_1 \cdots i_j}, \mathbb{Z})$$

We have the inclusions

$$\check{C}(\mathfrak{U}, \mathbb{Z}) \subset \check{C}(\mathfrak{U}, \mathbb{R}) \subset K^{j,0}(\mathfrak{U})$$

These form subcomplexes with the $\check{\partial}$ maps. Let ι denote the inclusion maps of either of the first two in $K^{j,0}(\mathfrak{U})$. Because the partition of unity functions are not locally constant, the cohomology of the subcomplexes does not vanish. Then the cohomologies of these subcomplexes are $\check{H}(\mathfrak{U}, \mathbb{R})$ and $\check{H}(\mathfrak{U}, \mathbb{Z})$, the Čech cohomologies of the covering $\mathfrak{U}$ with coeffiecients in $\mathbb{R}$ and $\mathbb{Z}$, respectively. When it is not necessary to indicate the group used, we will use G.

$\check{H}(\mathfrak{U}, G)$ caries a graded ring structure, as described in [BT]. The product comes from the map

$$\check{C}^j(\mathfrak{U}) \times \check{C}^k(\mathfrak{U}) \longrightarrow \check{C}^{j+k}(\mathfrak{U})$$

$$(f, f') \mapsto f \cdot f'$$

where

$$(f \cdot f')_{i_0 i_1 \cdots i_{k+j}} = (-1)^{jk} f_{i_0 \cdots i_j} f'_{i_j \cdots i_{j+k}},$$

for $f \in \check{C}^j(\mathfrak{U}, G),\ f' \in \check{C}^k(\mathfrak{U}, G),\ i_0 < i_1 < \cdots < i_{j+k}$

Theorem 4.2.1 *The maps*

$$(-1)^j r^{-1}(dh)^j \iota : \check{C}(\mathfrak{U}) \longrightarrow C^\infty(\mathfrak{U}, \Lambda^j)$$

induce a graded ring homomorphism

$$\theta_{\varphi,\mathfrak{U}} : \check{H}(\mathfrak{U}) \longrightarrow H(\mathbb{R}^n, C^\infty(X, \mathbb{R})),$$

for coefficients in either $\mathbb{Z}$ *or* $\mathbb{R}$ *for the Čech cohomology.*

Proof We have the following diagram

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \check{C}^2(\mathfrak{U},G) \arrow[u, "\check{\partial}"] \arrow[r, "\iota"] & K^{2,0}(\mathfrak{U}) \arrow[u, "\check{\partial}"] \arrow[r, "d"] \arrow[d, shift left, "h"] & K^{2,1}(\mathfrak{U}) \arrow[u, "\check{\partial}"] \arrow[r, "d"] \arrow[d, shift left, "h"] & \cdots \\
0 \arrow[r] & \check{C}^1(\mathfrak{U},G) \arrow[u, "\check{\partial}"] \arrow[r, "\iota"] & K^{1,0}(\mathfrak{U}) \arrow[u, shift left, "\check{\partial}"] \arrow[r, "d"] \arrow[d, shift left, "h"] & K^{1,1}(\mathfrak{U}) \arrow[u, shift left, "\check{\partial}"] \arrow[r, "d"] \arrow[d, shift left, "h"] & \cdots \\
0 \arrow[r] & \check{C}^0(\mathfrak{U},G) \arrow[u, "\check{\partial}"] \arrow[r, "\iota"] & K^{0,0}(\mathfrak{U}) \arrow[u, shift left, "\check{\partial}"] \arrow[r, "d"] & K^{0,1}(\mathfrak{U}) \arrow[u, shift left, "\check{\partial}"] \arrow[r, "d"] & \cdots \\
 & 0 \arrow[u] \arrow[r] & C^\infty(X,\Lambda^0\mathbb{R}^{n*}) \arrow[u, "r"] \arrow[r, "d"] & C^\infty(X,\Lambda^1\mathbb{R}^{n*}) \arrow[u, "r"] \arrow[r, "d"] & \cdots \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

and we can prove the claim by chasing though it.

First we need that $\ker\check{\partial} \subset \check{C}^j(\mathfrak{U},G)$ maps to $\ker d \subset C^\infty(X,\Lambda^j\mathbb{R}^{n*})$ under $\theta_{\varphi,\mathfrak{U}}$. Then we must show that if $\check{\partial}f \in \check{C}^j(\mathfrak{U},G)$, then its image under $\theta_{\varphi,\mathfrak{U}}$ is in $\operatorname{Im} d \subset C^\infty(X,\Lambda^j\mathbb{R}^{n*})$.

We begin by claiming that for any $f \in \ker\check{\partial} \subset \check{C}^j(\mathfrak{U},G)$, $(dh)^j\iota f \in \ker\check{\partial}$ and proving it by induction. For $j = 1$, notice that $\iota f \in \ker\check{\partial}$ because the inclusion map commutes with $\check{\partial}$. We proved earlier that $h\check{\partial}$ acts as the identity on such elements, so we have

$$
\begin{aligned}
\check{\partial}h\iota f &= \iota f \\
\Rightarrow d\check{\partial}h\iota f &= d\iota f \\
\Rightarrow \check{\partial}dh\iota f &= d\iota f.
\end{aligned}
$$

We know that ιf is constant on all U_i, so $d\iota f = 0$, and we have proven the case when $j = 1$. Now suppose true for $j-1$, that is, $(dh)^{j-1}\iota f \in \ker\check{\partial}$.

Then we have that

$$\begin{aligned}\check{\partial}h(dh)^{j-1}\iota f &= (dh)^{j-1}\iota f\\ \Rightarrow d\check{\partial}h(dh)^{j-1}\iota f &= d(dh)^{j-1}\iota f\\ \Rightarrow \check{\partial}dh(dh)^{j-1}\iota f &= 0.\end{aligned}$$

The last step follows due to the fact that the differentials commute and that $d^2 = 0$. Thus $(dh)^j\iota f \in \ker\check{\partial}$, and then $r^{-1}(dh)^j\iota f \in \ker\check{\partial}$ makes sense. We know r is one-to-one, so that $dr^{-1}(dh)^j\iota f = 0$ if and only if $rdr^{-1}(dh)^j\iota f = 0$. The maps r and d commute so we see that this is zero (r collapses with its inverse and $d^2 = 0$).

Now to show that if $\check{\partial}f \in \check{C}^j(\mathfrak{U}, G)$, then its image under $\theta_{\varphi,\mathfrak{U}}$ is in $\operatorname{Im} d \subset C^\infty(X, \Lambda^j\mathbb{R}^{n*})$. We will show for the case $j = 1$, the other cases are analagous. Take $\check{\partial}f \in \operatorname{Im}\check{\partial}$ and $x \in U_{i_0} \subset X$. Then

$$\begin{aligned}(-r^{-1}dh\iota\check{\partial}f)(x) &= -d\Big(\sum_i \rho_i(\iota\check{\partial}f)_{ii_0})\Big)\\ &= -d\Big(\sum_i \rho_i(f(U_{i_0}) - f(U_i))\Big)\\ &= -d\Big(\sum_i \rho_i(f(U_{i_0})) - \sum_i \rho_i f(U_i)\Big)\\ &= -d\Big(f(U_{i_0})\sum_i \rho_i - \sum_i \rho_i f(U_i))\\ &= -d\Big(f(U_{i_0}) - \sum_i \rho_i f(U_i)\Big)\\ &= d\Big(\sum_i \rho_i f(U_i)\Big).\end{aligned}$$

This d is the same that acts on $C^\infty(X, \Lambda^0\mathbb{R}^{n*})$, and $\sum_i \rho_i f(U_i)$ is in $C^\infty(X, \Lambda^0\mathbb{R}^{n*})$, so we have shown that if $\check{\partial}f \in \check{C}^1(\mathfrak{U}, G)$, then its image under $\theta_{\varphi,\mathfrak{U}}$ is in

$\operatorname{Im} d \subset C^\infty(X, \Lambda^1\mathbb{R}^{n*})$.

That $\theta_{\varphi,\mathfrak{U}}$ is additive is clear, so we need to show that it respects the graded ring structure. As this property is not explored in later computations, we leave the interested reader to see the proof given in [BT]. ■

Recall that Čech Cohomology of a space X is defined as the direct limit

$$H^k(X, G) = \lim_{\mathfrak{U}} H^k(\mathfrak{U}, G)$$

where the limit is over a directed set of refinements. If $\phi : \mathfrak{V} \to \mathfrak{U}$ are two covers together with the refinement map ϕ such that $V \subset \phi(V)$, then the map ϕ induces a map

$$\phi : \check{H}(\mathfrak{U}) \to \check{H}(\mathfrak{V})$$

over which we take the above direct limit. It is a fact (see [KP]) that

$$\theta_{\varphi,\mathfrak{V}} \circ \phi = \theta_{\varphi,\mathfrak{U}},$$

so that $\theta_{\varphi,\mathfrak{U}}$ induces a graded ring homomorhism $\theta_\varphi : \check{H}(X, G) \to H(\mathbb{R}^n, C^\infty(X, \mathbb{R}))$. This leads to the following definition from [KP], which was the goal of this section; it is what we wish to compute for tiling spaces.

Definition 4.2.3 *Let (X, φ) be an $\mathbb{R}^n$ action with a φ-invariant measure μ. The **Ruelle-Sullivan map** $\tau_{\varphi,\mu} : \check{H}(X, G) \to \Lambda\mathbb{R}^{n*}$ is defined by*

$$\tau_{\varphi,\mu}(a) = \langle C_\mu, \theta_\varphi(a) \rangle .$$

In particular, if we have $a \in \check{H}(\mathfrak{U}, G)$ where $\mathfrak{U}$ is an open cover, then $\tau_{\varphi,\mu}(a) = \langle C_\mu, \theta_{\varphi,\mathfrak{U}}(a) \rangle$. The philosophy, as stated in [KP], is that Čech cohomology together with the Ruelle-Sullivan map furnishes a better invariant for $\mathbb{R}^n$-actions.

4.3 Finding a Translation Invariant Measure on Ω_T when $n = 2$

So we have a way of connecting the cellular cohomology of Γ_k to the Čech cohomology of Ω_T. We now want to connect the Čech cohomology of a covering of Ω_T to the dynamical cohomology of Ω_T - this can be accomplished with our above adaptation of the Čech-deRham theorem, but first we need a translation invariant probability measure on Ω_T. Our construction will be for $n = 2$, a similar construction works for higher dimensions.

For a substitution tiling T with finite local complexity that forces its border, we can do the following - form the space Γ_0, which is a finite CW-Complex whose n-cells are copies of the prototiles $\{p_1, p_2, \ldots p_N\}$. The faces of the 2-cells are identified if they are adjacent in a tiling $\omega^k(T)$ for some k. Let A be the matrix whose (i, j)th entry is the number of times a translate of prototile p_i appears in $\omega(p_j)$. Then A is a matrix of non-negative numbers, and because ω is primitive, the matrix A is primitive in the sense that there is a $k > 0$ such that A^k is a matrix of positive numbers. We now invoke a version of the Perron-Frobenius theorem.

Theorem 4.3.1 *Let A be a primitive $m \times m$ matrix. Then A has a positive eigenvalue c with the following properties:*

1. *c is a simple root of the characteristic polynomial of A*
2. *c has an eigenvector v with only positive entries.*
3. *any other eigenvalue of A has modulus strictly less than c*
4. *any non-negative eigenvector of A is a positive multiple of v*

5. *if u is any non-zero vector in $\mathbb{R}^m$ with non-negative entries, then*

$$\lim_{n\to\infty} c^{-n} A^n u = \langle u, w\rangle \, v$$

where w is an eigenvector for A^t for which $\langle v, w\rangle = 1$

There is a proof of this in [BS]. In our case, the fifth condition says that if ω is applied repeatedly to a patch of tiles, then the proportion of the number of each tile converges to a vector v given in the theorem. More precisely, for any tiling T' in Ω_T, if for each $i = 1, \ldots, N$ we let $Q_i(R)$ denote the number of translates of prototile p_i contained in $T'(B_R(0))$, then

$$\lim_{R\to\infty} \frac{Q_i(R)}{\sum_{k=1}^{N} Q_k(R)} = \frac{v_i}{\sum_{k=1}^{N} v_k}.$$

We also have, and it is easy to prove, that if λ is our inflation constant, then the Perron eigenvalue of the substitution matrix is λ^2.

We can now use this to define a measure μ on Γ_0. For each $i = 1, \ldots, N$, let a be the vector in $\mathbb{R}^N$ such that a_i is the area of p_i. Find the Perron-Frobenius eigenvector v for A and scale it so that $\langle a, v\rangle = 1$. Then if E is a Borel set in Γ_0, define

$$\mu(E \cap p_i) = v_i L(E \cap p_i)$$

This gives a probability measure on Γ_0 because of the $\langle a, v\rangle = 1$ condition. We now construct a measure $\overline{\mu}$ on Ω_T from μ as follows.

Define $\pi : \Omega_T \to \Gamma_0$ by saying that $\pi(T') = (T', 0)_k$. That is, for any tiling T' we find the tile containing the origin and define $\pi(T')$ to be the point corresponding to the origin in the representation of this tile in the cell complex. Now π is easily seen to be continuous and onto, and

$$\pi \circ \omega(T') = (\omega(T'), 0)_k = \gamma_0(T', 0)_k = \gamma_0 \circ \pi(T')$$

so π is a topological semi-conjugacy.

Lemma 4.3.1 $\mu \circ \gamma_0^{-1} = \mu$.

Proof It suffices to check for a Borel set $E \subset p_i$. The set $\Gamma_0^{-1}(E)$ consists of copies of E scattered around the cell complex, each scaled by λ^{-1}. The number of such copies in p_j is the same as the number of translates of p_i in $\omega(p_j)$, which is just A_{ij}. Thus

$$\mu \circ \gamma_0^{-1}(E) = \sum_j A_{ij} v_j \lambda^{-2} \cdot L(E) = \lambda^2 v_i \cdot \lambda^{-2} \cdot L(E) = v_i \cdot L(E) = \mu(E)$$

where $L(E)$ indicates the area of E. ■

Now define $E_\epsilon \subset \Gamma_0$ as all points within ϵ of a 1-cell. If

$$P = \sum_i v_i \cdot perimeter(p_i),$$

then it's clear that $\mu(E_\epsilon) \leq P\epsilon$.

Lemma 4.3.2 *If $C(X)$ denotes the continuous complex-valued functions on any space X, then*

$$C(\Omega_0) \cong \lim_{\rightarrow} C(\Gamma_0) \xrightarrow{\gamma_0^*} C(\Gamma_0) \xrightarrow{\gamma_0^*} \cdots ,$$

where $\gamma_0^* f(x) = f(\gamma_0(x))$. As a consequence,

$$C(\Omega_T) \cong \lim_{\rightarrow} C(\Gamma_0) \xrightarrow{\gamma_0^*} C(\Gamma_0) \xrightarrow{\gamma_0^*} \cdots .$$

Proof The proof utilizes as its main tool the Stone-Weierstass theorem for algebras of continuous functions. Recall that if

$$D = \lim_{\rightarrow} C(\Gamma_0) \xrightarrow{\gamma_0^*} C(\Gamma_0) \xrightarrow{\gamma_0^*} \cdots$$

then D is the disjoint union $\coprod_{n=1}^{\infty} C(\Gamma_0)_n$ with $C(\Gamma_0)_n = C(\Gamma_0)$ for all n and with the equivalence relation saying that $f \in C(\Gamma_0)_n$ is equivalent to $g \in C(\Gamma_0)_m$ if there exists $s \geq m, n$ such that $\gamma_0^{*(s-n)} f = \gamma_0^{*(s-m)} g$. We also have that D is an algebra with the following operations. If $f \in C(\Gamma_0)_n$ and $g \in C(\Gamma_0)_m$, then $[f] + [g] = [\gamma_0^{(s-n)}(f) + \gamma_0^{(s-m)}(g)]$ and $[f][g] = [\gamma_0^{(s-n)}(f)\gamma_0^{(s-m)}(g)]$, where s is any integer with $s \geq m, n$.

Take $f \in D$, then $f \in C(\Gamma_0)_n$ for some n. Define a function f' on Ω_k by saying that $f'(\{x_i\}_{i=1}^{\infty}) = f(x_n)$. Because of the continuity of f and the projection maps on product spaces, we know that f' is continuous. Now, if $f \in C(\Gamma_0)_n$ and $g \in C(\Gamma_0)_m$ are equivalent, then for $x \in C(\Omega_k)$,

$$\begin{aligned}
f'(x) &= f(x_n) \\
&= f(\gamma_0^{(s-n)}(x_s)) \\
&= \gamma_0^{*(s-n)} f(x_s) \\
&= \gamma_0^{*(s-m)} g(x_s) \\
&= g(\gamma_0^{(s-m)}(x_s)) \\
&= g(x_m) \\
&= g'(x).
\end{aligned}$$

Now, if $f \in C(\Gamma_0)_n$ and $g \in C(\Gamma_0)_m$, then for an $s \geq m, n$ we have

$$\begin{aligned}
([f]+[g])'(x) &= [\gamma_0^{(s-n)}(f) + \gamma_0^{(s-m)}(g)]'(x) \\
&= \gamma_0^{(s-n)}(f)(x) + \gamma_0^{(s-m)}(g)(x) \\
&= f(x_n) + g(x_m) \\
&= (f' + g')(x).
\end{aligned}$$

A similar calculation shows that $'$ also respects products. If $f'(x) = 0$ for $f \in C(\Gamma_0)_n$ and all $x \in \Omega_k$, then $f(x_n) = 0$. This means that f is 0 on all of Γ_0, and hence f is zero. This shows that

$$' : D \to C(\Omega_k)$$

is an injection - let D' denote the image of D under this map. Then D is a subalgebra of $C(\Omega_k)$ which trivially contains the constant functions. To see that D' separates points, say we have $x \neq y \in \Omega_k$. Then there must be an n with $x_n \neq y_n$, and we can find a function in $C(\Gamma_0)_n$ that separates x_n and y_n, and its image will this separate x and y. Therefore D' is dense in $C(\Omega_k)$ by Stone-Weierstrass. Clearly D' is closed, so $D' = C(\Omega_k)$. The fact that $'$ is an injection gives us that

$$C(\Omega_0) \cong \varinjlim C(\Gamma_0) \xrightarrow{\gamma_0^*} C(\Gamma_0) \xrightarrow{\gamma_0^*} \cdots .$$

The fact that Ω_0 is homeomorphic to Ω_T when the substitution forces its border gives the result. ■

Consider μ as a linear funcitonal on each $C(\Gamma_0)$ by taking

$$\mu : C(\Gamma_0) \to \mathbb{C}$$

$$\mu(f) = \int_{\Gamma_0} f(x) d\mu(x).$$

Since $\mu \circ \gamma_0^{-1} = \mu$, μ takes the same value on equivalent elements in the direct limit, and so by continuity μ extends to a functional $\overline{\mu}$ on $C(\Omega_k)$. By Reisz Representation, $\overline{\mu}$ is realized as a measure on Ω_k such that when we have $f' \in C(\Omega_k)$,

$$\int_{\Omega_k} f'(x) d\overline{\mu}(x) = \int_{\Gamma_0} f(x) d\mu(x).$$

By definition of π and $\overline{\mu}$, we have that $\overline{\mu} \circ \pi^{-1}(E) = \mu(E)$ for any Borel set $E \subset \Gamma_0$.

Lemma 4.3.3 $\overline{\mu} \circ \omega^{-1} = \overline{\mu}$.

Proof If $E \subset \Gamma_0$ is a Borel set, then we have

$$\begin{aligned}
\overline{\mu} \circ \omega^{-1}(\pi^{-1}(E)) &= \overline{\mu} \circ \pi^{-1} \circ \gamma_0^{-1}(E) \\
&= \mu \circ \gamma_0^{-1}(E) \\
&= \mu(E) \\
&= \overline{\mu}(\pi^{-1}(E))
\end{aligned}$$

The rest follows from the definition of $\overline{\mu}$. ■

Lemma 4.3.4 *If $E \subset (\Gamma_0 - E_\epsilon)$ is a Borel set and $|x| < \epsilon$, then*

$$\overline{\mu}(\pi^{-1}(E) + x) = \overline{\mu} \circ \pi^{-1}(E).$$

Proof Since $E \subset (\Gamma_0 - E_\epsilon)$, it suffices to show for $E \subset p_i - E_\epsilon$ for some i because any $E \subset (\Gamma_0 - E_\epsilon)$ will be the union of such sets. If we think of

E as a subset of all translates of prototile p_i, then $\pi^{-1}(E)$ is the set of all tilings of $\mathbb{R}^n$ where the origin lies in E. If $|x| < \epsilon$, then $E + x$ will still be in p_i, and so $\pi^{-1}(E+x) = \pi^{-1}(E) + x$. Thus

$$\begin{aligned} \overline{\mu}(\pi^{-1}(E) + x) &= \overline{\mu}(\pi^{-1}(E + x)) \\ &= \mu(E + x). \end{aligned}$$

Since $E + x$ is still a subset of the same prototile, this is just $\mu(E)$ as it is Lebesgue measure times a constant, and Lebesgue measure is translation invariant. Thus $\overline{\mu}(\pi^{-1}(E)) = \mu(E) = \overline{\mu} \circ \pi^{-1}(E)$. ■

Theorem 4.3.2 *The measure $\overline{\mu}$ is translation invariant.*

Proof It suffices to take $E \subset \Gamma_0$ Borel, $x \in \mathbb{R}^2$, and show $\overline{\mu}(\pi^{-1}(E)+x) = \overline{\mu} \circ \pi^{-1}(E)$.

Let $\epsilon > 0$ and find $n \geq 0$ such that $|\lambda^{-n}x| < \epsilon$. Then we have

$$\begin{aligned} \overline{\mu}(\pi^{-1}(E) + x) &= \overline{\mu} \circ \omega^{-n}(\pi^{-1}(E) + x) \\ &= \overline{\mu}\Big(\omega^{-n}(\pi^{-1}(E)) + \lambda^{-n}x\Big) \\ &= \overline{\mu}\Big(\pi^{-1}(\gamma_0^{-n}(E)) + \lambda^{-n}x\Big). \end{aligned}$$

On the other hand,

$$\overline{\mu}(\pi^{-1}(E)) = \overline{\mu} \circ \omega^{-n}(\pi^{-1}(E)) = \overline{\mu}(\pi^{-1}(\gamma_0^{-n}(E))).$$

Let $E' = \gamma_0^{-n}(E)$ and $x' = \lambda^{-n}x$. Thus $|x'| < \epsilon$ and

$$\pi^{-1}(E') + x' = \Big[\big(\pi^{-1}(E') + x'\big) \cap \pi^{-1}(\Gamma_0 - E_\epsilon)\Big] \cup \Big[\big(\pi^{-1}(E') + x'\big) \cap \pi^{-1}(E_\epsilon)\Big].$$

The second set in this union is contained in $\pi^{-1}(E_\epsilon)$, so it has measure at most $P\epsilon$. Similarly,

$$\pi^{-1}(E') = \pi^{-1}\big(E' \cap (\Gamma_0 - E_{2\epsilon})\big) \cup \pi^{-1}(E' \cap E_{2\epsilon}.$$

Since $|x'| < \epsilon$, $\pi^{-1}\big(E' \cap (\Gamma_0 - E_{2\epsilon})\big) + x' \subset \pi^{-1}(\Gamma_0 - E_\epsilon)$ and $\pi^{-1}(E' \cap E_{2\epsilon} + x' \subset \pi^{-1}(E_{3\epsilon})$. Thus, intersecting $\pi^{-1}(E') + x'$ with $\pi^{-1}(\Gamma_0 - E_\epsilon)$ yields $\pi^{-1}\big(E' \cap (\Gamma_0 - E_{2\epsilon})\big) + x'$ unioned with a subset of $\pi^{-1}(E_{3\epsilon})$. The measure of the first is equal to $\mu\big(E' \cap (\Gamma_0 - E_{2\epsilon})\big)$ by 4.3.4. The measure of the second is at most $3P\epsilon$. Finally, $\mu\big(E' \cap (\Gamma_0 - E_{2\epsilon})\big)$ is within $2P\epsilon$ of $\mu(E') = \mu(E)$.

Thus, $\overline{\mu}(\pi^{-1}(E)+x)$ is within $6P\epsilon$ of $\mu(E) = \overline{\mu}(\pi^{-1}(E))$. As ϵ is arbitrary, this proves the result. ■

We can now use this measure to define a map from the dynamical cohomology to $\Lambda\mathbb{R}^{n*}$.

4.4 Computing $\langle C_{\overline{\mu}}, \theta_{\varphi,\mathcal{U}} \circ \alpha_1(\cdot)\rangle$.

In this section we compute the image of our map for elements of $H^1(\Gamma_0)$. It can be proved, but it is too long to include here, that the range of the map on $H^1(\Omega_T)$ can be computed from this by using the inverse limit structure of Γ_0. Also, from here on we will be considering cohomology with integer coefficients.

Our measure is defined as weighted Lebesgue measure on our cell complex - so we would like to know what the image of our maps looks like on each 2-cell individually. To do this, we first need to know some properties of the partition of unity functions.

To construct the functions, first consider the x-axis in $\mathbb{R}^2$. Then there exists a smooth function $\rho_\epsilon : \mathbb{R}^2 \to [0,1]$ with the following properties:

1. $\rho_\epsilon(x,y) = 1$ for all $(x,y) \in \mathbb{R}^2$ with $y \geq \epsilon$.
2. $\rho_\epsilon(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$ with $y \leq -\epsilon$.
3. $\rho_\epsilon(x,y) > 0$ for all $(x,y) \in \mathbb{R}^2$ with $\epsilon > y > -\epsilon$.
4. The function is anti-symmetric about the x-axis – $\rho_\epsilon(x,-y) = 1 - \rho_\epsilon(x,y)$.
5. If $R = [a,b] \times [-\epsilon, \epsilon]$, then
$$\int_R \nabla \rho_\epsilon = (b-a)[0,1]^t.$$
6. $|\nabla \rho_\epsilon| \leq K\epsilon^{-1}$ for some positive real constant K.

Now this function is defined in terms of the x-axis, but we can see that we can define similar functions in terms of any line ℓ in $\mathbb{R}^2$. That is to say, $\rho_{\ell,\epsilon}$ would be a smooth function on $\mathbb{R}^2$ with values between 1 and 0 that takes value 1 on one side of an ϵ-band around ℓ, 0 on the other side, and slopes up smoothly in the ϵ-band in such a way that the integral condition is satisifed, see Figure 4.3. In the figure, $v = \int_R \nabla \rho_{\ell,\epsilon}$ and is the vector which points perpendicular to to the direction of ℓ from the region where $\rho_{\ell,\epsilon} = 0$ to where it is 1 and whose magnitude is the length c.

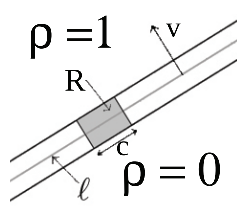

Figure 4.3: Properties of $\rho_{\ell,\epsilon}$

Extending this, say we have two rays ℓ_1 and ℓ_2 in the plane which meet at a point. This path $\ell_1 \cup \ell_2$ then divides the plane in 2 pieces. Then the function $\rho_{\ell_1,\epsilon}\rho_{\ell_2,\epsilon}$ will take the value 1 on the concave side of the ϵ-band around $\ell_1 \cup \ell_2$ and the value 0 on the other side, see Figure 4.4. This function has the properties of the $\rho_{\ell,\epsilon}$, namely, that $\int_R \rho_{\ell_1,\epsilon}\rho_{\ell_2,\epsilon} = v$, where v has length equal to the length of the region R provided R does not intersect with the dotted region; the bound on the gradient follows as well from the product rule. We can also see that if we wanted a smooth function which is 1 on the convex side and 0 on the concave side with the same properties, then we could take the function $1 - \rho_{\ell_1,\epsilon}\rho_{\ell_2,\epsilon}$.

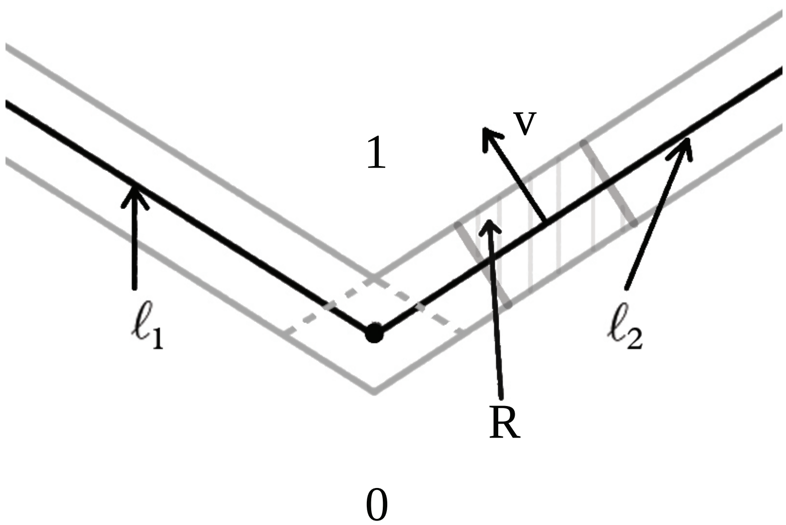

Figure 4.4: $\rho_{\ell_1,\epsilon}\rho_{\ell_2,\epsilon}$

Now, say we have a star-shaped polygon p (convex or not) in the plane, and take an ϵ-band around it. Suppose we want to find a function which is 1 in the interior of the polygon, 0 outside it, and which is smooth in the

ϵ-band. As in Section 4.1, we can pick a point in the interior of p about which is it starshaped and connect it with a line to the middle of each edge and form the regions r_{v_i} for each vertex v_i, see Figure 4.5.

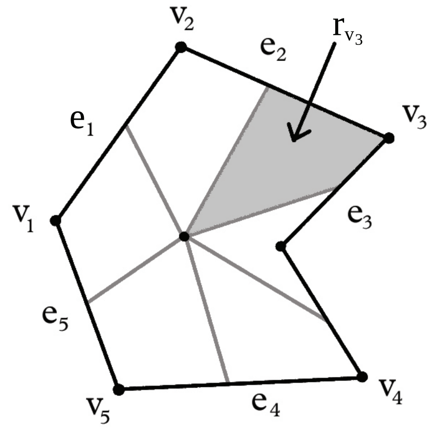

Figure 4.5: Splitting of a polygon into regions.

Then we can define our function $\rho_{p,\epsilon}$ piecewise on $B_\epsilon(r_{v_i})$ by using the ρ functions as earlier. For example, in Figure 4.5,

$$\rho_{p,\epsilon}|_{B_\epsilon(r_{v_3})} = \big(\rho_{e_3,\epsilon}\rho_{e_2,\epsilon}\big)|_{B_\epsilon(r_{v_3})}.$$

If we define $\rho_{p,\epsilon}$ to be 0 outisde of $B_\epsilon(p)$, then it is clear that the functions will match up where the regions meet (taking 1 - $\rho_{e_i,\epsilon}\rho_{e_j,\epsilon}$ in areas where the edges e_i, e_j meet concavely). This function will share the integration properties of the $\rho_{e,\epsilon}$ on the appropriate regions around edge segments.

Finally, recall our open cover $\mathcal{V} = \{V_i\}_{i=1}^L$ of Ω_T, where the set V_i consists of all tilings which are translates of tilings which have the pattern $T(v_i) - v_i$ at the origin translated by vectors which keep the origin in r_{v_i}. For each V_i, define a function $\rho_i^0 : \Gamma_0 \to [0,1]$ as follows. If $T(v_i) - v_i$ is the vertex pattern associated with V_i, then Figure 4.6 shows $O(v_i)$ (defined back in Section 4.1). Now $T(v_i) - v_i$ can be viewed as a subset of Γ_0 by seeing it as the union of the 2-cells associated with the tiles in vertex pattern $T(v_i) - v_i$. This in turn can be seen as a subset of $\mathbb{R}^n$, so we define $\rho_i^0 := \rho_{p_i,\epsilon}$, our function from the previous paragraph. This function is not defined on Γ_0 because what happens near the edges of the 2-cells will be different for each vertex pattern it is in. Now, for any tiling $T' \in \Omega_T$, define $\rho_i(T') = \rho_i^0(T', 0)_0$. After normalizing, this is a partition of unity subordinate to $\mathcal{V}$.

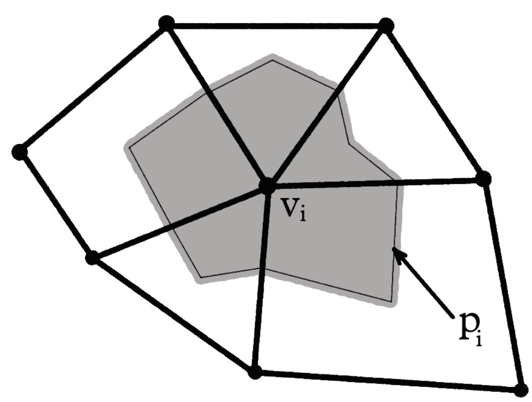

Figure 4.6: $O(v_i)$ with the polygon p_i.

Now, say we have a cocycle f in $H^1(\Gamma_0)$ for a tiling T of $\mathbb{R}^2$ which forces its border. Then f is a function from the finite set of 1-cells in Γ_0 to $\mathbb{Z}$, and can thus be thought of as a vector in $\mathbb{Z}^m$ whose entires sum to zero around any 2-cell, where m is the number of 1-cells in Γ_0. When we map to Čech cocycles, we want to know what value the image of f will take on the pairwise intersections of patterns around a given 2-cell. As described above, it takes the same value that f does on the edge connecting two patterns, respecting orientation.

Say we have the patch $T(\sigma)$ as shown (see Figure 4.7) in our tiling T. Suppose also we have an $f \in H^1(\Gamma_0)$ with the values $f(e_{12}) = 1$, $f(e_{23}) = 1$, $f(e_{13}) = 2$. Then

$$\begin{aligned}
h\alpha_1 f(V_1) &= \sum_i \alpha_1 \rho_i f(V_{i1}) \\
&= \rho_2 \alpha_1 f(V_{21}) + \rho_3 \alpha_1 f(V_{31}) \\
&= -\rho_2 - 2\rho_3 \\
h\alpha_1 f(V_2) &= \sum_i \alpha_1 \rho_i f(V_{i2}) \\
&= \rho_1 \alpha_1 f(V_{12}) + \rho_3 \alpha_1 f(V_{32}) \\
&= \rho_1 - \rho_3 \\
h\alpha_1 f(V_3) &= \sum_i \alpha_1 \rho_i f(V_{i3}) \\
&= \rho_1 \alpha_1 f(V_{13}) + \rho_2 \alpha_1 f(V_{23}) \\
&= 2\rho_1 + \rho_2
\end{aligned}$$

Viewing σ as a subset of $\mathbb{R}^2$, these are smooth real-valued functions on the open set $\cup_{x\in\sigma} B_\epsilon(x)$. The d map is then just the gradient of these functions. These functions do not agree on σ, but their gradients do, except possibly at the points of 3-way intersection (see [BT]). Refering to the above example $h\alpha_1 f(V_1) = -\rho_2 - 2\rho_3$ has value –1 at and around the vertex V_2, –2 at and around the vertex V_3 and has value 0 at and around vertex V_1, see Figure

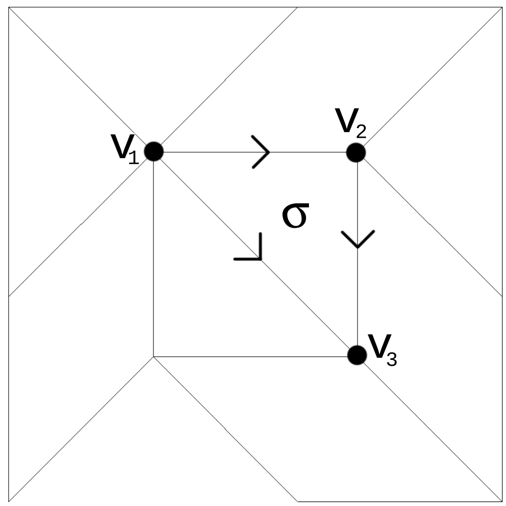

Figure 4.7: $T(\sigma)$

4.9. It slopes smoothly from one value to another at the places of 2-way intersection between vertex patterns. Thus we have that $-dh\alpha_1 f(V_1)$ is 0 at and around the three vertices and is non-zero only on the places of 2-way intersection, where it takes vector values depending on the direction of the gradient, see Figure 4.10. Now, if we look at $h\alpha_1 f(V_2) = \rho_1 - \rho_3$ instead, we see that it has value 1 at and around V_1, value -1 at and around V_3 and value 0 at and around V_2. Thus it is just $-\rho_2 - 2\rho_3 - 1$ and, differing just by a constant, has the same gradient. The same holds for $h\alpha_1 f(V_3) = 2\rho_1 + \rho_2$

For our f, we can then do this for each 2-cell in Γ_0, and get smooth $\Lambda\mathbb{R}^{*2}$ functions on all the 2-cells. Aside from a small neighborhood around the edges, these define functions on Γ_0 which we can integrate over our measure μ on Γ_0. In short, to compute the image of a cocycle in $H^1(\Gamma_0)$, we do the following.

1. Take $f \in H^1(\Gamma_0)$. Then for each 1-cell (edge) e in Γ_0, $f(e) \in \mathbb{Z}$.

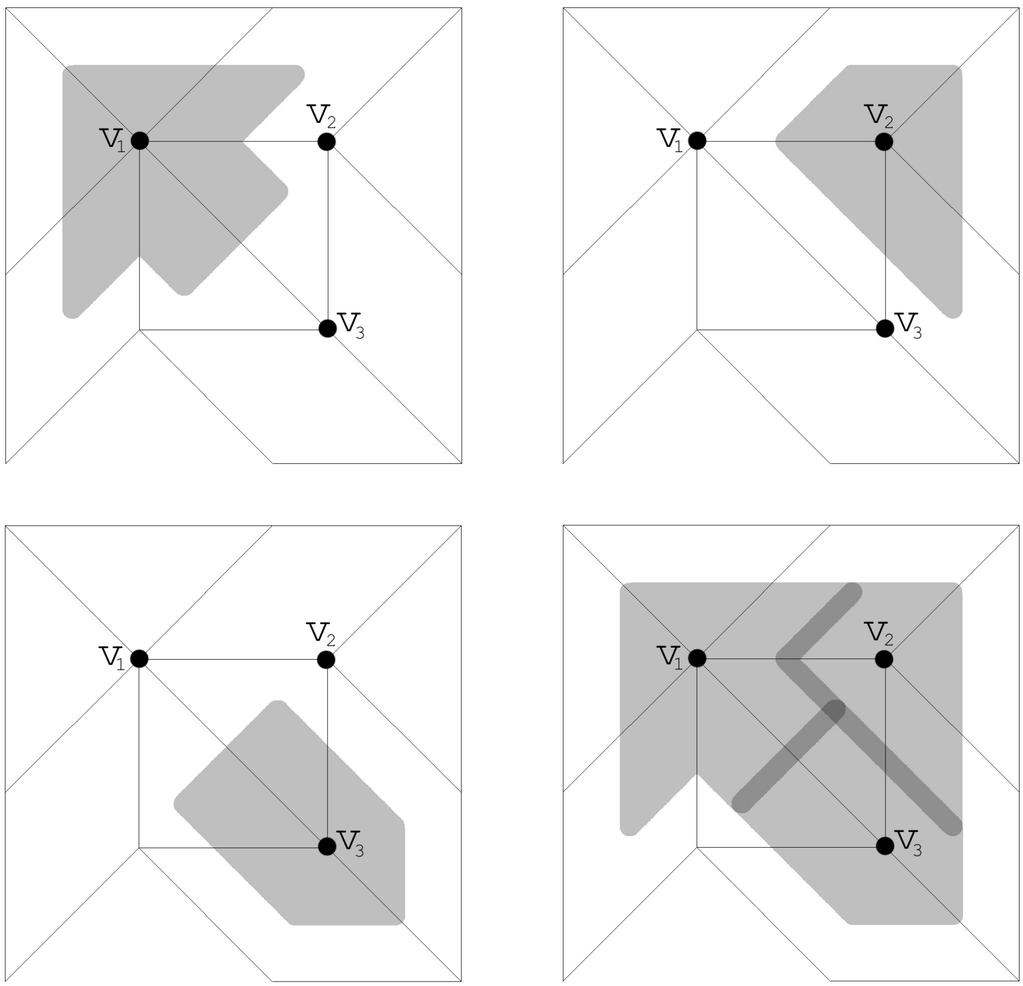

Figure 4.8: Three Vertex Patterns and their Intersections

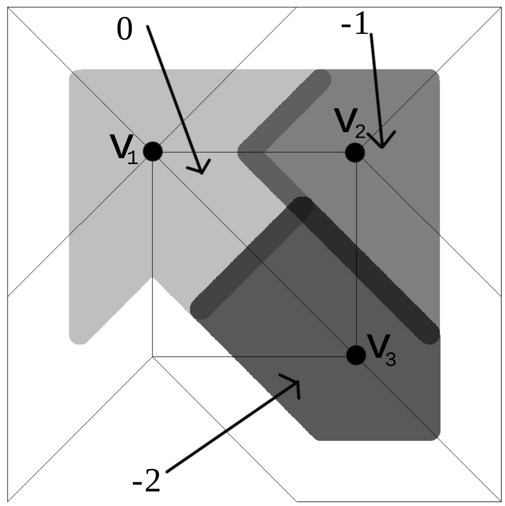

Figure 4.9: $h\alpha_1 f(V_1)$

Represent Γ_0 visually as a cell complex, as in 5.3.

2. As outlined in Section 4.1, pick a point in the interior of each 2-cell about which it is star-shaped, and mark it with a dot. For each edge e for which $f(e) \neq 0$, place a dot in the interior of e.

3. We now have a collection of dots in the interiors of some 2-cells and 1-cells. For each dot on an edge e, connect it to each 2-cell for which e is a face with a line. Call the union of these lines $l(f)$, see Figure 4.11. The image of f will be non-zero on

$$L(f) = \bigcup_{x \in l(f)} B_\epsilon(x).$$

In Figure 4.11, we see that this area is shaded on the tile p_i.

4. Let $l(f)_{j,i}$ be the line from the point in the interior of edge e_j to the point in the interior of p_i. Restricting to p_i and integrating over Lebesgue

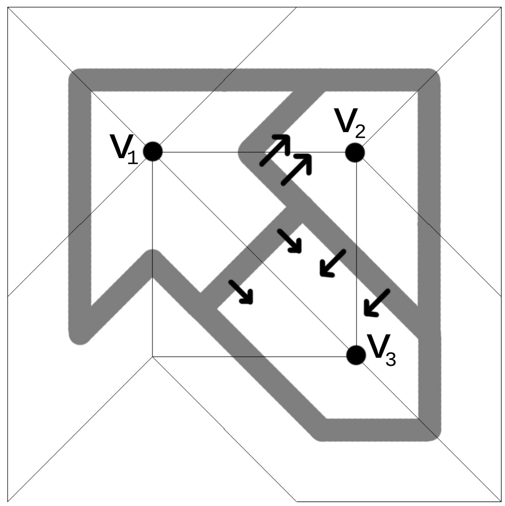

Figure 4.10: $-dh\alpha_1 f(V_1)$

measure, $\langle C_{\overline{\mu}}, \theta_{\varphi,\mathcal{U}} \circ \alpha_1(\cdot)\rangle$ will then be

$$\sum_j [p_i : e_j] f(e_j) |l(f)_{j,i}| u_j^{\perp}$$

Where $|l(f)_{j,i}|$ stands for the length of the line segment and $u_j^{\perp}$ is the unit vector $u_x dx + u_y dy$ which is perpendicular to $l(f)_{j,i}$ whose direction agrees with the orientation of p_i.

5. The full integral $\langle C_{\overline{\mu}}, \theta_{\varphi,\mathcal{U}} \circ \alpha_1(f)\rangle$ will then be the weighted sum of these taken over each 2-cell. The weights are the v_i - the components of the Perron-Frobenius eigenvector v from when we defined the measure. That is to say,

$$\langle C_{\overline{\mu}}, \theta_{\varphi,\mathcal{U}} \circ \alpha_1(f)\rangle = \sum_i v_i \Bigg(\sum_j [p_i : e_j] f(e_j) |l(f)_{j,i}| u_j^{\perp} \Bigg)$$

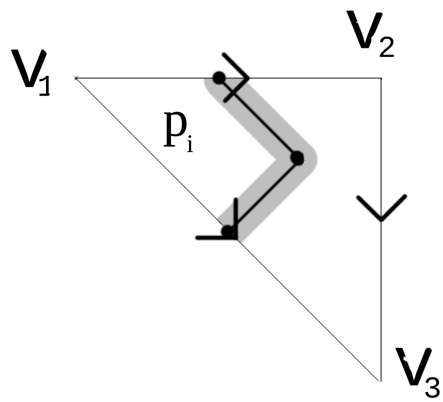

Figure 4.11: $l(f)$ and $L(f)$ on p_i

Theorem 4.4.1 *Given the setup above,*

$$\langle C_{\overline{\mu}}, \theta_{\varphi,\mathcal{U}} \circ \alpha_1(f) \rangle = \sum_i v_i \left(\sum_j [p_i : e_j] f(e_j) |l(f)_{j,i}| u_j^{\perp} \right)$$

Proof Let $\{p_1, p_2, \ldots p_{N_2}\}$, $\{e_1, e_2, \ldots e_{N_1}\}$ and $\{\nu_1, \nu_2, \ldots \nu_{N_0}\}$ denote the 2-, 1-, and 0-cells in the cell complex for Γ_0 respectively. Let $\mathcal{V}$ be our open cover of Ω_T by vertex patterns, and let φ_x denote the action of translating a tiling in Ω_T by $x \in \mathbb{R}^2$. Let $f \in H^1(\Gamma_0)$. Then

$$\begin{aligned}
-r \circ \theta_{\varphi,\mathcal{V}} \circ \alpha_1(f)_{i_0} &= dh\iota\alpha_1(f)_{i_0} \\
&= d(h(\alpha_1 f)_{i_0}) \\
&= d\left(\sum_i \rho_i (\alpha_1 f)_{i i_0} \right) \\
&= d\left(\sum_i \rho_i (\sum_{i \to i_0} [t_{i i_0} : e] f(e)) \right)
\end{aligned}$$

where $\sum_{i\to i_0}[t_{ii_0}:e]f(e))$ indicates that the vertex patterns V_i and V_{i_0} meet in a tile t_{ii_0}, and we sum around the tile according to the orientation of the cell. As before, if two patterns meet in two different tiles, then it must be that the vertices at the middle of the patterns are connected by an edge e - if this is the case then we sum around the tile t for which $[t:e]$ is postitive (all 2-cells are assumed to have the same orientation). Because f is a cocycle, the sum will be $\pm f(e)$ - positive if the orientation on e runs from i to i_0 and negative if otherwise.

We can split the sum into two cases:

$$-r\circ\theta_{\varphi,\mathcal{V}}\circ\alpha_1(f)_{i_0} = d\Bigg(\sum_{i\nsim i_0}\rho_i(\sum_{i\to i_0}[t_{ii_0}:e]f(e))\Bigg) + d\Bigg(\sum_{i\sim i_0}\rho_i f(e_{i\to i_0})\Bigg),$$

where we write $i\sim i_0$ if there an edge in common between the patterns V_i and V_{i_0}, and $i\nsim i_0$ otherwise. Also, $f(e_{i\to i_0})$ is the value of f at the edge from the vertex at the middle of pattern i and the vertex at the middle of pattern i_0, where it is understood that we take the negative of the value if the orientation runs from i_0 to i.

For any indices m,k, if $m\sim k$, then on the set V_{mk}, $\sum_{i\sim m}\rho_i f(e_{i\to m}) = \sum_{i\sim k}\rho_i f(e_{i\to k})$ except on a set whose measure is proportional to ϵ^2 (this is the the area around the chosen points at the centers of tiles and around the edges, it's easy to see from the discussion at the beginnning of Section 4.4 that this is indeed the case). Thus there is a function F in $C^\infty(\Omega_T,\Lambda^1\mathbb{R}^{2*})$ whose image under r agrees with $d\Bigg(\sum_{i\sim i_0}\rho_i f(e_{i\to i_0})\Bigg)$ except on a set of measure proportional to ϵ^2, and it is described as follows.

For each prototile $p_k\in\mathbb{R}^2$, look at $l(f)\cap p_k$, the line segments constructed in steps 1-3 on pages 65-67. As stated there, F will be non-zero on the ϵ

neighborhood around this collection of line segments. The exact form of F is given piecewise on each line segment $l(f)_i$ as

$$d\rho_{l_i,\epsilon}[p_k : e_i]$$

whose vectors agree with the orientation of p_k. The function F on p_k is then the sum of these over all line segments. This is a function on Γ_0, but can be seen as a function on Ω_T because of our map from Ω_T to Γ_0. Now, $\theta_{\varphi,\mathcal{V}} \circ \alpha_1(f)$ agrees with F except on a set of measure proportional to ϵ^2 around the edges of the prototiles, because $\theta_{\varphi,\mathcal{V}} \circ \alpha_1(f)$ may take different values along the edges depending on what prototiles were around it in the tiling while F was defined on the prototiles independent of tiles around it. Because of the bounds on the gradients given at the beginning of this section, we have that

$$|\langle C_{\overline{\mu}}, \theta_{\varphi,\mathcal{U}} \circ \alpha_1(f)\rangle - \langle C_{\overline{\mu}}, F\rangle| < P\epsilon$$

where P is some positive real constant. Now,

$$\langle C_{\overline{\mu}}, F\rangle = \sum_i v_i \left(\sum_j [p_i : e_j] f(e_j) |l(f)_{j,i}| u_j^{\perp} \right)$$

where v_i is the weight on tile p_i, $[p_i : e_j]$ is the incidence number between p_i and e_j, $|l(f)_{j,i}|$ is the length of the line segment in p_i adjacent to e_j and $u_j^{\perp}$ is the unit vector perpendicular to $l(f)_{j,i}$ whose direction agrees with the orientation of p_i. Now since ϵ is arbitrary, we have proved the formula. ■

Chapter 5

Computations

Now that we have a definition of the Ruelle-Sullivan Map, we would like to compute it for some examples. The two that will be looked at are the Octagonal Tiling and the Penrose Tiling. The first cellular cohomology group of these two tilings are both isomorphic to $\mathbb{Z}^5$. We would like to see if our map can distinguish between these obviously different tilings.

5.1 The Octagonal Tiling

The octagonal tiling is a substitution tiling with 20 prototiles. It has finite local complexity, and the substitution shown in Figure 5.1 (and taken from [KP2]) forces its border. The prototiles are those pictured as well as their flips about the horizontal and rotations though $n\pi/4$. The substitution rule extends by symmetry. Since we have a 2-dimensional tiling, the Cohomology chain will look like

$$0 \xrightarrow{0} F(\Gamma_{00}, \mathbb{Z}) \xrightarrow{\partial_0} F(\Gamma_{01}, \mathbb{Z}) \xrightarrow{\partial_1} F(\Gamma_{02}, \mathbb{Z}) \xrightarrow{0} 0$$

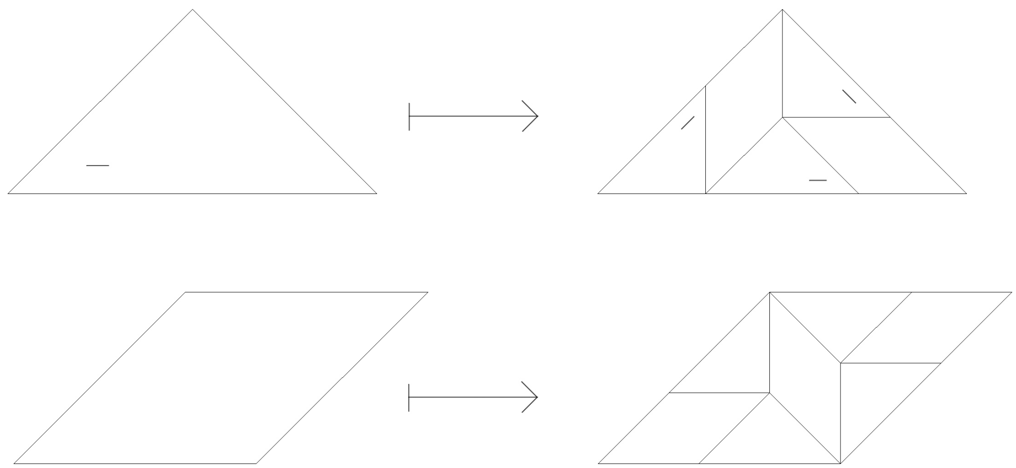

Figure 5.1: Substitution Rule for Octagonal Tiling

If we start by giving any vertex a label, say x, then go though and give to all the vertices that could touch our first vertex the same label, and repeat with the vertices so given our label, we see that all the vertices will be given the same label, thus $F(\Gamma_{00}, \mathbb{Z}) \cong \mathbb{Z}$, with a generator being the function that takes x to 1[1]. This means that ∂_0 is the zero map, because $(\partial_0 f)(e) = f(t(e)) - f(i(e))$ for each edge, where $t(e)$ and $i(e)$ denote the terminal vertex and initial vertex of e, respectively. These vertices are equal, so this is always 0. Thus, $H^0(\Gamma_0) = 0$ and $H^1(\Gamma_0) \cong \ker \partial_1$.

Consider our cell complex, Figure 5.3. Since we have 16 edges, ie, $\Gamma_{01} = \{\mathtt{1}, \mathtt{2}, \ldots, \mathtt{16}\}$, each element g of $F(\Gamma_{01}, \mathbb{Z})$ can be viewed as a vector $[g(\mathtt{1}), \ldots, \mathtt{g}(\mathtt{16})]^{\mathtt{t}} \in \mathbb{Z}^{16}$. Similarly, if g is in $F(\Gamma_{02}, \mathbb{Z})$, then is can also be viewed as a vector $[g(\mathbf{1}), g(\mathbf{2}), \ldots, g(\mathbf{20})]^t \in \mathbb{Z}^{20}$. Thus the map ∂_1 can be

[1]This also means that our cell complex is not a regular CW-complex, as the homeomorphisms that take the edges to the unit interval do not extend to their boundaries.

viewed as a linear transformation from $\mathbb{Z}^{16}$ to $\in \mathbb{Z}^{20}$, ie, a 20×16-matrix of integers. If we call this matrix A_{∂_1}, then $(A_{\partial_1})_{ij}$ is the incidence number of edge j and row i. This matrix is

$$\begin{bmatrix}
1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}$$

This matrix has rank 11. It therefore has a kernel of dimension 5. Its kernel is generated by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

As discussed at the end of last chapter, we can represent the image of our cocycles on the cell complex as shown in Figures 5.4, 5.5 and 5.6. The shaded areas denote where the function is non-zero.

Figure 5.7 shows how v_1 looks represented on a larger patch of the tiling. v_2 and v_3 look similar when represented in this way, just rotated by appropriate multiples of $\pi/4$ (ie, v_2 looks like v_1 rotated by $\pi/4$, v_3 looks like v_1 rotated by $2\pi/4$ etc.). v_4 looks different however, see Figure 5.8. If we were to represent v_5 on the tiling, it would be non-zero on the ϵ neighbourhoods of all the lines connecting the centers of the tiles to the centers of the edges to which it is adjacent.

To compute the Ruelle-Sullivan Map, we need our measure. We first take our substitution matrix

$$
\begin{bmatrix}
2&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1\\
0&2&0&1&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0\\
1&0&2&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1\\
0&1&0&2&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0\\
0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&0&1&0&0&0\\
1&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&0&0\\
0&1&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&0\\
0&0&1&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&0&1&0&1&0&0&0&0&0&1&1&0&0&0&0&0&0&0\\
1&0&0&0&1&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&1&0&0&0&1&0&1&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&1&0&1&0&0&0&0&0&0&1&0&0&0&0\\
1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0\\
0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1&0\\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1\\
0&0&0&1&0&0&0&0&0&0&0&1&1&0&0&0&0&0&1&0\\
1&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1\\
0&1&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&0&0&0\\
0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&0&0\\
0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&0
\end{bmatrix}
$$

Which has Perron-Frobenius eigenvalue $3 + 2\sqrt{2}$ with eigenvector

$$u_{oct} = \xi[2\sqrt{2}, 2\sqrt{2}, 2\sqrt{2}, 2\sqrt{2}, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]^t$$

for any $\xi \in \mathbb{R}$. The area vector is, if we let the side of the rhomb be b,

$$a_{oct} = \frac{b^2}{2}[\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]^t$$

Thus our condition $a_{oct} \cdot u_{oct} = 1$ on the measure leaves us with

$$\frac{b^2\xi}{2}\big(8(\sqrt{2})^2 + 16\big) = 32\frac{b^2\xi}{2} = 1$$

Thus the eigenvector we want is

$$P := \frac{1}{16b^2}[2\sqrt{2}, 2\sqrt{2}, 2\sqrt{2}, 2\sqrt{2}, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]^t.$$

So

$$\tau v_1 = \sum_{\sigma_i \in \Gamma_{01}} \int_{\sigma_i} (-1)\big(dh\alpha_1\big)(v_1)$$

Back when defining our open cover, we chose points in the interiors of the cells. For this example, pick the points to bisect the 1-cells, and the points in the 2-cells as shown. Because we are integrating the partition of unity functions, the integration is easy. On σ_1, for example:

$$\begin{aligned}\int_{\sigma_1}(-1)\big(dh\alpha_1\big)(v_1) &= (\frac{1}{\sqrt{2}}dx - \frac{1}{\sqrt{2}}dy)bP_1 \\ &= \frac{1}{8b}(dx - dy)\end{aligned}$$

And on σ_6

$$\begin{aligned}\int_{\sigma_6}(-1)\big(dh\alpha_1\big)(v_1) &= \Big(\big(\frac{1}{\sqrt{2}}dx - \frac{1}{\sqrt{2}}dy\big)\frac{b}{2\sqrt{2}} + \big(\frac{1}{\sqrt{2}}dx + \frac{1}{\sqrt{2}}dy\big)\frac{b}{2\sqrt{2}}\Big)P_6 \\ &= \frac{1}{32b}dx\end{aligned}$$

In fact, on each of the triangles where the function is non-zero, the integral computes to $\frac{1}{32b}dx$, while on the other rhomb σ_4, it computes to $\frac{1}{8b}(dx+dy)$, bringing the final sum to $\frac{1}{2b}dx$ for the space. The direction of this vector makes sense if one views the vector field on a patch of the tiling - see Figure

5.7. Similarly,

$$
\begin{aligned}
\sum_{\sigma_i \in \Gamma_{01}} \int_{\sigma_i} (-1)\big(dh\alpha_1\big)(v_1) &= \frac{1}{2b} dx \\
\sum_{\sigma_i \in \Gamma_{01}} \int_{\sigma_i} (-1)\big(dh\alpha_1\big)(v_2) &= \frac{1}{2b}\Big(\frac{1}{\sqrt{2}} dx + \frac{1}{\sqrt{2}} dy\Big) \\
\sum_{\sigma_i \in \Gamma_{01}} \int_{\sigma_i} (-1)\big(dh\alpha_1\big)(v_3) &= \frac{1}{2b} dy \\
\sum_{\sigma_i \in \Gamma_{01}} \int_{\sigma_i} (-1)\big(dh\alpha_1\big)(v_4) &= \frac{1}{4b}\Big(1dx + (1+\sqrt{2})dy\Big) \\
\sum_{\sigma_i \in \Gamma_{01}} \int_{\sigma_i} (-1)\big(dh\alpha_1\big)(v_5) &= 0
\end{aligned}
$$

This may be alarming. The first three computations result in vectors which are rotations of each other by multiples of $\frac{\pi}{4}$. It is curious then, that the rotations of the fourth generator do not appear, as the octagonal tiling is symmetric with respect to rotations of multiples of $\frac{\pi}{4}$. Well, it turns out that if we take our generating set to be $\{v_4 - v_1, v_2, v_3, v_4, v_5\}$ then the image of $v_4 - v_1$ is the rotation of the image of v_4 by $\frac{\pi}{4}$.

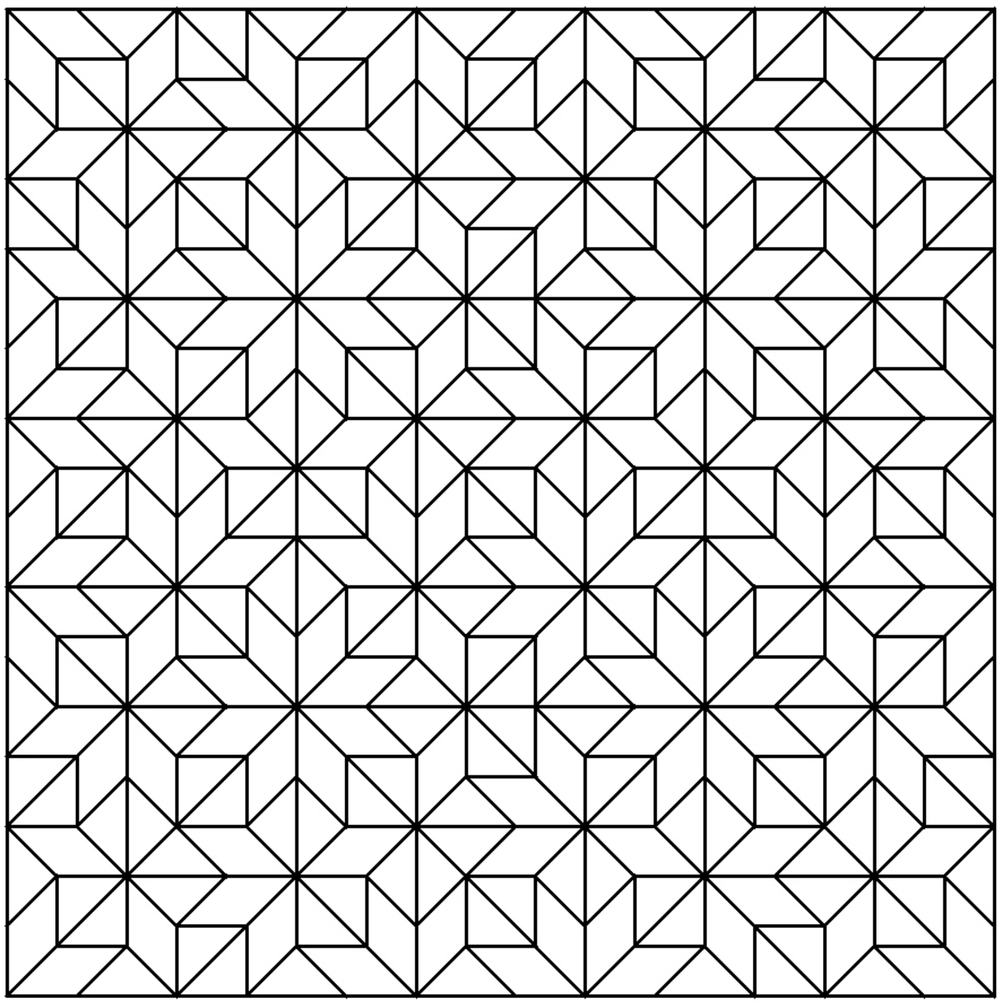

Figure 5.2: A Patch of an Octagonal Tiling

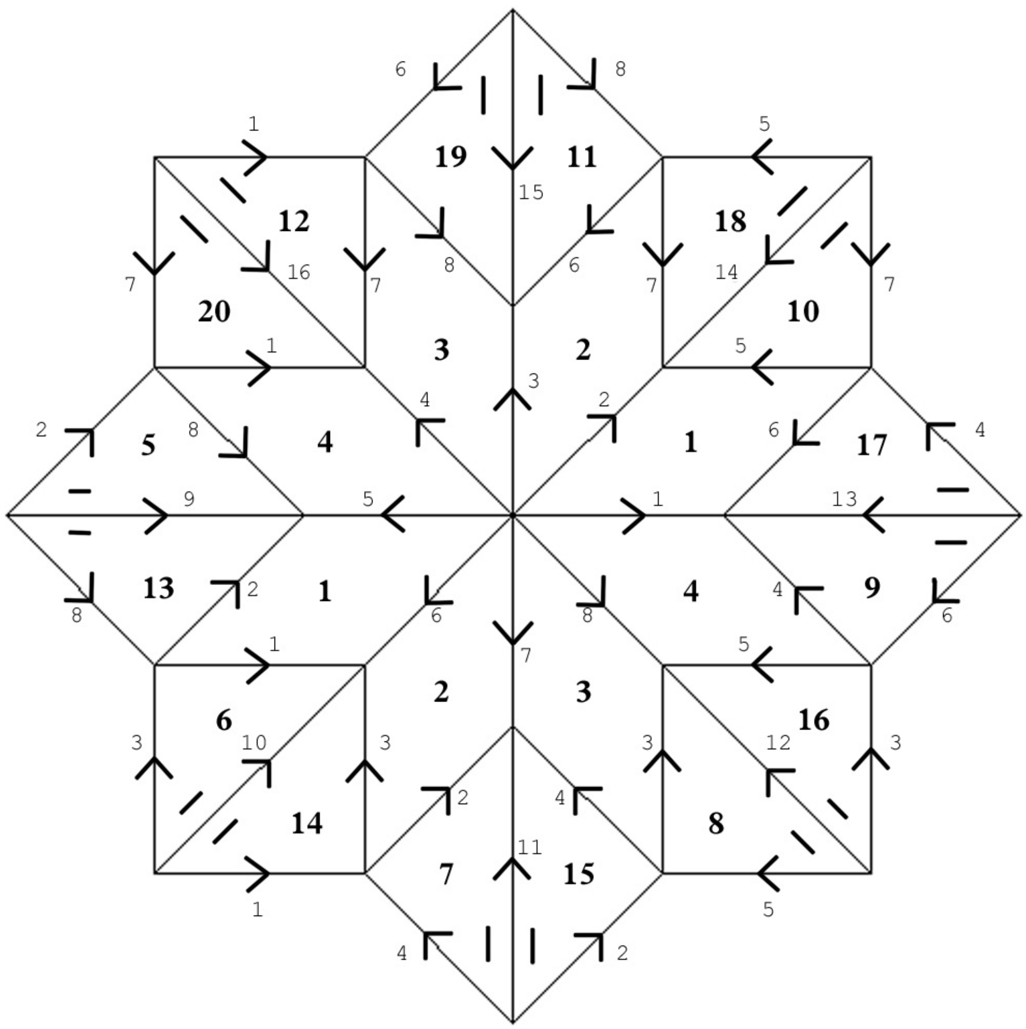

Figure 5.3: Cell Complex Generated by the Octagonal Tiling

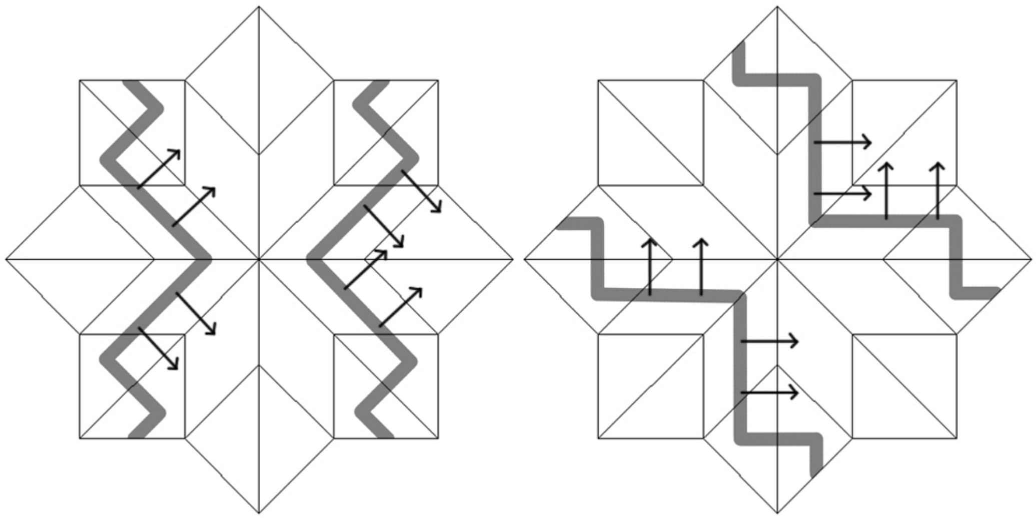

Figure 5.4: Representation of v_1, v_2 on the Cell Complex

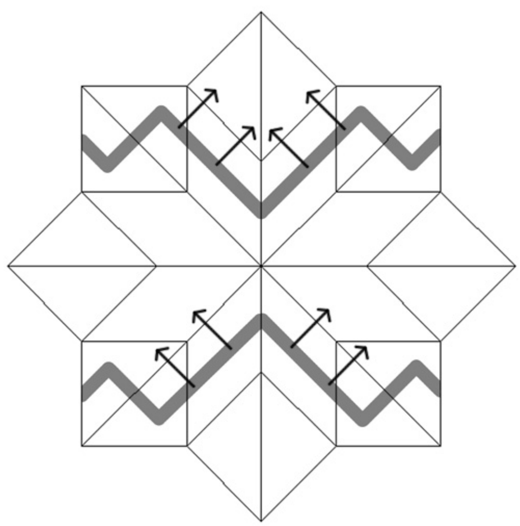

Figure 5.5: Representation of v_3 on the Cell Complex

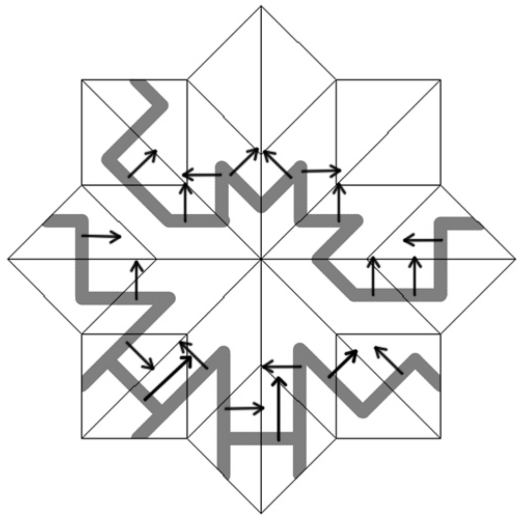

Figure 5.6: Representation of v_4 on the Cell Complex

∇

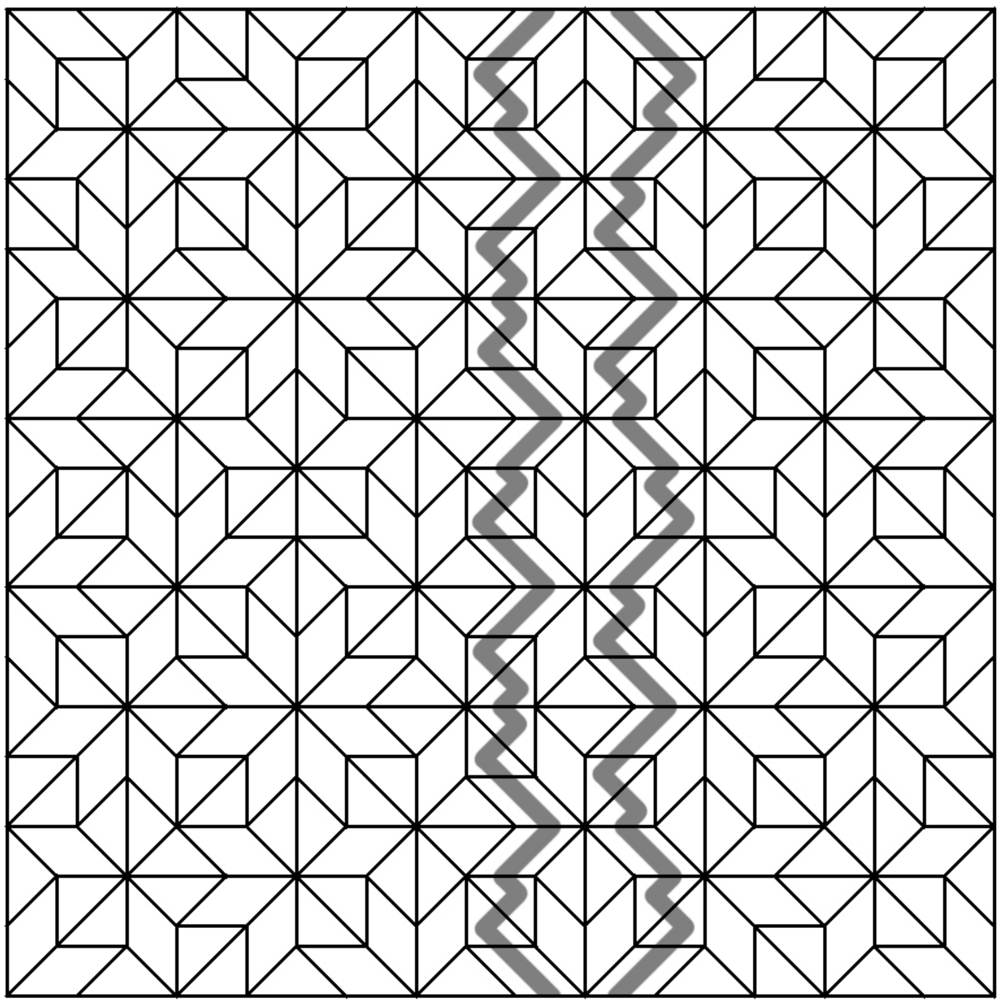

Figure 5.7: v_1 Represented on a Larger Patch

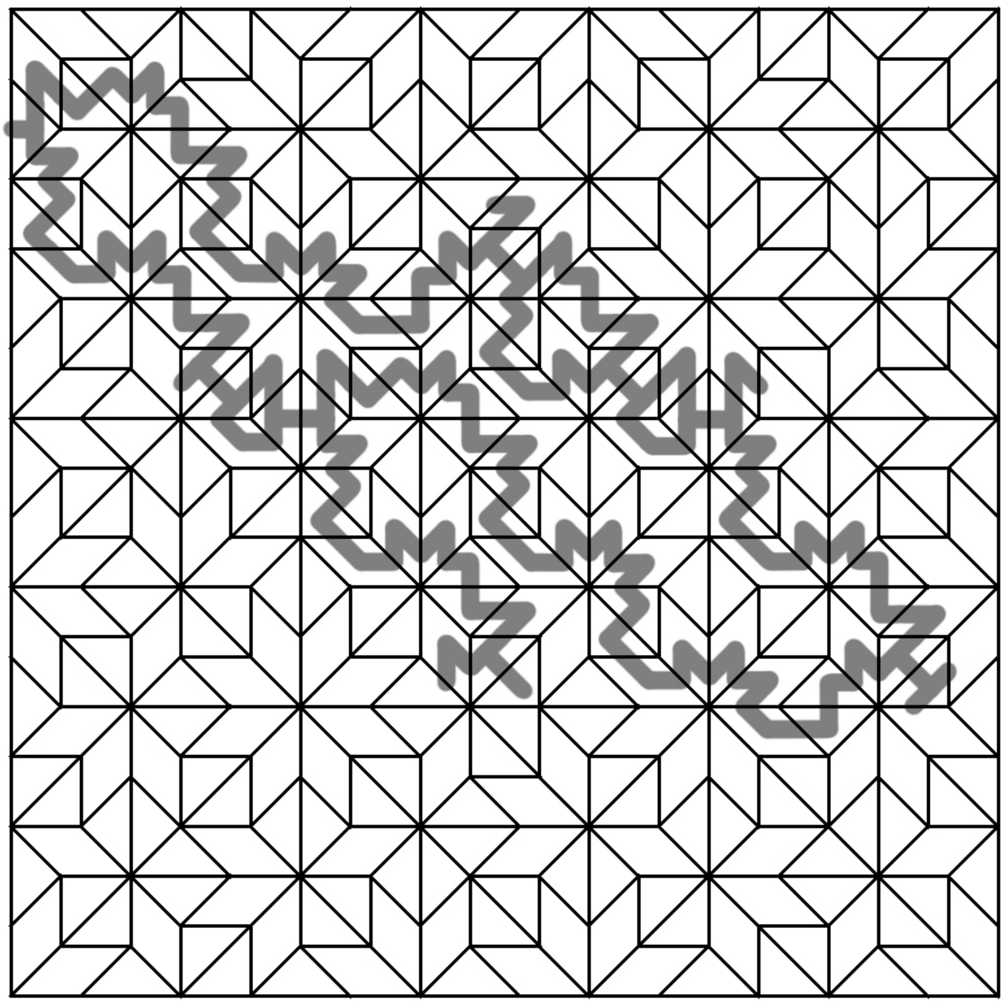

Figure 5.8: v_4 Represented on a Larger Patch

5.2 The Penrose Tiling

The famouse kites-and-darts tiling of Penrose give a nice example of an apreiodic tiling with finite local complexity. Splitting up the kites and darts into triangles allows up to define a substitution rule to produce such tilings, see Figure 5.9. This substitution forces its border (see [AP]).

This is the substitution matrix for the Penrose tiling.

$$\left[\begin{array}{cccccccccccccccccccccccccccccccccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
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\end{array}\right]$$

Let γ denote the Golden Ratio, $\gamma = \frac{1+\sqrt{5}}{2}$. Then the above matrix has Perron-Frobenius eigenvalue γ^2, with eigenvector

$$u_{pen} = \xi[\underbrace{1, \ldots, 1}_{20}, \underbrace{\gamma, \ldots \gamma}_{20}]^t$$

for any $\xi \in \mathbb{R}$. If b is the length of the medium length edge (ie, edge 1), then

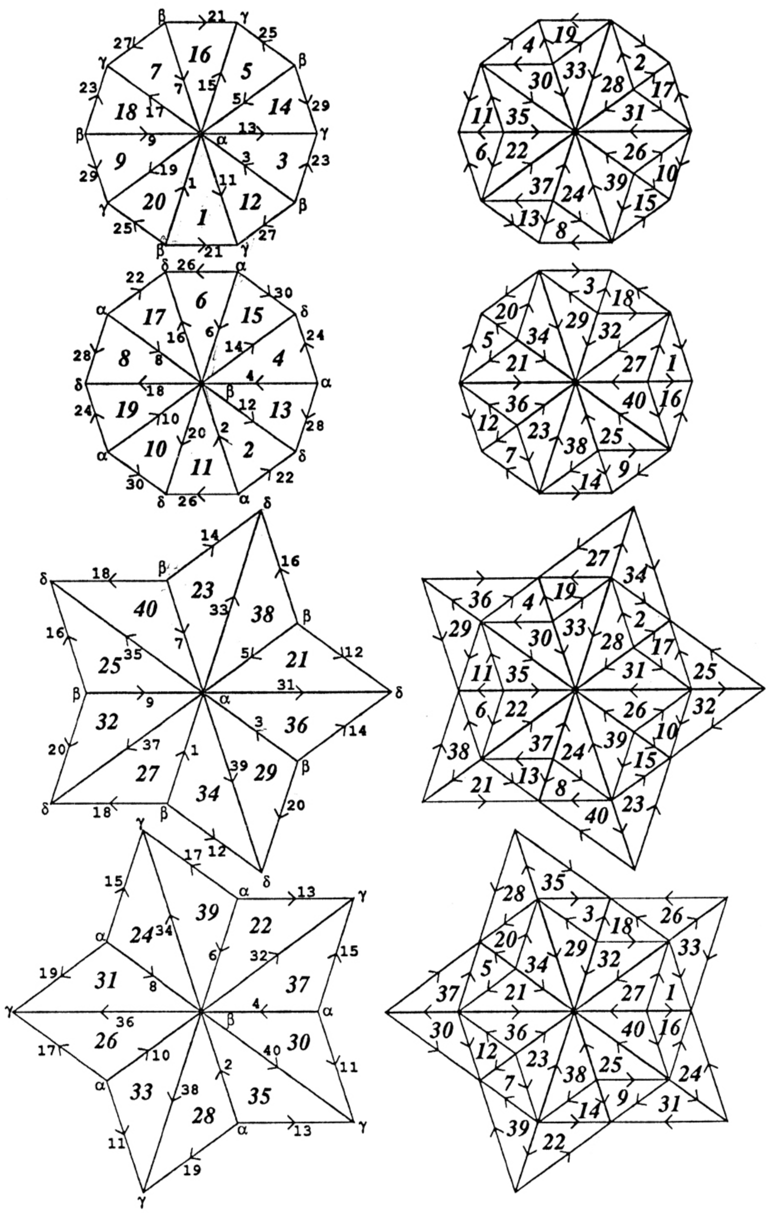

Figure 5.9: Cell Complex and Substitution for the Penrose Tiling.

the area vector is

$$a_{pen} = \frac{b^2}{4\gamma^2}\sqrt{4\gamma^2 - 1}[\underbrace{1, \ldots, 1}_{20}, \underbrace{\gamma, \ldots \gamma}_{20}]^t$$

We want to choose ξ so that $a_{pen} \cdot u_{pen} = 1$, thus

$$\xi \frac{b^2}{4\gamma^2}\sqrt{4\gamma^2 - 1}(20 + 20\gamma^2) = 1$$

$$\xi = \frac{\gamma^2}{5b^2(1 + \gamma^2)\sqrt{4\gamma^2 - 1}}$$

In Figure 5.10, the four lengths are equal to

$$\begin{aligned} L_1 &= \frac{b}{4\gamma}\sqrt{4\gamma^2 - 1} \\ L_2 &= \frac{b}{4\gamma} \\ L_3 &= \frac{\gamma b}{4} \\ L_4 &= \frac{b}{4}\sqrt{4 - \gamma^2} \end{aligned}$$

The boundary matrices can be read off Figure 5.9, and they yield the following generators of $H^1(\Gamma_0)$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad v_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

To calculate the image of v_1 under our map, we integrate over the cell complex:

$$\begin{aligned}\int_{\sigma_i}(-1)\big(dh\alpha_1\big)(v_1) = & \ \big(\sqrt{4\gamma^2-1}dx+dy\big)\tfrac{b}{4\gamma}\xi; \quad i=1,6,11,16\\ &= \big(\sqrt{4\gamma^2-1}u_{\frac{8\pi}{10}}+u_{\frac{3\pi}{10}}\big)\tfrac{b}{4\gamma}\xi; \quad i=5,10,15,20\\ &= \tfrac{\gamma^2 b\xi}{4}u_{\frac{7\pi}{10}}+\tfrac{\gamma b\xi}{4}\sqrt{4-\gamma^2}u_{\frac{2\pi}{10}}; \quad i=22,27,32,37\\ &= \tfrac{\gamma^2 b\xi}{4}u_{\frac{\pi}{10}}+\tfrac{\gamma b\xi}{4}\sqrt{4-\gamma^2}u_{\frac{6\pi}{10}}; \quad i=24,29,34,39.\end{aligned}$$

When worked out this gives

$$\sum_{\sigma_i\in\Gamma_{0_1}}\int_{\sigma_i}(-1)\big(dh\alpha_1\big)(v_1)=\frac{2}{5b}u_{\frac{2\pi}{5}}$$

Where again $u_{\frac{2\pi}{5}}$ denotes the unit vector in the direction of $\frac{2\pi}{5}$ from the horizontal. Similarly,

$$\begin{aligned}\int_{\sigma_i}(-1)\big(dh\alpha_1\big)(v_2) &= \frac{2}{5b}u_{\frac{3\pi}{5}}\\ \int_{\sigma_i}(-1)\big(dh\alpha_1\big)(v_3) &= \frac{2}{5b}u_{\frac{4\pi}{5}}\\ \int_{\sigma_i}(-1)\big(dh\alpha_1\big)(v_4) &= \frac{2}{5b}u_{\pi}\\ \int_{\sigma_i}(-1)\big(dh\alpha_1\big)(v_5) &= 0\end{aligned}$$

Here we see how the Ruelle-Sullivan map separates the two tilings – the image of the map is a set of vectors which have rotational symmetry correst-ponding to the rotational symmetry in the tiling. Indeed, this was our goal – to extract more information from the cohomology groups to help distinguish

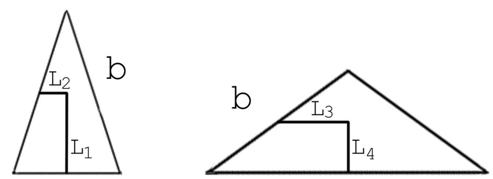

Figure 5.10: Lengths Needed for Computations.

between fundamentally different tilings. It seems that the Ruelle-Sullivan on tiling spaces captures the symmetry present, at least in these two examples.

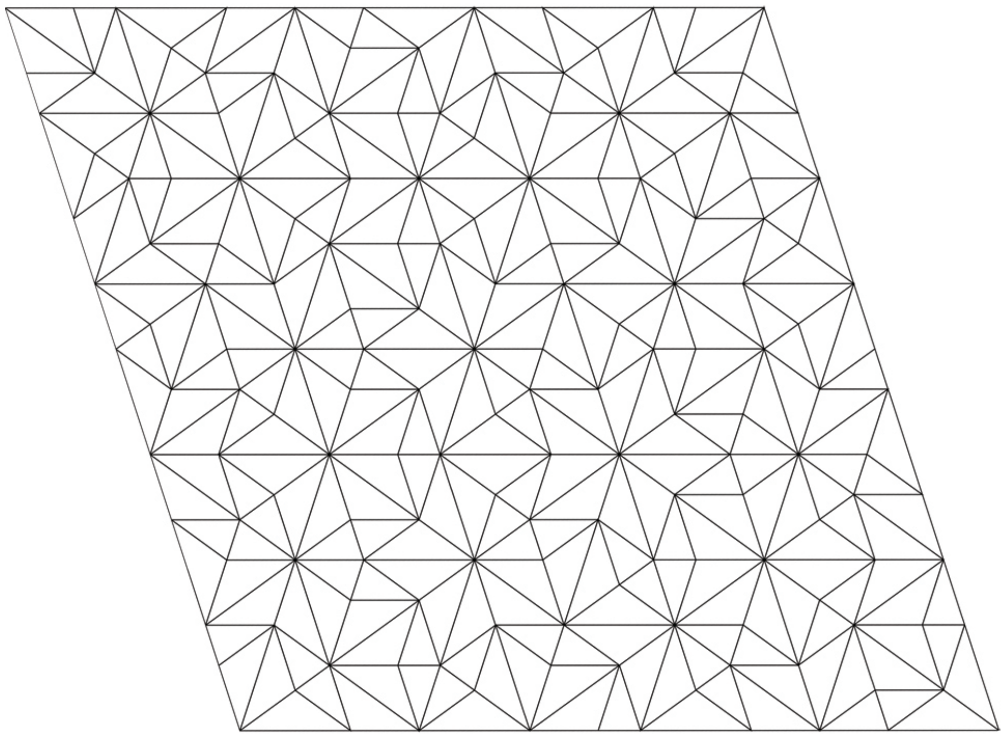

Figure 5.11: Patch of a Penrose Tiling

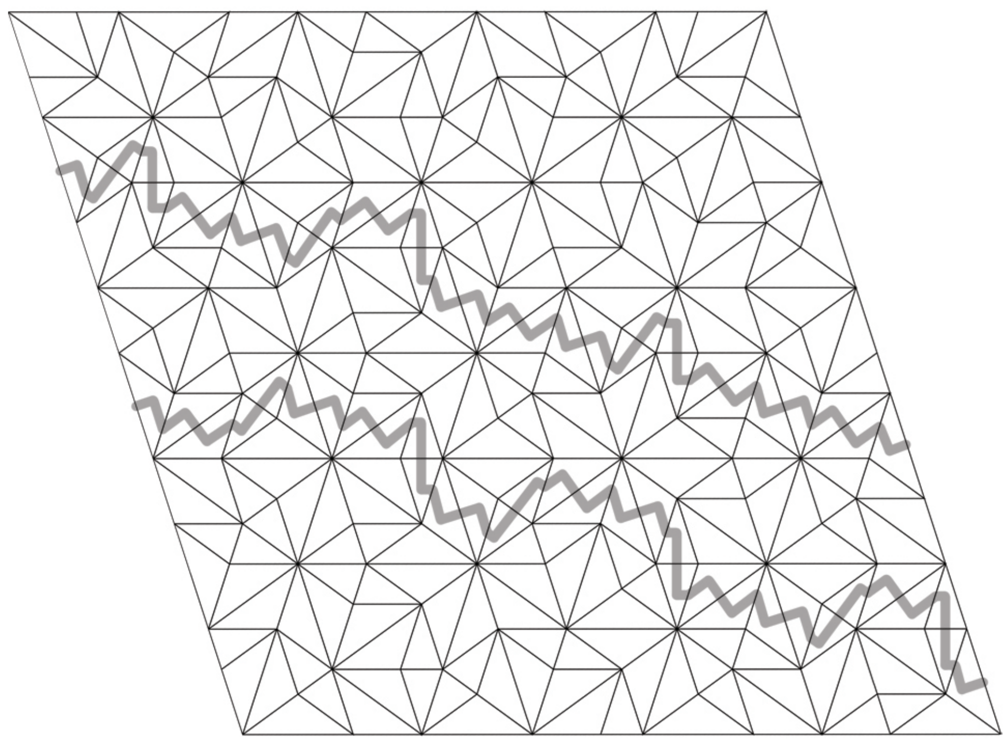

Figure 5.12: First Generator of $H^1(\Gamma_0)$ Represented on a Larger Patch.

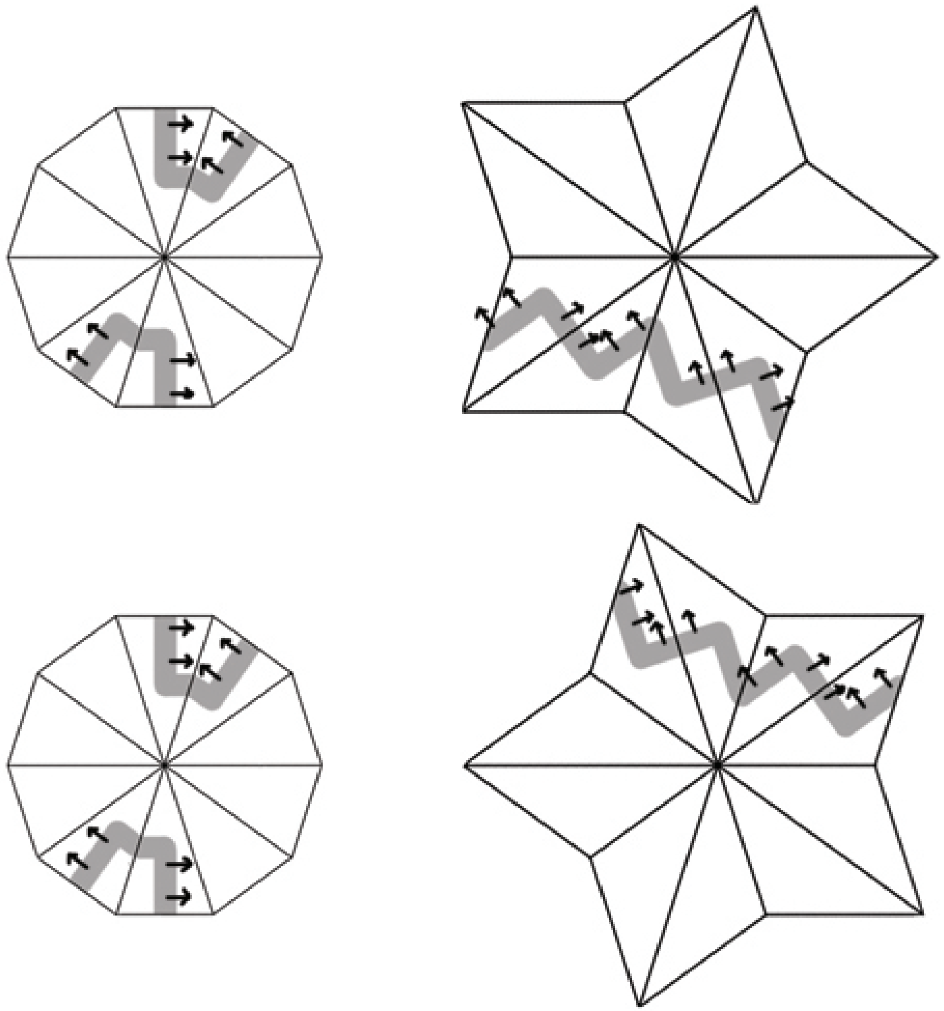

Figure 5.13: First Generator of $H^1(\Gamma_0)$ represented on Penrose Cell Complex

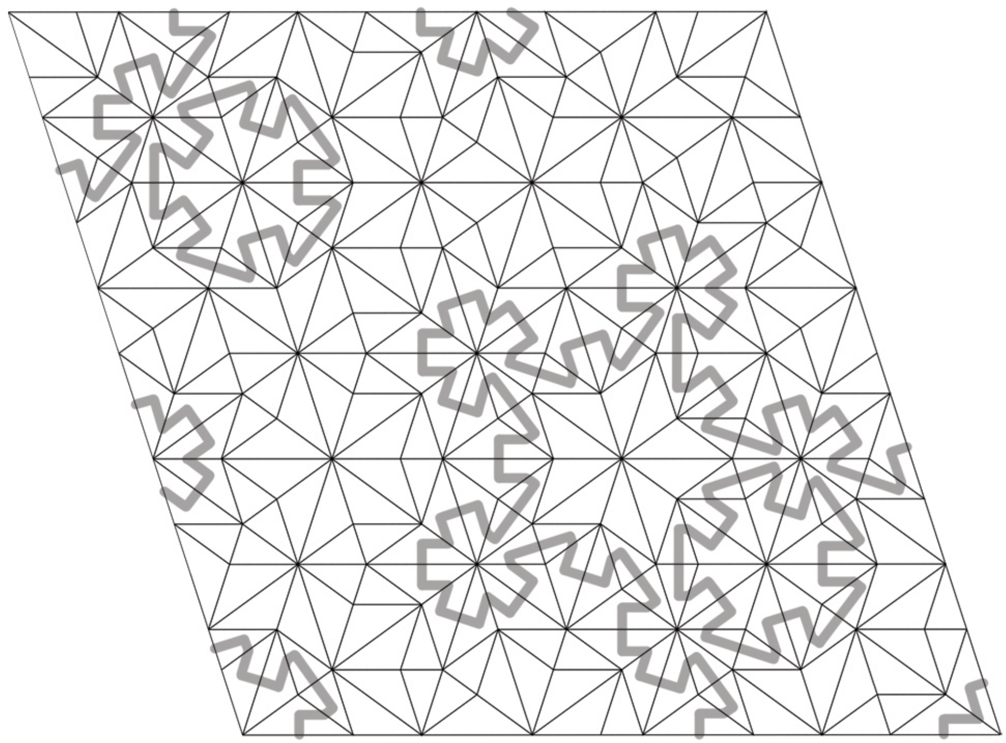

Figure 5.14: Fifth Generator of $H^1(\Gamma_0)$ Represented on a Larger Patch.

Chapter 6

Conclusion

We began with an aperiodic substitution tiling T of $\mathbb{R}^n$ and after making some standard assumptions about the substitution we formed a cell complex Γ_0. We found a map from the cellular cohomology of the cell complex to the Čech cohomology of a certain cover of Ω_T, and then mapped this cohomology group to the dynamical cohomology group through an adaptation of the Čech-deRham theorem. We then showed that this group could be mapped in a homomorphic way to $\Lambda\mathbb{R}^{*n}$ as in [KP].

We showed that this map indeed distinguishes between the two different tilings given in Chapter 5. This is consistent with the sentiment given in [KP] - that the cohomology groups together with the Ruelle-Sullivan map will furnish a better invariant for tiling spaces. It is in this way that the Ruelle-Sullivan map aids in the study of aperiodic order.

The fact that the Penrose tiling admits a generator of cohomology which maps to 0 while the octagonal tiling does not seems to suggest that there is more to learn about these tilings - this we feel warrants further study.

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