Actions of Finite Groups on Substitution Tilings and Their Associated C*-algebras

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Thesis Submitted to the Faculty of Graduate and Postdoctoral Studies
In partial fulfilment of the requirements for the degree of Doctor of Philosophy in Mathematics [1]

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[1]The Ph.D. program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

Abstract

The goal of this thesis is to examine the actions of finite symmetry groups on aperiodic tilings. To an aperiodic tiling with finite local complexity arising from a primitive substitution rule one can associate a metric space, transformation groupoids, and C*-algebras. Finite symmetry groups of the tiling act on each of these objects and we investigate appropriate constructions on each, namely the orbit space, semidirect product groupoids, and crossed product C*-algebras respectively. Of particular interest are the crossed product C*-algebras; we derive important structure results about them and compute their K-theory.

Acknowledgements

First and foremost, I would like to thank my wife and the love of my life Anna. This would not have been possible without her love, support, encouragement and friendship. I am also eternally grateful to my wonderful supervisor Thierry Giordano for being generous with excellent advice (both mathematical and otherwise) and financial support throughout my PhD studies. I would also like to thank my family: my mother Susan, my brother Bob, my grandparents Noeline (Nana) and Peter, and my stepfather TJ. In particular I must thank my Nana, who taught me at a young age the importance of continual learning, inquisitiveness, curiosity, and compassion.

Mathematically, I would like to thank Ian Putnam, Michael Whittaker, Daniel Gonçalves, David Handelman, and Siegfried Echterhoff for many extremely helpful conversations about this thesis. I would also like to thank NSERC for financial support for the first half of my PhD studies.

Dedication

In memory of my father Ray, who was taken from us before I could finish this degree. We love you, dad.

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Chapter 1

Introduction

Mathematical research on aperiodic tilings traces back to Wang [69] who considered the problem of whether sets of square tiles could tile the plane given a set of rules about which edges could be adjacent. Despite his conjecture that no such set could tile the plane aperiodically, an aperiodic set of 20426 such squares was found by his student Berger in [12]. The question of the smallest number of tile shapes needed to cover the plane aperiodically was soon considered, and in the 70's Penrose [46] discovered an aperiodic set consisting of two rhombs which when given certain matching rules could *only* tile the plane aperiodically. Even though every such tiling was aperiodic, they also had a strong repetitivity property – every patch in such a tiling appeared infinitely often and such appearances occurred with regularity.

In 1984, D. Shechtman et al [64] discovered a crystalline substance which had a diffraction pattern with five-fold rotational symmetry; this was a surprising result because symmetry of this kind is not possible from a periodic lattice. Penrose tiles and their three-dimensional analogues were seen as likely models for such materials (now known as *quasicrystals*), and so interest in aperiodic tilings grew.

In his book on noncommutative geometry [16], Connes presented the set of Penrose tilings as an example of a noncommutative space. He used the fact that Penrose tilings are self-similar to write down a Bratteli diagram and associate to each Penrose tiling an infinite path through the diagram. He showed that a tiling T could be taken to another tiling T' by an isometry of the plane if and only if their associated paths through the Bratteli diagram eventually coincide (that is, they are *tail equivalent*). The space of paths in a Bratteli diagram under tail equivalence naturally gives rise to a C*-algebra, an *approximately finite* or *AF* C*-algebra as studied and classified by Elliott [22].

Connes' construction showed that one could fruitfully study aperiodic tilings using C*-algebras. This point of view was further explored by Kellendonk and Putnam in [33], [34], [35], [36], and [52]. However, their point of view was also a dynamical one. Interest in *symbolic dynamics* (the study of the space of bi-infinite sequences on a finite alphabet together with the map that shifts such sequences one place to the left) was growing due to its many applications to cryptography and coding. An aperiodic tiling can be seen as a two-dimensional generalization of a bi-infinite sequence, so Kellendonk and Putnam considered only translation on tilings. In considering these dynamics, interesting topological spaces arose and were studied by computing their invariants, notably their cohomology [1]. Cohomology turned out to be quite a useful and computable invariant for studying tilings, and is discussed thoroughly by Sadun in [62] and computed for large classes of examples by Gähler, Hunton and Kellendonk in [25] and [26].

The elements of the C*-algebra $C_r^*(\mathcal{R}_{\text{punc}})$ Kellendonk and Putnam considered were seen to have physical significance by Bellissard in [6]. If an aperiodic tiling is seen as a model for a quasicrystal with atoms located at the vertices of tiles, certain elements of $C_r^*(\mathcal{R}_{\text{punc}})$ can be thought of as observables associated to a particle moving through this quasicrystal, see for example [7], [8] and [9]. In [6] Bellissard proved the remarkable result that $K_0(C_r^*(\mathcal{R}_{\text{punc}}))$ labels the gaps in the spectrum of the Schrödinger operator for such a particle. An extension of this result (the so-called *gap-labeling conjecture*) was proved by Bellissard along with Gambaudo and Benedetti in [10] and also independently by Kaminker and Putnam in [31] and by Benameur and Oyono-Oyono in [11].

Physics aside, this C*-algebra is interesting in its own right. It is simple, separable, and nuclear, and hence is of interest in the current program initiated by Elliott to classify all such algebras. In the case of a one-dimensional tiling, $C_r^*(\mathcal{R}_{\text{punc}})$ is strongly Morita equivalent to the crossed product of the Cantor set by a minimal homeomorphism, and so by a result of Putnam [53], is classifiable by its K-theory. To this end, Anderson and Putnam [1] calculated the K-theory of this C*-algebra, and in [52] Putnam proved that the order on projections for $C_r^*(\mathcal{R}_{\text{punc}})$ was determined by its unique trace. Phillips [47] generalized this result and also showed that $C_r^*(\mathcal{R}_{\text{punc}})$ has real rank zero and stable rank one by using a canonical AF subalgebra of $C_r^*(\mathcal{R}_{\text{punc}})$, which we denote AF_ω.

What initially garnered interest in Penrose tilings from physicists was their rotational symmetry. The goal of this thesis is to examine the actions of rotational and dihedral symmetry groups on $C_r^*(\mathcal{R}_{\text{punc}})$. The interaction of finite groups with spaces of tilings has been explored by Ormes, Radin and Sadun in [45] and Rand in [58]. In [45], the authors used the naturally arising rotation groups to arrive at a finer invariant than the cohomology by computing the cohomology one representation at a time. In her thesis, Rand [58] incorporates the rotation groups into the cohomology. From the quasicrystal perspective, symmetries in diffraction spectra were considered by Mermin in [41] and later by Lenz and Moody in [37].

The material in Chapters 2, 3, and 4 of this thesis is background material, almost all of which is present in the literature. In Chapter 2 we record terminology and notation used in the study of tilings, and define the tiling space Ω. We summarize the assumptions we place on our tilings in Remark 2.5.8, and under these assumptions Ω is a compact metric space. The elements of Ω are tilings and $\mathbb{R}^d$ acts minimally by translation. We also define a subspace $\Omega_{\text{punc}} \subset \Omega$ that is homeomorphic to a Cantor set. In Chapter 3 we provide background on locally compact groupoids, the primary object upon which the C*-algebras of interest are built. We also describe the r-discrete groupoid $\mathcal{R}_{\text{punc}}$ as defined by Kellendonk in [35]. In Chapter 4 we provide some background on C*-algebras, and define C*-algebras associated to dynamical systems (Section 4.2) and to r-discrete groupoids (Section 4.3). We then describe the C*-algebra $C_r^*(\mathcal{R}_{\text{punc}})$, its AF subalgebra AF_ω and summarize its properties as originally described by Kellendonk, Putnam, and Phillips in [33], [34], [35], [47], and [52].

Chapters 5 and 6 are the heart of the present work. Chapter 5 examines the action of a finite symmetry group G on Ω, $\mathcal{R}_{\text{punc}}$ and $C_r^*(\mathcal{R}_{\text{punc}})$ for a tiling of $\mathbb{R}^2$. We begin by showing that the orbit space Ω/G has the structure of an inverse limit of CW complexes; this is a result stated for rotation groups in [45] and analogous to the main result in [1] that Ω has this structure. We then describe precisely which tilings in Ω are fixed under a given group element. In Section 5.5 we describe the crossed product $C_r^*(\mathcal{R}_{\text{punc}}) \rtimes G$. Under the assumption that no tile in our tiling is fixed by a nontrivial element of G, we show that $C_r^*(\mathcal{R}_{\text{punc}}) \rtimes G$ is simple, real rank zero, stable rank one, has a unique trace, and that the order on its projections is determined by this trace. We also examine the crossed product of AF_ω by symmetry groups, and show that in the case of the Penrose tiling and $G = D_{10}$ that $AF_\omega \rtimes G$ is isomorphic to the AF algebra Connes originally considered in [16].

In Section 5.7 we prove that if $C_r^*(\mathcal{R}_{\text{punc}})$ has tracial rank zero (a conjecture of Phillips) then $C_r^*(\mathcal{R}_{\text{punc}}) \rtimes G$ has tracial rank zero as well. We prove this by showing that the action of G on $C_r^*(\mathcal{R}_{\text{punc}})$ has the tracial Rokhlin property assuming $C_r^*(\mathcal{R}_{\text{punc}})$ has tracial rank zero. We also prove that when restricted to AF_ω, the action of G has the strict Rokhlin property.

In Chapter 6 we compute the K-theory of these crossed products for some example tilings using techniques of [15] and [19].

Chapter 2

Tilings

In this chapter we define what we mean by a tiling and much of the terminology common in the study of tilings. We also introduce the assumptions we will apply to all the tilings studied in this work.

2.1 Tilings

A **tile** is a subset of $\mathbb{R}^d$ homeomorphic to the closure of the open unit ball $B_1(0)$. A **partial tiling** is a collection of tiles whose interiors are pairwise disjoint. A finite partial tiling will be called a **patch**. The **support** of a partial tiling is the union of its tiles; the support of a partial tiling T is denoted $\mathrm{supp}(T)$. We define a **tiling** to be a partial tiling whose support is $\mathbb{R}^d$. Given $U \subset \mathbb{R}^d$ and a partial tiling T, $T(U)$ is all the tiles that intersect U, that is, $T(U) = \{t \in T \mid t \cap U \neq \emptyset\}$. For $x \in \mathbb{R}^d$, $T(\{x\})$ is frequently abbreviated as $T(x)$. Two partial tilings T and T' are said to **agree on** U if $T(U) = T'(U)$.

Given a vector $x \in \mathbb{R}^d$, we can take any subset $U \subset \mathbb{R}^d$ and form its translate by x, namely $U + x = \{u + x \mid u \in U\}$. Thus, given a tiling T we can form another tiling by translating every tile by x. We denote this new tiling by

$$T + x = \{t + x \mid t \in T\}.$$

A set of tiles $\mathcal{P} = \{p_1, p_2, \ldots, p_N\}$ is called a set of **prototiles** for a tiling T if for each tile $t \in T$ there exists $p_i \in \mathcal{P}$ and $x \in \mathbb{R}^d$ such that $t = p_i + x$. Prototiles may carry labels to distinguish possibly translationally equivalent tiles. We also insist that the prototiles, when viewed as subsets of $\mathbb{R}^d$, have the origin in their interior. This allows us to define a designated point in each tile. If t is a tile in a tiling and $t = p + x_t$ for some $p \in \mathcal{P}$ we say that the **puncture** of t is x_t.

A tiling for which $T + x = T$ for some nonzero $x \in \mathbb{R}^d$ is called **periodic**. A tiling for which no such nonzero vector exists is called **aperiodic**.

Example 2.1.1 One famous example of an aperiodic tiling is the **Penrose rhomb tiling**. In Figure 2.1 we show a patch of such a tiling.

Example 2.1.2 Another example of an aperiodic tiling is the **Ammann-Beenker tiling**, also known as the **octagonal tiling**. In Figure 2.2 we show a patch of such a tiling.

2.2 Tiling Spaces

Given a set $\mathcal{T}$ of tilings, we would like to define a metric on $\mathcal{T}$.

Definition 2.2.1 *Suppose that $\mathcal{T}$ is a set of tilings of $\mathbb{R}^d$ and that $T, T' \in \mathcal{T}$. We define the distance between T and T' to be*

$$\begin{aligned} d(T,T') &= \inf\{1, \varepsilon \mid \exists\ x, x' \in \mathbb{R}^n \ni |x|, |x'| < \varepsilon, \\ &\qquad (T - x)(B_{1/\varepsilon}(0)) = (T' - x')(B_{1/\varepsilon}(0))\}, \end{aligned}$$

where $B_{1/\varepsilon}(0)$ denotes the open ball radius $\frac{1}{\varepsilon}$ centred at 0 in $\mathbb{R}^d$. This is called the **tiling metric**.

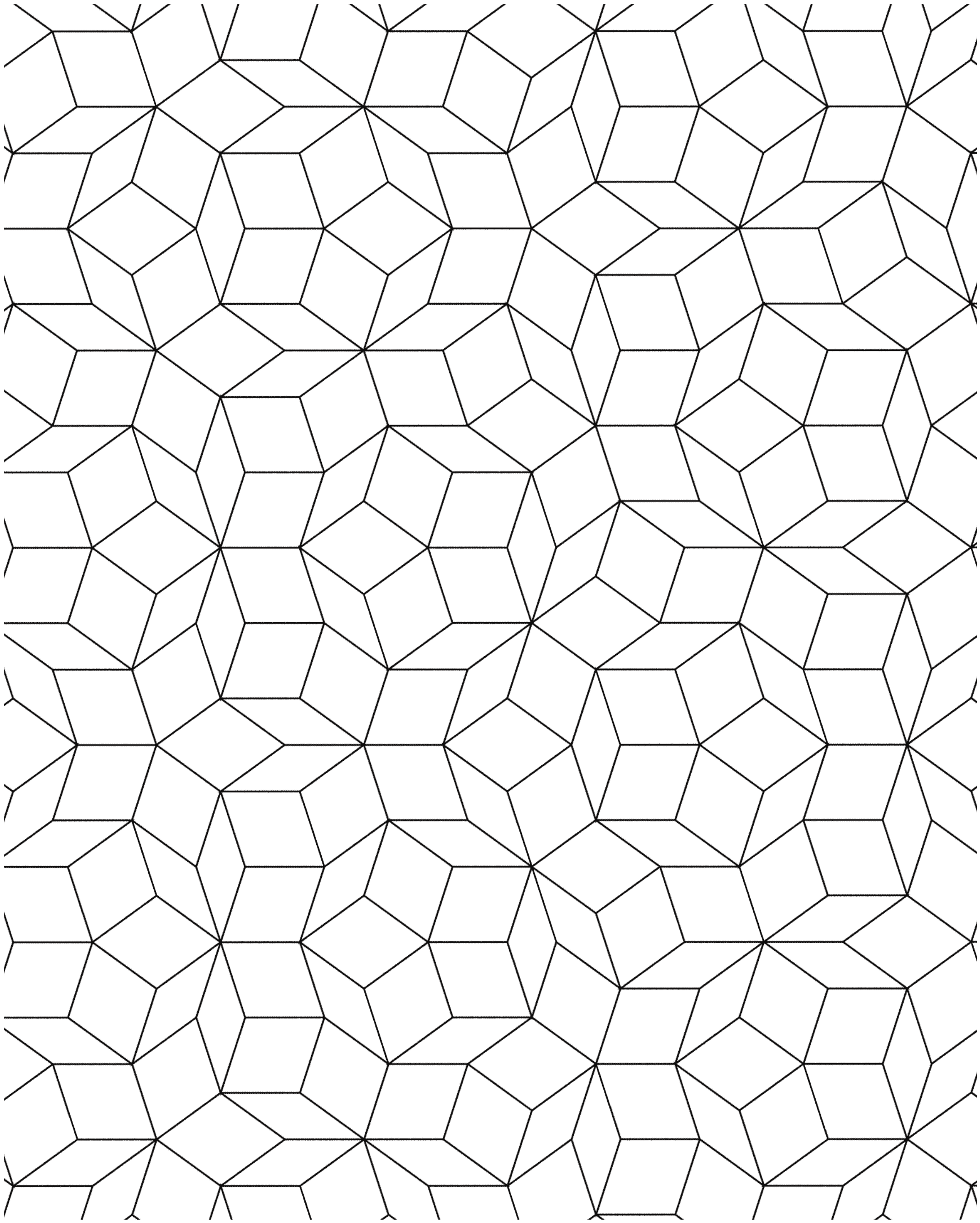


Figure 2.1: Patch of a Penrose rhomb tiling.

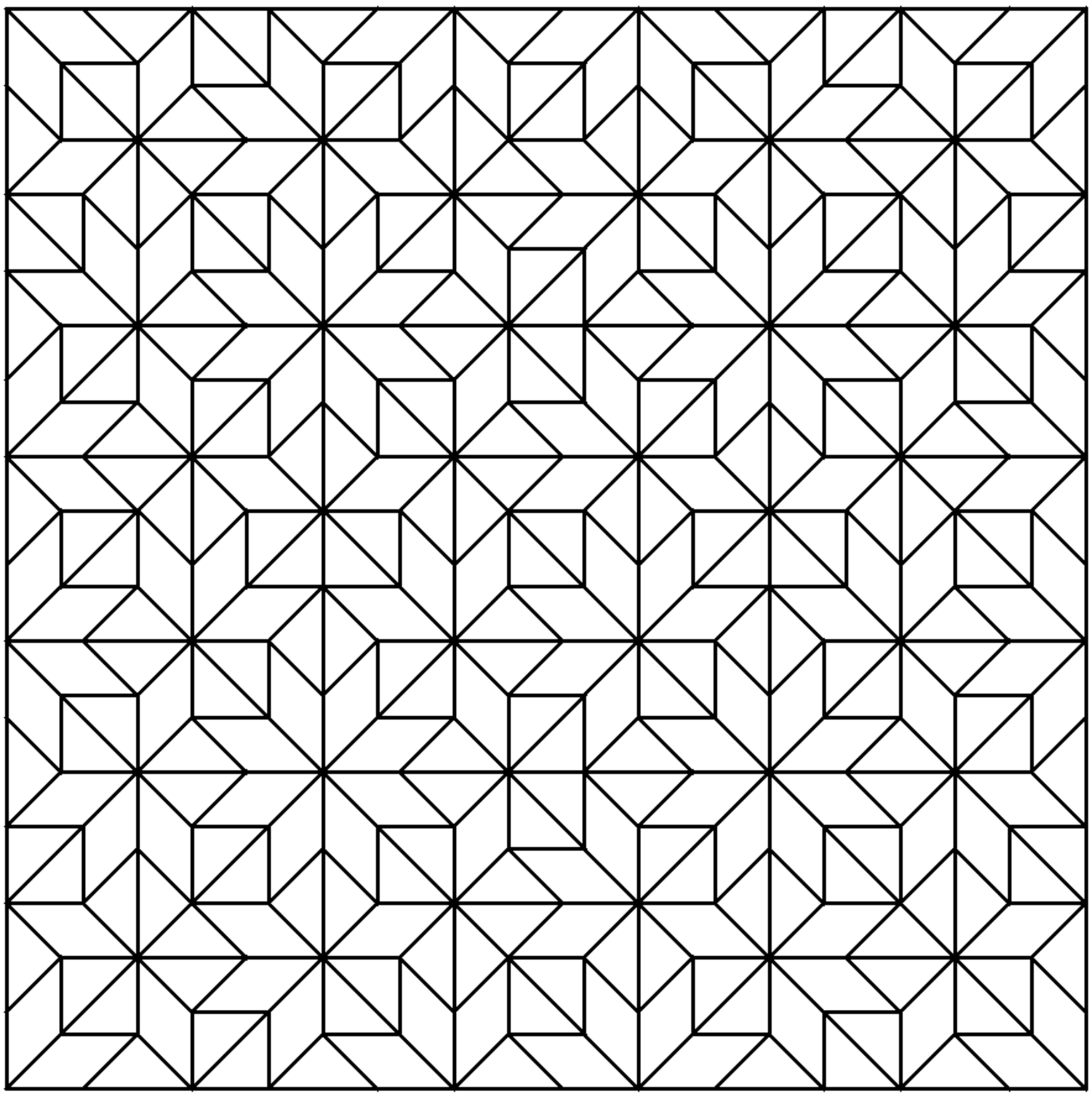


Figure 2.2: Patch of an Ammann-Beenker (octagonal) tiling.

This definition is standard, see for example [56] or [66]. For a proof that this defines a metric, see for example [68], Proposition 2.3.1. We denote the ball of radius r centered at T in this metric to be $B_r^{\Omega}(T)$. Two tilings are close in this metric if they agree on a large ball around the origin up to a small translation.

Definition 2.2.2 *Let T be a tiling and consider the set of tilings $T + \mathbb{R}^d = \{T + x \mid x \in \mathbb{R}^d\}$. We define Ω_T to be the completion of this set in the tiling metric, and call this space the* **continuous hull** *of T (also called the* **tiling space** *associated with T).*

The following is well-known, but we include a proof for completeness.

Lemma 2.2.3 *The elements of Ω_T are all tilings. That is, given a Cauchy sequence $\{T_i\}_{i\in\mathbb{N}}$ of elements in $T + \mathbb{R}^d$ one can find a tiling T' (not necessarily in $T + \mathbb{R}^d$) such that $\{T_i\}_{i\in\mathbb{N}}$ converges to T'.*

Proof: We note first that if S is a tiling, then for any $r > 0$ and $x \in \mathbb{R}^d$ we have

$$S(B_r(0)) + x = (S + x)(B_r(x)). \tag{2.2.1}$$

Suppose that $\{T_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence of tilings in the tiling metric. For each $n > 3$, find $K_n \in \mathbb{N}$ such that $d(T_i, T_j) < \frac{1}{n}$ whenever $i, j \geq K_n$. We may always pick the K_n so that they are increasing with n. Thus for each such n if $i, j \geq K_n$ there exist $x_{i,n}, x_{j,n} \in \mathbb{R}^d$ with $|x_{i,n}|, |x_{j,n}| < \frac{1}{n}$ such that

$$(T_i - x_{i,n})(B_n(0)) = (T_j - x_{j,n})(B_n(0)).$$

Define $P_n = (T_i - x_{i,n})(B_n(0))$. Now if $s > n > 3$ we have that for all $i, j \geq K_s \geq K_n$,

$$(T_i - x_{i,s})(B_s(0)) = (T_j - x_{j,s})(B_s(0)) = P_s.$$

These equations and Equation 2.2.1 imply that

$$T_i(B_s(x_{i,s})) = P_s + x_{i,s}$$

$$T_i(B_n(x_{i,n})) = P_n + x_{i,n}$$

Hence $P_n + x_{i,n}$ and $P_s + x_{i,s}$ are both patches in T_i. Furthermore, since $|x_{i,n}|, |x_{i,s}| < \frac{1}{3}$, we must have that $B_n(x_{i,n}) \subset B_s(x_{i,s})$, and so we have

$$P_n + x_{i,n} \subset P_s + x_{i,s}.$$

This is true for any $i \geq K_s$, so we define

$$T' = \bigcup_{l=4}^{\infty}(P_l + x_{K_l,l}).$$

Since T' is built from successive patches of inner radius at least l, it is a tiling. We claim that $\{T_i\}_{i\in\mathbb{N}}$ converges to T'. Let $\varepsilon > 0$ and find $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. Then for all $n > N$

$$\begin{aligned} P_n + x_{K_n,n} &\subset T' \\ (T_n - x_{K_n,n})(B_n(0)) + x_{K_n,n} &\subset T' \\ (T_n - x_{K_n,n})(B_n(0)) &\subset T' - x_{K_n,n} \\ \Longrightarrow (T_n - x_{K_n,n})(B_n(0)) &= (T' - x_{K_n,n})(B_n(0)) \end{aligned}$$

From previous, $|x_{K_n,n}| < \frac{1}{n} < \varepsilon$ and so $d(T', T_n) < \varepsilon$. ∎

For an alternative formulation of Ω_T, see [1] or [57]. There the authors produce a set of tilings and show that any translational orbit is dense.

Definition 2.2.4 *A tiling T is said to have* **finite local complexity** *if for every $r > 0$, the set $\{T(B_r(x)) \mid x \in \mathbb{R}^d\}$ contains only finitely many different patches modulo translation.*

If T admits a finite set of prototiles which are polygons, and tiles meet full-edge to full-edge in T, then T has finite local complexity, see [56]. This will be the case for all of our examples.

Lemma 2.2.5 ([56], Lemma 2) *If T has finite local complexity, then Ω_T is compact.*

We will consider only tilings with finite local complexity.

Definition 2.2.6 *A tiling T is said to be* **strongly aperiodic** *if Ω_T contains no periodic tilings.*

This definition is not vacuous; for example, consider the tiling T consisting of a unit grid in $\mathbb{R}^2$ with the four squares around the origin removed and replaced with a 2×2 square. All translates of this tiling are aperiodic but $\{T + (n, 0)\}_{n \in \mathbb{N}}$ is a Cauchy sequence of tilings converging to the usual grid.

We let $\Omega_{\text{punc}} \subset \Omega_T$ be the set of tilings in Ω that have a puncture at the origin. In other words, $T \in \Omega_{\text{punc}}$ if and only if $T(0) \in \mathcal{P}$. The space Ω_{punc} is sometimes called the **punctured hull** of the tiling.

Lemma 2.2.7 ([33], p. 187) *If T is strongly aperiodic and has finite local complexity, then its punctured hull Ω_{punc} is homeomorphic to the Cantor set.*

The space Ω_{punc} has a neighbourhood base consisting of sets of the following form: for a patch P and tile $t \in P$, define

$$U(P, t) = \{T \in \Omega_{\text{punc}} \mid P - x_t \subset T\}.$$

Each of these sets is clopen and the set of all such $U(P, t)$ forms a neighbourhood base for Ω_{punc}, see [33]. If $y \in \mathbb{R}^d$, then the sets $U(P, t)$ and $U(P + y, t + y)$ are identical. Hence when describing $U(P, t)$ we may use any patch which is a translate of P, along with the tile corresponding to t.

2.3 Local Equivalence of Tilings

In this section, we discuss a sufficient condition for the existence of a homeomorphism commuting with translation (by elements of $\mathbb{R}^d$) between two tiling spaces Ω_T and

Ω_S. Roughly, we want to say that two tilings T and S are equivalent if there is a way to construct T from S only using local data, and vice versa. The simplest example of constructing one tiling from another using local data is obtained from decomposing one or more of the prototiles into smaller tiles and extending this decomposition to the whole tiling.

For example, consider the Penrose rhomb tiling of Figure 2.1. The prototiles are two rhombs, and their rotates by multiples of ${}^{\pi}/_{5}$. We form a new tiling by splitting each rhomb down a diagonal, see Figure 2.3. One can see that in this case, the process is invertible – just delete the diagonals. The resulting tiling is known as a tiling by **Robinson triangles**. The following definition is due to Baake et al. [3].

Definition 2.3.1 *Let T and S be tilings. Then we say that T is* **locally derivable** *from a tiling S if there exists a radius $r > 0$ such that for all $x \in \mathbb{R}^d$ whenever we have $S(B_r(x)) = S(B_r(y)) + (x - y)$ then $T(x) = T(y) + x - y$. If T is locally derivable from S and S is locally derivable from T, then we say that S and T are* **mutually locally derivable***, or* **MLD***.*

Lemma 2.3.2 ([36], Section III) *If S and T are MLD, then there exists a homeomorphism $\varphi : \Omega_T \to \Omega_S$ such that $\varphi(T' + x) = \varphi(T') + x$ for all $T' \in \Omega_T$ and $x \in \mathbb{R}^d$. In other words, the dynamical systems $(\Omega_T, \mathbb{R}^d)$ and $(\Omega_S, \mathbb{R}^d)$ are topologically conjugate.*

2.4 Substitution Tilings

In this section, we present a well-studied class of tilings, the so-called substitution tilings. Substitution tilings are constructed using self-similarity on a finite set of tiles. Tilings of this type have been considered by many authors; for example see [33], [34], [35], and [66].

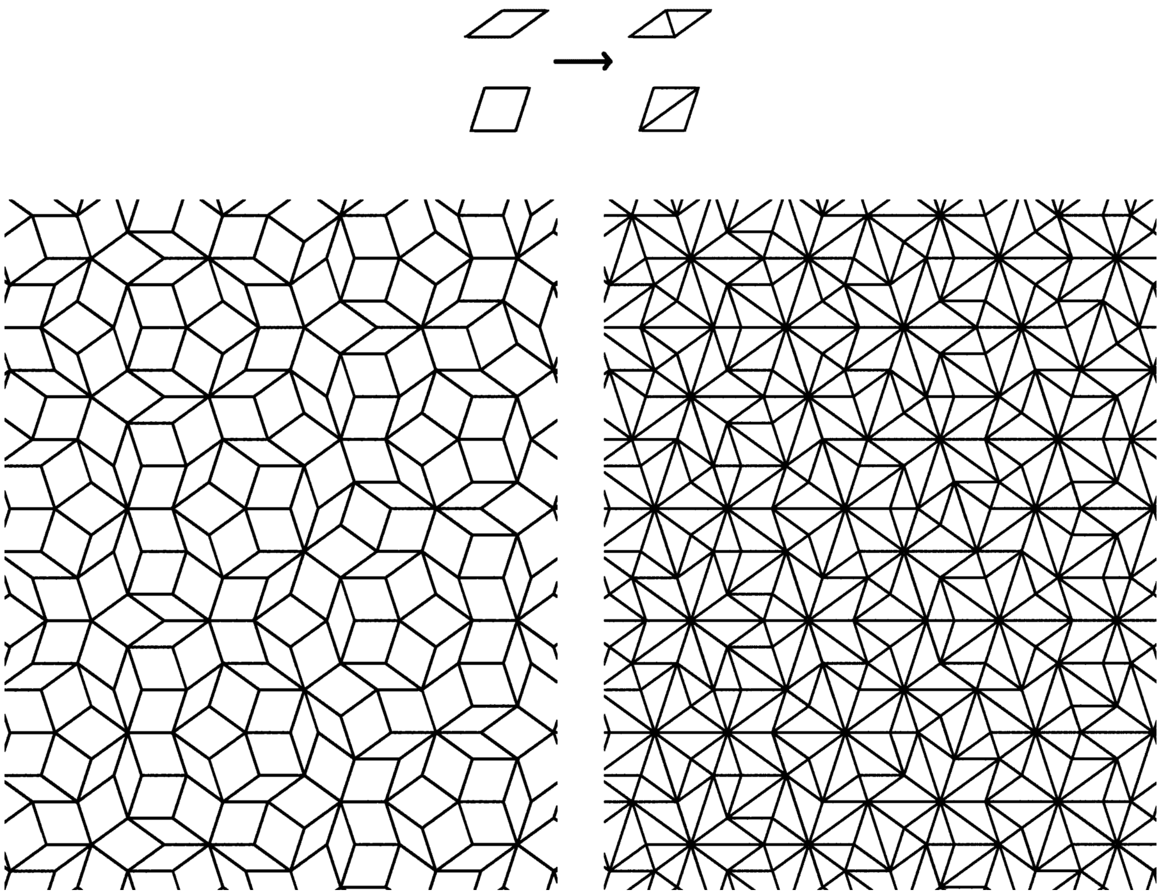


Figure 2.3: The Penrose rhomb tiling is locally equivalent to a tiling by Robinson Triangles.

Definition 2.4.1 *Let $\mathcal{P} = \{p_1, p_2, \ldots, p_N\}$ be a set of tiles. A* **substitution rule** *ω with scaling factor $\lambda > 1$ on $\mathcal{P}$ is a rule which associates with $p \in \mathcal{P}$ a patch $\omega(p)$ with support λp whose tiles are all translates of elements of $\mathcal{P}$.*

We may extend a substitution rule to translates of elements of $\mathcal{P}$ via the formula $\omega(p + x) = \omega(p) + \lambda x$. Thus the substitution may be applied to patches – if P is a patch then

$$\omega(P) = \{\omega(t) \mid t \in P\}.$$

In particular, the substitution may be iterated.

Definition 2.4.2 *A substitution rule ω is said to be* **primitive** *if there exists $m \in \mathbb{N}$ such that for every $p_i, p_j \in \mathcal{P}$, $\omega^m(p_i)$ contains a translate of p_j. If ω is primitive, we call $(\omega, \mathcal{P})$ a* **primitive substitution tiling system**.

Primitivity of a substitution gives us the self-similarity needed to construct a tiling.

Construction 2.4.3 (Construction of a tiling from a primitive substitution rule). Let $(\mathcal{P}, \omega)$ be a primitive substitution rule. Let $p \in \mathcal{P}$, and find $k \in \mathbb{N}$ such that $\omega^k(p)$ contains a tile in the interior of its support. Since the substitution is primitive, there exist $x \in \mathbb{R}$ and $m \in \mathbb{N}$ such that $p + x \in \omega^{k+m}(p)$ and $p + x$ is contained in the interior of the $\text{supp}(\omega^{k+m}(p)) = \lambda^{k+m}p$. The function

$$f : \lambda^{k+m}p \to p + x$$

$$z \mapsto \lambda^{-(k+m)}(z + x)$$

is continuous and onto, and $p + x \subset \lambda^{k+m}p$. Hence by the Brouwer fixed point theorem (see for example [44], Theorem 55.6), f has a fixed point in its interior; call it z_0. That is, z_0 satisfies

$$\begin{aligned} \lambda^{k+m}z_0 &= z_0 + x \\ \implies \qquad x &= \lambda^{k+m}z_0 - z_0 \end{aligned}$$

We have

$$\begin{aligned} p + x &\in \omega^{k+m}(p) \\ p + \lambda^{k+m} z_0 - z_0 &\in \omega^{k+m}(p) \\ p - z_0 &\in \omega^{k+m}(p) - \lambda^{k+m} z_0 \\ p - z_0 &\in \omega^{k+m}(p - z_0). \end{aligned}$$

Hence,

$$\{p - z_0\} \subset \omega^{k+m}(p - z_0) \subset \cdots \subset \omega^{i(k+m)}(p - z_0) \subset \omega^{(i+1)(k+m)}(p - z_0) \subset \ldots$$

is an increasing nested sequence of patches. Further, since $p - z_0$ is in the interior of $\omega^{k+m}(p - z_0)$, the supports of these patches are an increasing nested sequence of sets in $\mathbb{R}^d$ whose union is $\mathbb{R}^d$. Thus

$$T = \bigcup_{i=1}^{\infty} \omega^{i(k+m)}(p - z_0)$$

is a tiling.

With this choice of T, it is clear that $\mathcal{P}$ is a set of prototiles for T. If every tiling arising this way from $(\mathcal{P}, \omega)$ has finite local complexity, then we say that $(\mathcal{P}, \omega)$ has finite local complexity. This is satisfied if, for example, the elements of $\mathcal{P}$ are polygons and meet full-face to full-face in $\omega^n(p)$ for all $n \in \mathbb{N}$ and $p \in \mathcal{P}$.

Definition 2.4.4 *A tiling T is said to be* **repetitive** *if for every patch $P \subset T$ there is an $r > 0$ such that for every $x \in \mathbb{R}^d$ there is a translate of P contained in $T(B_r(x))$.*

Repetitivity is a strong regularity condition on a tiling. If a tiling T has repetitivity, every patch in T appears infinitely often and such appearances occur with regularity.

Lemma 2.4.5 ([66], Lemma 2.2) *If T is formed by a primitive substitution as in Construction 2.4.3, and T has finite local complexity, then T is repetitive.*

The following is a corollary to the main result of [56] but is stated in the following form by Solomyak.

Lemma 2.4.6 ([66], Lemma 1.2) *The tiling T is repetitive if and only if for every $T' \in \Omega_T$ we have $\Omega_{T'} = \Omega_T$.*

In light of Lemma 2.4.6, if T is repetitive we drop the T and refer to the tiling space as simply Ω. When we form the space Ω from a substitution in this way, it has another useful characterization: Ω is the space of all tilings T such that every patch $P \subset T$ is contained in $\omega^n(p)$ for some $n \in \mathbb{N}$ and $p \in \mathcal{P}$ (see [1], Section 2 for details).

The substitution ω can be applied to any tiling in Ω. In [1], the following is proved:

Lemma 2.4.7 *Consider the space Ω formed from a primitive substitution tiling system with finite local complexity. Then*

1. $\omega(\Omega) = \Omega$
2. $\omega : \Omega \to \Omega$ *is continuous.*

Proof: See [1] Proposition 2.2 and Proposition 3.1. ∎

Definition 2.4.8 *The substitution ω is said to be* **locally invertible** *if there exists $r > 0$ such that whenever we have a tiling $T \in \Omega, t, t' \in \omega(T)$ and*

$$(\omega(T) - x_t)(B_r(0)) = (\omega(T) - x_{t'})(B_r(0)),$$

then this implies that $(T - \lambda^{-1}x_t)(0) = (T - \lambda^{-1}x_{t'})(0)$.

In words, ω has this property if there is a radius r such that the substituted tile that a tile $t \in \omega(T)$ belongs to is uniquely determined by the pattern $(\omega(T) - x_t)(B_r(0))$. Kellendonk notices ([35], discussion after Lemma 4) that this implies local invertibility of ω^n.

Lemma 2.4.9 *Consider the tiling space Ω of a tiling created from a primitive substitution tiling system with finite local complexity. The following are equivalent:*

1. *The space Ω contains no periodic tilings.*
2. *The map ω is injective when restricted to Ω.*
3. *The map ω is locally invertible.*

Proof: See [1] Proposition 2.3 and [66] Lemma 2.7. ∎

This provides a way of checking aperiodicity.

Example 2.4.10 Figure 2.4 illustrates a substitution on a set of prototiles

$$\mathcal{P}_{\text{Pen}} = \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{40}\},$$

all of which are Robinson triangles. Only four prototiles are shown; the others are obtained by rotation. Let r denote the counterclockwise rotation of $\mathbb{R}^2$ by ${}^{\pi}/_{5}$ and let $\mathbf{2} = r\mathbf{1}$, $\mathbf{12} = r\mathbf{11}$, and so on. It is easy to check that this substitution is primitive and has finite local complexity. Because tilings by Robinson triangles and Penrose rhombs are MLD, this substitution will be what we refer to as the **Penrose substitution**.

Example 2.4.11 The octagonal tiling of Figure 2.2 can be obtained via substitution rule as well; see Figure 2.5. There are 20 prototiles; 4 rhombs and 16 triangles obtained by rotating the given tiles by multiples of ${}^{\pi}/_{4}$, as well as reflecting the triangle over the x-axis and rotating it by multiples of ${}^{\pi}/_{4}$.

In both these examples, the substitution rule is given on a subset of prototiles and then extended by symmetry. In other words, there is a finite group G which acts on $\mathcal{P}$ and there exists a subset $\mathcal{S}_G \subset \mathcal{P}$ such that $G\mathcal{S}_G = \mathcal{P}$. The substitution is defined on $\mathcal{S}_G$, and if $p = gq$ for some $q \in \mathcal{S}_G$ and $g \in G$, we define

$$\omega(p) := \omega(gq) = g\omega(q).$$

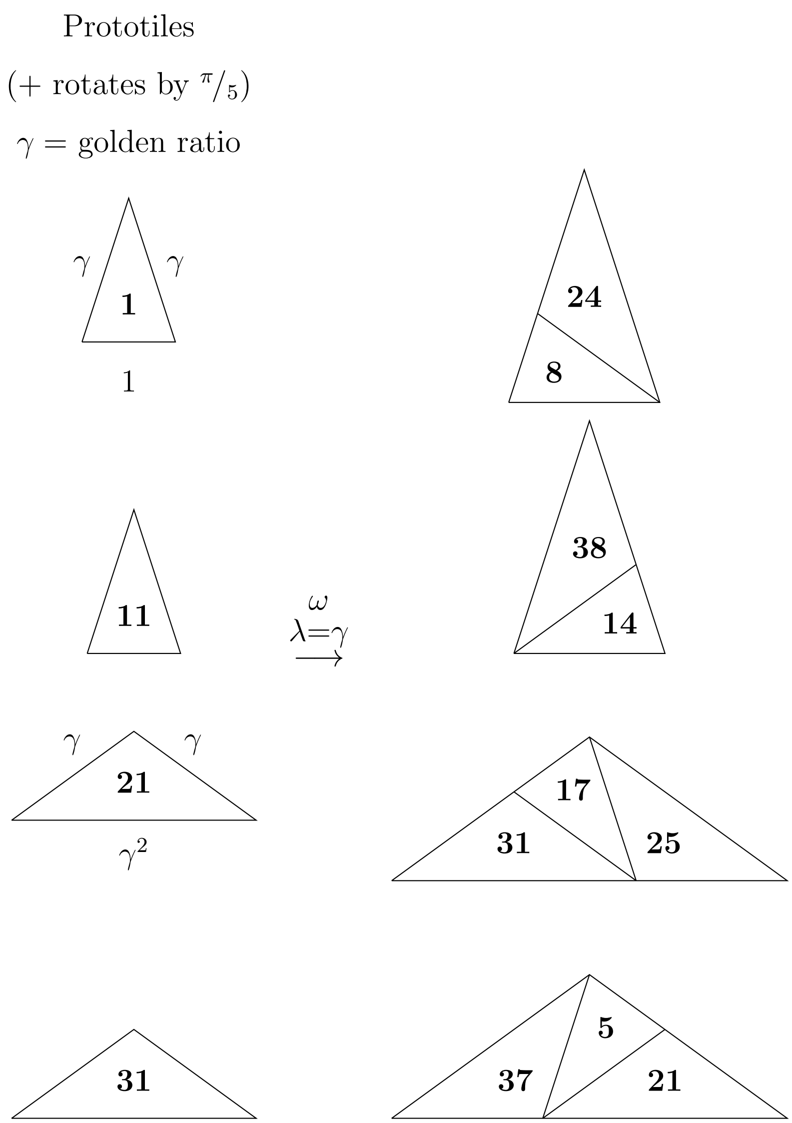


Figure 2.4: Substitution rule on Robinson triangles.

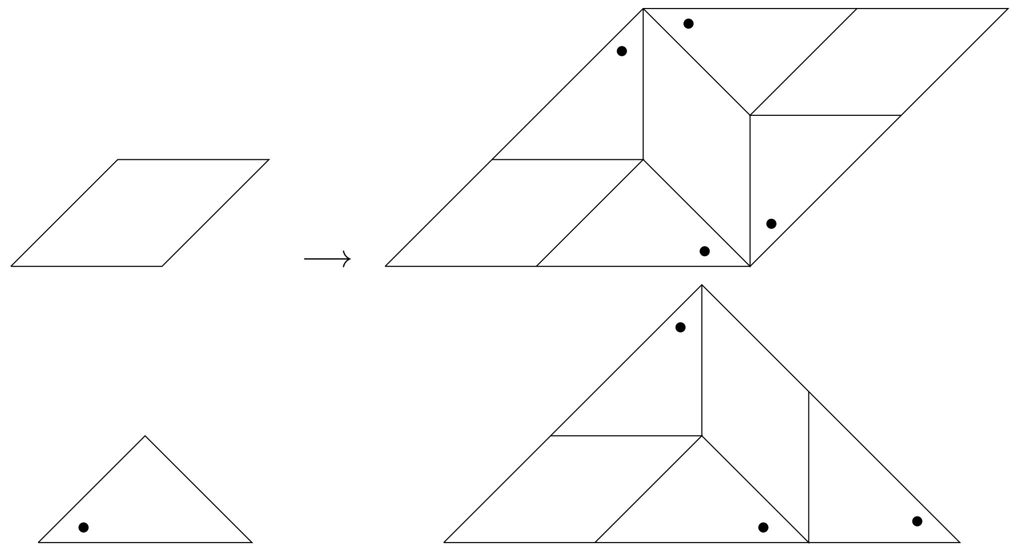


Figure 2.5: Substitution rule for the octagonal tiling.

In the case of the Penrose tiling above, we can take G to be the dihedral group D_{10} generated by r (the counterclockwise rotation by $^{\pi}/_{5}$) and f (the reflection over the x-axis). These elements satisfy the relations

$$r^{10} = f^2 = e, \qquad frf = r^{-1}.$$

In this case, we can take $\mathcal{S}_{D_{10}} = \{\mathbf{1}, \mathbf{21}\}$. Another feature of this action is that D_{10} acts freely on $\mathcal{P}_{\text{Pen}}$, that is, if $gp = p$ for some $g \in D_{10}$ and $p \in \mathcal{P}_{\text{Pen}}$, then $g = e$. We note that for the subgroup $\langle r \rangle$ we have $\mathcal{S}_{\langle r \rangle} = \{\mathbf{1}, \mathbf{11}, \mathbf{21}, \mathbf{31}\}$ and the action of $\langle r \rangle$ also free.

For the octagonal tiling, we can take G to be the dihedral group D_8 generated by r (the counterclockwise rotation by $^{\pi}/_{4}$) and f (the reflection over the x-axis). As before, these elements satisfy the relations

$$r^{10} = f^2 = e, \qquad frf = r^{-1}.$$

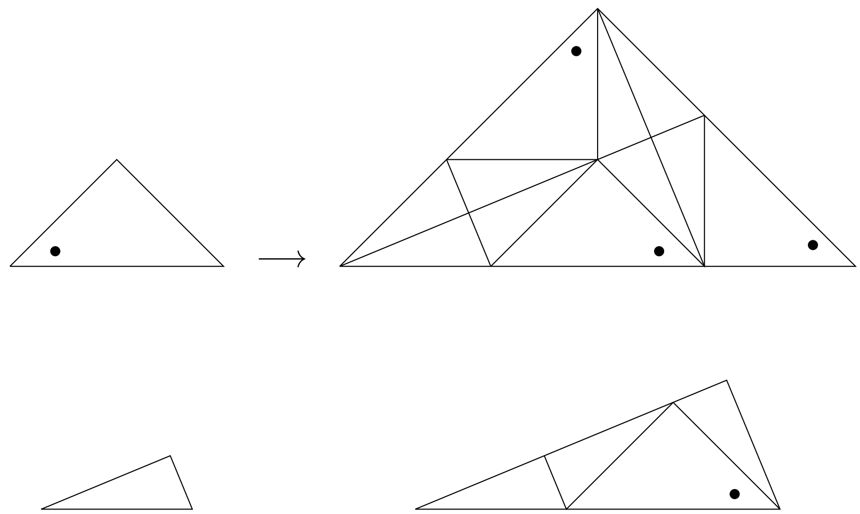


Figure 2.6: Substitution rule for the octagonal tiling after breaking symmetry.

Then $\mathcal{S}_{D_8}$ consists of a rhomb and a triangle. In this case, the action of D_8 on $\mathcal{P}$ is not free, as the rhomb is fixed by r^4, rf, and $r^5 f$. However, one can always arrange for a free action of a group on the prototiles, as the following example shows.

Example 2.4.12 Figure 2.5 shows the substitution for the octagonal tiling on a rhomb and a triangle. To break the symmetry of the rhomb, we divide it into 4 triangles by cutting along both its diagonals. The resulting substitution rule is given in Figure 2.6.

2.5 The Tiling Space as an Inverse Limit

In [1], Anderson and Putnam obtain a tractable description of Ω for a substitution tiling. This is a landmark in the theory, and has since been generalized to classes of tilings other than substitution tilings; see [10], [45], [58] and [61].

We begin with standard definitions from topology.

Definition 2.5.1 ([29], discussion on page 5) *An* **open n-cell** *is a space homeomor-*

phic to the open unit ball in $\mathbb{R}^n$ – the open 0-cell is the singleton space. A space X is called a **finite CW complex** *(or simply a* **CW complex** *or a* **cell complex***) if X can be written as an increasing union*

$$X^0 \subset X^1 \subset \cdots \subset X^{k-1} \subset X^k = X$$

where X^0 is a finite set whose points are regarded as 0-cells, and $X^m \setminus X^{m-1}$ is a finite disjoint union of open m-cells for all m, such that for each m-cell in $X^m \setminus X^{m-1}$ there exists a continuous map f_e from the closed unit ball in $\mathbb{R}^m$ into X^m such that f_e restricted to the open unit ball is a homeomorphism onto e.

Furthermore, if $X = \cup X^n$ and $Y = \cup Y^n$ are CW complexes, a continuous map $f : X \to Y$ is called **cellular** *if $f\left(X^k\right) \subset Y^k$ for all k. A continuous map $g : X \to Y$ is called a* **CW map** *if whenever e is an m-cell in $X^m \setminus X^{m-1}$ we have $g(e)$ is an m-cell in $Y^m \setminus Y^{m-1}$.*

Let (P, ω) be a primitive substitution tiling system. We assume, as is the case in all our examples, that our prototiles are polytopes. Consider

$$Y = \{(x, p) \in \mathbb{R}^d \times \mathcal{P} \mid x \in p\}$$

i.e., the disjoint union of the prototiles. We define an equivalence relation on this set as follows: we declare (x, p) and (y, q) to be equivalent if there is a tiling T in Ω such that, for some $z_p, z_q \in \mathbb{R}^d$ we have $p + z_p, q + z_q \in T$ and $z_p + x = z_q + y$. In words, we treat the prototiles as disjoint sets and then glue them together wherever they could possibly meet up in any tiling. If $\mathcal{R}$ is the equivalence relation generated by the above, we let

$$\Gamma = Y/\mathcal{R}.$$

When our prototiles are polytopes meeting full-face to full-face, then in [1] it is shown that Γ is a d-dimensional CW complex whose d-cells are the prototiles. The substitution induces a map γ on Γ in the obvious way – if x is in some prototile p, then $\omega(p)$

is a patch consisting of translates of prototiles, so λx lies inside at least one translate of a prototile $p_i + y$. So then $\lambda x - y$ is in p_i, and we define $\gamma((x,p)) = (\lambda x - y, p_i)$. Even though λx could lie in more than one tile, and hence the image could be in more than one prototile, this map is well-defined precisely because such points are identified. It is proved in [1] that γ is continuous.

Recall that if X is a compact Hausdorff space and $\varphi : X \to X$ is a continuous surjection, then the inverse limit $\mathscr{X} = \varprojlim(X \xleftarrow{\varphi} X)$ is the subspace of $\prod_{n\in\mathbb{N}} X$ of all sequences $(x_n)_{n\in\mathbb{N}}$ such that $\varphi(x_{n+1}) = x_n$ for all $n \in \mathbb{N}$, with the relative topology from the product topology. For an open set $U \subset X$ and $n \in \mathbb{N}$, let $B^{\mathscr{X}}_{U,n}$ denote the set

$$B^{\mathscr{X}}_{U,n} = \{(x_i)_{i\in\mathbb{N}} \mid x_i \in \varphi^{n-i}(U), i = 0, 1, \dots, n\}.$$

The collection of sets $B^{\mathscr{X}}_{U,n}$ forms a basis for the topology on $\mathscr{X}$.

We let

$$\Omega_0 = \varprojlim\left(\Gamma \xleftarrow{\gamma} \Gamma\right) = \{(x_1, x_2, \dots) \mid x_i \in \Gamma, \gamma(x_i) = x_{i-1}\}$$

and define $\omega_0 : \Omega_0 \to \Omega_0$ as the left shift (or from another perspective, application of γ to each coordinate). In [1] the authors prove that ω_0 is a homeomorphism. In the presence of the following condition, they get even more.

Definition 2.5.2 *A substitution tiling system $(\mathcal{P}, \omega)$ is said to* **force its border** *if there exists an $n \in \mathbb{N}$ such that for all $p \in \mathcal{P}$ if we have that whenever $\omega^n(p) + x \in T$ and $\omega^n(p) + x' \in T'$ then we can conclude that*

$$T\left(\text{supp}(\omega^n(p) + x)\right) - x = T'\left(\text{supp}(\omega^n(p) + x')\right) - x'.$$

In words, a substitution forces its border if there exists an n such that the tiles touching the patch $\omega^n(p)$ are the same no matter where in any given tiling one sees a translate of it. The Penrose substitution of Figure 2.4 and the octagonal substitution of Figure 2.6 both force their border. In the case of the Penrose one can see that it satisfies this property by the following argument. Suppose that $\{t_1, t_2\}$ is a two-tile patch in a Penrose tiling such that $t_1 \cap t_2$ is homeomorphic to $[0,1]$. There are only finitely many such patches. Consider the patch

$$P = \{t \in T \mid t \in \omega(\{t_1, t_2\}), t \cap \mathrm{Int}(\lambda(t_1 \cap t_2))\}.$$

Informally, P is the set of all tiles that touch the edge $\lambda(t_1 \cap t_2)$ after performing the substitution. Then, if f_{t_1,t_2} denotes the reflection of $\mathbb{R}^2$ over line through the origin with the same direction as $t_1 \cap t_2$, one can check that $f_{t_1,t_2}P = P$. In other words, substituting an edge in a Penrose tiling results in a patch which is reflection-symmetric over the substituted edge. Since we must care for the tiles which intersect the ends of the edges too, we end up getting that the Penrose tiling forces its border with $n = 4$. The octagonal tiling shares this property. An example of a tiling which does not force its border is the Chair tiling; for a discussion of this tiling see [1], Example 10.5.

Theorem 2.5.3 ([1], Theorem 4.3) *If $(\mathcal{P}, \omega)$ forces its border, then the dynamical systems (Ω, ω) and (Ω_0, ω_0) are topologically conjugate, that is, there exists a homeomorphism $\varphi : \Omega \to \Omega_0$ such that $\varphi \circ \omega = \omega_0 \circ \varphi$.*

Mapping a tiling T to a sequence is straightforward - to get the first point in the sequence you look at the position of the origin inside whatever tile sits around the origin in T and find the corresponding point in Γ. To find the next point you do the same thing for $\omega^{-1}(T)$ and so on.

Remark 2.5.4 The need to force the border is not as restrictive as it looks. The following argument from ([1], §4) explains why. From a substitution tiling system $(\mathcal{P}, \omega)$ we form a new one $(\mathcal{P}', \omega')$ as follows: for each prototile $p \in \mathcal{P}$, look at the set of all patches $\Omega(p) = \{T(p) \mid T \in \Omega\}$. By finite local complexity, this set is finite. We let

$$\mathcal{P}' = \{(p, P) \mid p \in \mathcal{P}, P \in \Omega(p)\}.$$

In words, we create a labeled copy of p for each patch consisting of tiles that intersect p that could possibly surround it in any tiling in Ω. The substitution extends to this in the natural way. If we let Γ_1 and γ_1 be the CW complex and map formed as above but from $(\mathcal{P}', \omega')$, and form

$$\Omega_1 = \lim_{\leftarrow} \left(\Gamma_1 \stackrel{\omega'}{\leftarrow} \Gamma_1 \right)$$

then, with ω_1 the shift map on the above, (Ω_1, ω_1) is always topologically conjugate to (Ω, ω). This procedure is called *collaring*.

Example 2.5.5 The CW complex of the Penrose tiling is given in Figure 2.7. This is the same as [1] Figure 6 with a correction - they have the short edge on triangles **17** and **2** labeled as 27, and we have it labeled correctly as 22 (here bold numbers indicate 2-cells and text numbers indicate 1-cells).

Example 2.5.6 The CW complex of the octagonal tiling is given in Figure 2.8.

Example 2.5.7 After breaking symmetry on the octagonal tiling, we get the CW complex in Figure 2.9. It is easy to see that the two CW complexes given in Figure 2.8 and Figure 2.9 are homeomorphic.

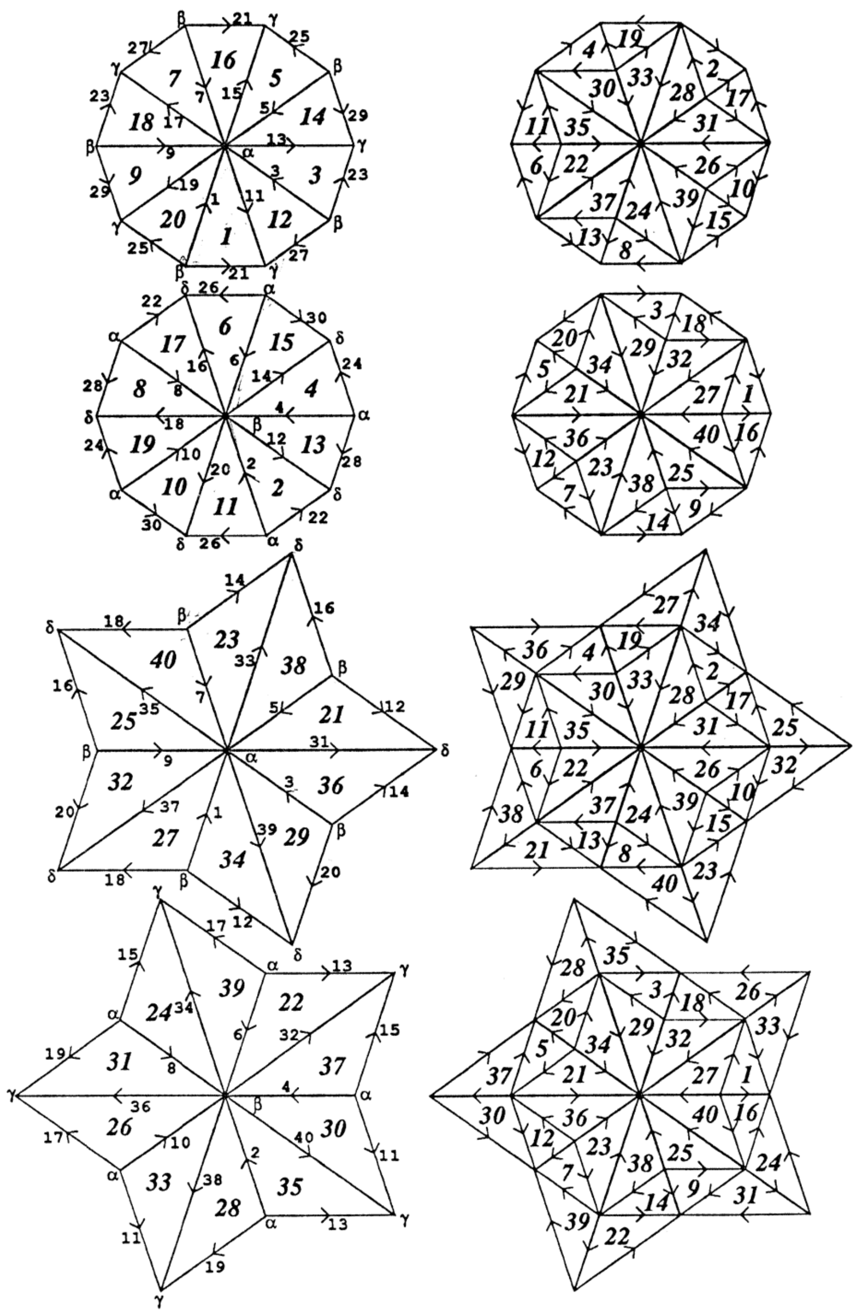


Figure 2.7: CW complex for the Penrose tiling.

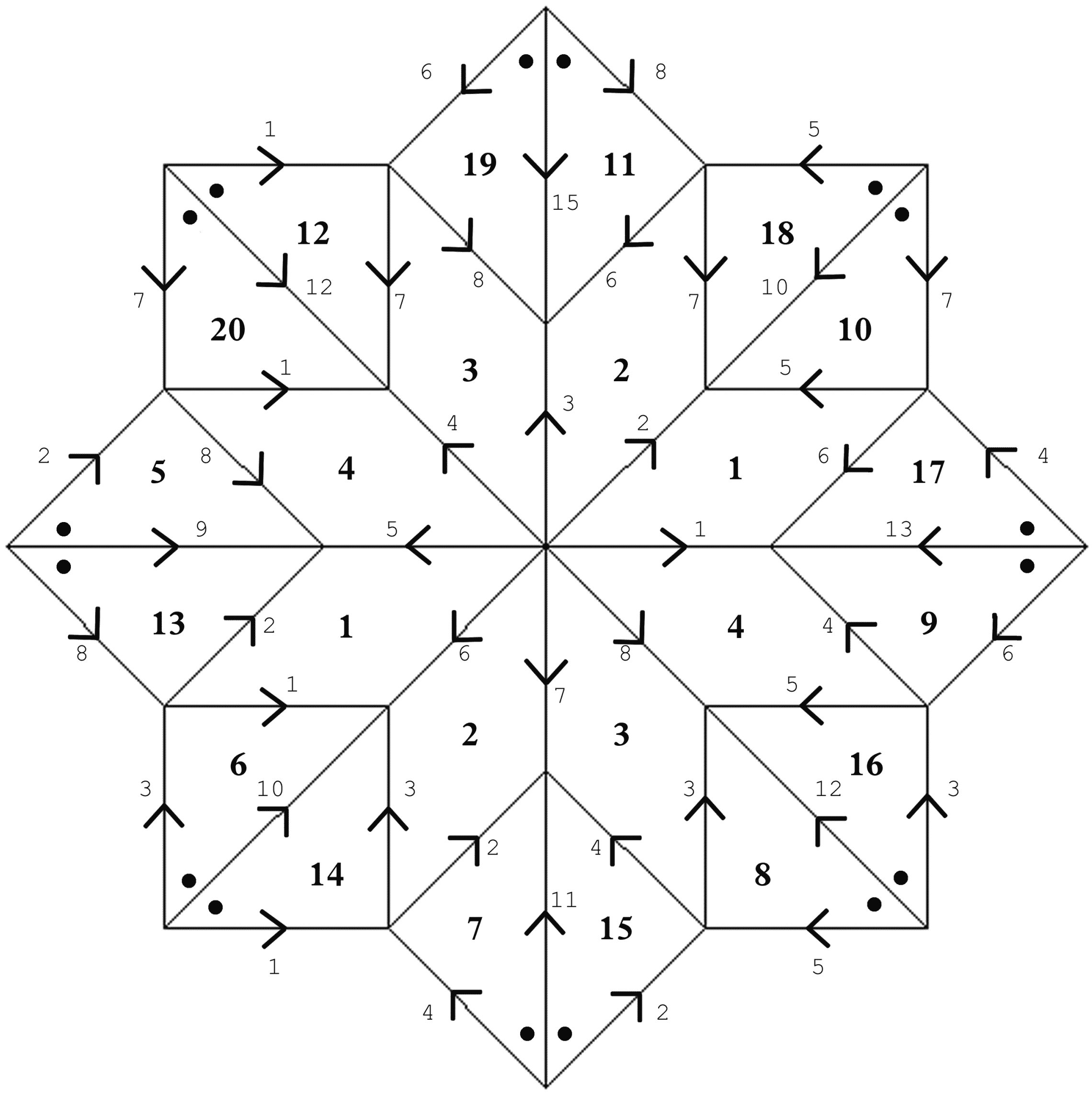


Figure 2.8: CW complex for the octagonal tiling.

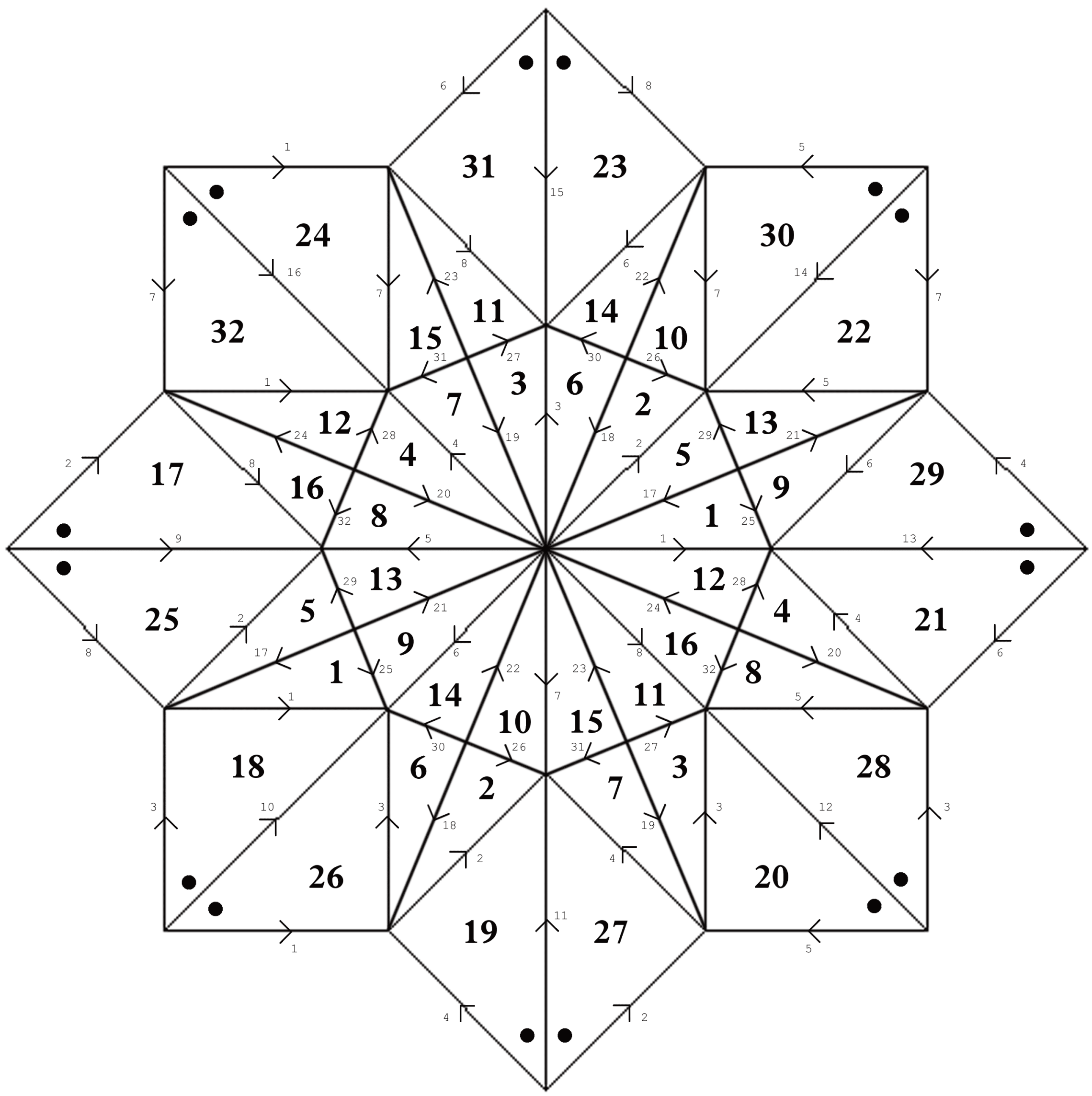


Figure 2.9: CW complex for the octagonal tiling after breaking symmetry.

Remark 2.5.8 For the rest of this work, we assume that

1. $(\mathcal{P}, \omega)$ is a primitive substitution tiling system (Definition 2.4.2)

such that

2. every tiling arising from $(\mathcal{P}, \omega)$ via Construction 2.4.3 has finite local complexity (Definition 2.2.4).

If Ω is the tiling space formed from any such tiling, it does not depend on the choice of tiling chosen. We also assume that

3. Ω contains no periodic tilings, and

4. $(\mathcal{P}, \omega)$ forces its border (Definition 2.5.2).

When these are satisfied, Ω is compact (Lemma 2.2.5) and every $\mathbb{R}^d$ orbit in Ω is dense (Lemma 2.4.6). The map $\omega : \Omega \to \Omega$ is a homeomorphism (Lemmas 2.4.7, 2.4.9) and ω is locally invertible (Definition 2.4.8). The subspace $\Omega_{\text{punc}} \subset \Omega$ is homeomorphic to a Cantor set (Lemma 2.2.7).

Chapter 3

Groupoids and Equivalence Relations

In this chapter we recall definitions and terminology used in the study of groupoids, especially in how they relate to the study of C*-algebras. Many of the definitions in Sections 3.1 and 3.2 are from Renault's Lecture Notes on groupoid C*-algebras [59].

3.1 Groupoids

Definition 3.1.1 *A* **groupoid** *is a set $\mathscr{G}$, a subset $\mathscr{G}^{(2)} \subset \mathscr{G} \times \mathscr{G}$ called the set of composable pairs, a partially defined product from $\mathscr{G}^{(2)} \to \mathscr{G}$ with $(\gamma, \eta) \to \gamma\eta$ and an inverse map from $\mathscr{G} \to \mathscr{G}$ with $\gamma \to \gamma^{-1}$ such that the following relations are satisfied:*

1. $(\gamma^{-1})^{-1} = \gamma$,
2. $(\gamma, \eta), (\eta, \xi) \in \mathscr{G}^{(2)}$ *implies* $(\gamma\eta, \xi), (\gamma, \eta\xi) \in \mathscr{G}^{(2)}$ *and* $(\gamma\eta)\xi = \gamma(\eta\xi)$,
3. $(\gamma^{-1}, \gamma) \in \mathscr{G}^{(2)}$ *and if* $(\gamma, \eta) \in \mathscr{G}^{(2)}$ *then* $\gamma^{-1}(\gamma\eta) = \eta$,
4. $(\gamma, \gamma^{-1}) \in \mathscr{G}^{(2)}$ *and if* $(\xi, \gamma) \in \mathscr{G}^{(2)}$ *then* $(\xi\gamma)\gamma^{-1} = \xi$.

If $\gamma \in \mathscr{G}$, $s(\gamma) = \gamma^{-1}\gamma$ is the **source** of γ and $r(\gamma) = \gamma\gamma^{-1}$ is its **range**. The pair (γ, η) is composable if and only if the range of η is the source of γ. We call the set $\mathscr{G}^0 = d(\mathscr{G}) = r(\mathscr{G})$ the **unit space** of $\mathscr{G}$.

A groupoid $\mathscr{G}$ is said to be **principal** if the map (r, s) from $\mathscr{G}$ into $\mathscr{G}^{(0)} \times \mathscr{G}^{(0)}$ is injective; it is said to be **transitive** if the map (r, s) is onto.

Definition 3.1.2 *A* **subgroupoid** $\mathscr{H} \subset \mathscr{G}$ *is a subset of* $\mathscr{G}$ *that is closed under products and inverses such that* $\mathscr{H}^{(0)} = \mathscr{G}^{(0)}$.

Definition 3.1.3 *Let* $\mathscr{H}$ *and* $\mathscr{G}$ *be groupoids. A map* $\phi : \mathscr{H} \to \mathscr{G}$ *is a* **homomorphism** *if whenever* $(\gamma, \eta) \in \mathscr{G}^{(2)}$, *we have that* $(\phi(\gamma), \phi(\eta)) \in \mathscr{H}^{(2)}$, *and in this case* $\phi(\gamma\eta) = \phi(\gamma)\phi(\eta)$. *If* ϕ *is bijective, we call it an* **isomorphism**. *If* $\phi : \mathscr{G} \to \mathscr{G}$ *is an isomorphism, we call it an* **automorphism**. *For a groupoid* $\mathscr{G}$, *we denote the group of all automorphisms on* $\mathscr{G}$ *by* $\mathrm{Aut}(\mathscr{G})$.

One sees that homomorphisms take units to units and inverses to inverses.

If A and B are subsets of $\mathscr{G}$ and $u, v \in \mathscr{G}^0$ we may form the following:

- $A^{-1} = \{\gamma^{-1} \mid \gamma \in A\}$
- $AB = \{\xi \in \mathscr{G} \mid \gamma \in A, \eta \in B \text{ such that } \xi = \gamma\eta\}$
- $\mathscr{G}^u = r^{-1}(u)$, $\mathscr{G}^A = r^{-1}(A)$
- $\mathscr{G}_v = s^{-1}(v)$, $\mathscr{G}_B = s^{-1}(B)$
- $\mathscr{G}_u^v = \mathscr{G}^u \cap \mathscr{G}_v$, $\mathscr{G}_B^A = \mathscr{G}^A \cap \mathscr{G}_B$
- $\mathscr{G}(u) = \mathscr{G}_u^u$, which is a group, is called the **isotropy group** at u.

The relation $u \sim v \Leftrightarrow \mathscr{G}_v^u \neq \emptyset$ is an equivalence relation on the unit space. Its equivalence classes are called **orbits** and the orbit of u is denoted $[u]$. A groupoid is transitive if and only if it has a single orbit.

Definition 3.1.4 *Let $\mathscr{G}$ be a groupoid. We say that a set $S \subset \mathscr{G}$ is a* $\mathscr{G}$**-set** *if the restrictions of r and s to S are injective. Equivalently, S is a $\mathscr{G}$-set if and only if SS^{-1} and $S^{-1}S$ are both subsets of $\mathscr{G}^{(0)}$.*

Definition 3.1.5 *Let $\mathscr{G}$ be a groupoid and $E \subset \mathscr{G}^{(0)}$. Then the* **saturation** *of E, denoted $[E]$, is*

$$[E] = r(s^{-1}(E)) = r(\mathscr{G}_E).$$

We say that E is **invariant** *if $[E] = E$.*

Example 3.1.6 Let X be a set and let $\mathscr{R}$ be an equivalence relation on X. Then $\mathscr{R}$ is endowed with a natural groupoid structure on setting

1. $((x, y), (w, z)) \in \mathscr{R}^{(2)} \Leftrightarrow y = w$ and $(x, y)(y, z) = (x, z)$

2. $(x, y)^{-1} = (y, x)$

In this case, from the definition of the range and source maps, $r(x, y) = (x, x)$ and $s(x, y) = (y, y)$ for all $(x, y) \in \mathscr{R}$. Thus the unit space $\mathscr{R}^{(0)}$ is given by the diagonal $\Delta_X = \{(x, x) \mid x \in X\}$, and upon identifying the diagonal with X we see that the range and source maps are the projections onto the first and second coordinate respectively. It is easy to see that every principal groupoid can be seen as an equivalence relation on its unit space. If $u \in X$, then its orbit $[u]$ is its equivalence class, and if $E \subset X$ its saturation $[E]$ is the union of the equivalence classes of elements of E.

Example 3.1.7 Let X be a set and let G be a group acting on X. That is, there is a map $G \times X \to X$ with $(g, x) \mapsto gx$ such that $ex = x$ for all $x \in X$ and $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$. We endow $X \times G$ with a groupoid structure by saying (x, g) is composable with (y, h) if and only if $y = gx$, and in this case

$$(x, g)(gx, h) = (x, hg).$$

The switching of the order of the group elements arises because we are assuming G to be acting on the left. One can easily see that $(x,g)^{-1} = (gx, g^{-1})$, that $r(x,g) = (x,e)$ and $s(x,g) = (gx, e)$. This groupoid is called the **transformation group groupoid** which we denote (X, G). The unit space consists of all the pairs (x, e), and so can be identified with X.

Renault [59] gives us the following definition of a groupoid arising from a group action on a groupoid. We will refer to this construction often.

Definition 3.1.8 *Let $\mathscr{G}$ be a groupoid, G be a group, and let $\alpha : G \to \mathrm{Aut}(\mathscr{G})$ be a homomorphism. We write $\gamma \cdot g = \alpha_{g^{-1}}(\gamma)$ for $g \in G$ and $\gamma \in \mathscr{G}$. The* **semidirect product** *$\mathscr{G} \rtimes_\alpha G$ is the groupoid $\mathscr{G} \times G$ where*

1. *(γ, g) and (ξ, h) are composable if and only if $\xi = \eta \cdot g$ with γ and η composable,*

2. *$(\gamma, g)(\eta \cdot g, h) = (\gamma\eta, gh)$, and*

3. *$(\gamma, g)^{-1} = (\gamma^{-1} \cdot g, g^{-1})$.*

In this case, $r(\gamma, g) = (r(\gamma), e)$ and $s(\gamma, g) = (s(\gamma) \cdot g, e)$. In light of this, the unit space of $\mathscr{G} \rtimes_\alpha G$ may be identified with the unit space of $\mathscr{G}$.

We note that the notation $\gamma \cdot g = \alpha_{g^{-1}}(\gamma)$ is in fact more than notation – this indeed defines a right action of the group G on $\mathscr{G}$.

Example 3.1.9 Example 3.1.7 can be seen as a semidirect product. Let $\mathscr{R} = \{(x,x) \mid x \in X\}$ be the trivial equivalence relation on X. If G acts on X then it acts on $\mathscr{R}$ in the obvious way, and the semidirect product $\mathscr{R} \rtimes G$ is isomorphic to the transformation group groupoid (X, G).

3.2 Topological Groupoids

Definition 3.2.1 *If $\mathscr{G}$ is a groupoid with a topology and $\mathscr{G}^{(2)}$ is given the product topology, then we say $\mathscr{G}$ is a* **topological groupoid** *if the inverse map $\mathscr{G} \to \mathscr{G}$ and the product $\mathscr{G}^{(2)} \to \mathscr{G}$ are both continuous.*

In the case of an equivalence relation $\mathscr{R}$ on a set X, the following are immediate consequences of the above definition. The range and source maps are continuous and $\gamma \mapsto \gamma^{-1}$ is a homeomorphism. If $\mathscr{R}$ is Hausdorff, then $\mathscr{R}^{(0)}$ is closed in $\mathscr{R}$. If $\mathscr{R}^{(0)}$ is Hausdorff, then $\mathscr{R}^{(2)}$ is closed in $\mathscr{R} \times \mathscr{R}$. Moreover, $\mathscr{R}^{(0)}$ is both a subspace of $\mathscr{R}$ and a quotient of $\mathscr{R}$ by the map r. We also observe that in a Hausdorff topology a single point $\{x\}$ is closed for all $x \in \mathscr{R}^{(0)}$ and from this we have that $r^{-1}\{x\}$ is closed in $\mathscr{R}$.

Definition 3.2.2 *A* **Haar system** *on a locally compact Hausdorff groupoid $\mathscr{G}$ is a family $\{\mu^u\}_{u \in \mathscr{G}^{(0)}}$ of non-negative measures on $\mathscr{G}$ such that*

1. $\mathrm{supp}(\mu^u) = \mathscr{G}^u, u \in \mathscr{G}^{(0)}$,
2. *for $f \in C_c(\mathscr{G})$, the function*
$$u \to \int f d\mu^u$$
on $\mathscr{G}^{(0)}$ is continuous and compactly supported, and
3. *for $\gamma \in \mathscr{G}, \gamma\mu^{s(x)} = \mu^{r(\gamma)}$, that is, $\int f(\gamma\eta) d\mu^{s(\gamma)}(\eta) = \int f(\eta) d\mu^{r(\gamma)}(\eta)$.*

A groupoid need not admit a Haar system. The following is a case where possible Haar systems are easy to describe.

Definition 3.2.3 *A locally compact Hausdorff groupoid $\mathscr{G}$ is called* r**-discrete** *if $\mathscr{G}^{(0)}$ is open in $\mathscr{G}$.*

Many of the groupoids considered in this thesis will be r-discrete, and so their properties will be important to us. Renault shows in [59] that if $\mathscr{G}$ is a locally compact Hausdorff r-discrete groupoid we have the following:

1. for any $u \in \mathscr{G}^{(0)}$, $\mathscr{G}^u$ and $\mathscr{G}_u$ are discrete,

2. if $\mathscr{G}$ has a Haar system, then the Haar system consists of counting measures, and

3. $\mathscr{G}$ has a Haar system if and only if r and s are local homeomorphisms.

Lemma 3.2.4 ([59], Proposition I.2.8) *Let $\mathscr{G}$ be a locally compact Hausdorff groupoid. Then the following are equivalent:*

1. *$\mathscr{G}$ is r-discrete and admits a Haar system,*

2. *$r : \mathscr{G} \to \mathscr{G}^{(0)}$ is a local homeomorphism,*

3. *$\mathscr{G}$ has a neighbourhood base consisting of open $\mathscr{G}$-sets.*

We will be concerned with the semidirect product of r-discrete groupoids by finite groups. The following proposition is slightly more general.

Proposition 3.2.5 *Let $\mathscr{G}$ be a locally compact Hausdorff r-discrete groupoid which admits a Haar system, let G be a discrete group, and let $\alpha : G \to \mathrm{Aut}(\mathscr{G})$ be a homomorphism. Then the semidirect product $\mathscr{G} \rtimes_\alpha G$ is locally compact, Hausdorff, r-discrete and admits a Haar system.*

Proof: We recall from the definition of the semidirect product groupoid (Definition 3.1.8) that the unit space of $\mathscr{G} \rtimes_\alpha G$ is also $\mathscr{G}^{(0)}$ and that $r(\gamma, g) = r(\gamma)$ for all $\gamma \in \mathscr{G}$ and $g \in G$. For $\gamma \in \mathscr{G}$ and $g \in G$ we will also use the notation $\gamma \cdot g := \alpha_{g^{-1}}(\gamma)$ introduced in Definition 3.1.8. The groupoid $\mathscr{G} \rtimes_\alpha G$ is given the product topology, so it is locally compact and Hausdorff. Let $(\gamma, g) \in \mathscr{G} \rtimes_\alpha G$, and find a neighbourhood U of γ in $\mathscr{G}$ such that $r|_U : U \to r(U)$ is a homeomorphism. Then $U \times \{g\}$ is open in $\mathscr{G} \rtimes_\alpha G$ and $r(U \times \{g\}) = r(U)$, and so r is a local homeomorphism. Hence by Lemma 3.2.4, $\mathscr{G} \rtimes_\alpha G$ admits a Haar system consisting of counting measures. ∎

The following definition is of a special class of r-discrete principal groupoids of particular interest.

Definition 3.2.6 *Let $\mathscr{R}$ be an equivalence relation on a compact metrizable space X. We say the topology $\mathcal{T}$ on $\mathscr{R}$ is an* **étale topology** *for $\mathscr{R}$ if the following conditions are satisfied.*

1. *$(\mathscr{R}, \mathcal{T})$ is σ-compact;*
2. *$\Delta_X = \{(x, x) \mid x \in X\}$ is open in $(\mathscr{R}, \mathcal{T})$;*
3. *every point (x_0, y_0) in $\mathscr{R}$ has an open neighbourhood U in $(\mathscr{R}, \mathcal{T})$ such that $r(U)$ and $s(U)$ are open in X and both $r : U \to r(U)$ and $s : U \to s(U)$ are homeomorphisms;*
4. *if U and V are open sets in $(\mathscr{R}, \mathcal{T})$ then UV is open in $(\mathscr{R}, \mathcal{T})$;*
5. *if U is an open set in $(\mathscr{R}, \mathcal{T})$ then U^{-1} is open in $(\mathscr{R}, \mathcal{T})$.*

In this case, $\mathscr{R}$ is called an **étale equivalence relation***.*

The following definition describes many of the groupoids we encounter when studying tilings.

Definition 3.2.7 ([47], Definition 1.1) *A topological groupoid $\mathscr{G}$ equipped with a Haar system is called a* **Cantor groupoid** *if the following conditions are satisfied:*

1. *$\mathscr{G}$ is Hausdorff, locally compact, and second countable.*
2. *The unit space $\mathscr{G}^{(0)}$ is compact, totally disconnected, metrizable, and has no isolated points (so is homeomorphic to the Cantor set).*
3. *$\mathscr{G}^{(0)}$ is open in $\mathscr{G}$.*
4. *The Haar system consists of counting measures.*

Definition 3.2.8 *A Cantor groupoid $\mathscr{G}$ is called* **approximately finite** *(AF for short), if it is an increasing union of a sequence of compact open principal Cantor subgroupoids, each of which contains the unit space $\mathscr{G}^{(0)}$.*

Phillips [47] proves that an AF Cantor groupoid is AF in the sense of Definition 1.1 in Chapter 3 of [59].

Definition 3.2.9 ([47], Definition 2.1) *Let $\mathscr{G}$ be a Cantor groupoid and let $K \subset \mathscr{G}^{(0)}$ be a compact subset. We say K is* **thin** *if for every n there exist compact $\mathscr{G}$-sets $S_1, S_2, \ldots, S_n \subset \mathscr{G}$ such that $s(S_k) = K$ and the sets $r(S_1), r(S_2), \ldots, r(S_n)$ are pairwise disjoint.*

Definition 3.2.10 ([47], Definition 2.2) *Let $\mathscr{G}$ be a Cantor groupoid. We say that $\mathscr{G}$ is an* **almost AF Cantor groupoid** *if we have the following:*

1. *There exists an open AF subgroupoid $\mathscr{G}_0 \subset \mathscr{G}$ which contains the unit space such that whenever K is a compact subset of $\mathscr{G} \setminus \mathscr{G}_0$, we have that $s(K)$ is thin in the sense of Definition 3.2.9.*

2. *For every closed invariant subset $E \subset \mathscr{G}^{(0)}$, and every nonempty relatively open subset $U \subset E$, there is a $\mathscr{G}$-invariant Borel probability measure μ on $\mathscr{G}^{(0)}$ such that $\mu(U) > 0$*

Equivalence relations are principal groupoids. A slightly weaker notion than principal will become useful to us later.

Definition 3.2.11 ([59], Definition II.4.3) *We say that a locally compact Hausdorff groupoid $\mathscr{G}$ is* **essentially principal** *if for every invariant closed subset F of its unit space, the set of u in F for which $\mathscr{G}(u) = \{u\}$ is dense in F.*

Lemma 3.2.12 ([47], Lemma 2.6) *If $\mathscr{G}$ is an almost AF Cantor groupoid, then it is essentially principal.*

3.3 Equivalence of Groupoids

In this section we present an important notion of equivalence for topological groupoids. For the most part, we follow the development of [42].

Definition 3.3.1 ([42], Definition 2.12) *Let $\mathscr{G}$ be a groupoid and let X be a set. We say $\mathscr{G}$* **acts** *on X (on the left), and that X is a* **left $\mathscr{G}$-space**, *if there is a surjection $r : X \to \mathscr{G}^{(0)}$ and a map $(\gamma, x) \mapsto \gamma x$ from $\mathscr{G} * X := \{(\gamma, x) \mid s(\gamma) = r(x)\}$ to X such that*

1. *$r(\gamma x) = r(\gamma)$ for all $(\gamma, x) \in \mathscr{G} * X$;*
2. *if $(\gamma_1, x) \in \mathscr{G} * X$ and $(\gamma_2, \gamma_1) \in \mathscr{G}^{(2)}$, then $(\gamma_2\gamma_1, x), (\gamma_2, \gamma_1 x) \in \mathscr{G} * X$ and $\gamma_2(\gamma_1 x) = (\gamma_2\gamma_1)x$; and*
3. *$r(x)x = x$ for all $x \in X$.*

   Right actions and right $\mathscr{G}$-spaces are defined similarly, but s is used to denote the map X to $\mathscr{G}^{(0)}$ and we write $X * \mathscr{G} = \{(x, \gamma) \mid s(x) = r(\gamma)\}$.

We are interested in the case where X and $\mathscr{G}$ have topologies.

Definition 3.3.2 ([42], Remark 2.30) *Let $\mathscr{G}$ be a locally compact Hausdorff groupoid which acts on a locally compact Hausdorff space X on the left. Then we say that the action is* **continuous** *if the map $r : X \to \mathscr{G}^{(0)}$ is continuous and open and the map $(\gamma, x) \mapsto \gamma x$ from $\mathscr{G} * X$ to X is continuous. Similarly, if $\mathscr{G}$ acts on the right of X, the action is continuous if the map $s : X \to \mathscr{G}^{(0)}$ is continuous and open and the map $(x, \gamma) \mapsto \gamma x$ from $X * \mathscr{G}$ to X is continuous.*

Definition 3.3.3 ([42], Definition 5.25) *Let $\mathscr{G}$ be a locally compact Hausdorff groupoid which acts continuously on a locally compact Hausdorff space X on the left. Then we say the action is* **proper** *if the map Φ from $\mathscr{G}*X$ to $X \times X$ defined by $\Phi(\gamma, x) = (\gamma x, x)$*

is proper in the usual sense, i.e., for each compact subset $K \subset X \times X, \Phi^{-1}(K)$ is compact in $\mathscr{G} * X$.

If the action is free in addition to being proper, we say that X is a **principal $\mathscr{G}$-space**.

Properness for right actions is defined analogously.

For left actions, we write $\mathscr{G}\backslash X$ for the quotient space of X under the relation $x \sim y$ if and only if there is a $\gamma \in \mathscr{G}$ such that $\gamma x = y$. For right actions, we write $X/\mathscr{G}$ for the analogous space.

Proposition 3.3.4 ([42], Definition 5.27) *Let $\mathscr{G}$ be a locally compact Hausdorff groupoid which acts continuously on a locally compact Hausdorff space X on the left. Then the quotient map from X to $\mathscr{G}\backslash X$ is open. If the action is proper, then $\mathscr{G}\backslash X$ is Hausdorff.*

Definition 3.3.5 ([42], Definition 5.32) *Let $\mathscr{G}$ and $\mathscr{H}$ be locally compact Hausdorff groupoids. We say that a locally compact Hausdorff space X is a* **$(\mathscr{G}, \mathscr{H})$-equivalence** *if*

1. *X is a principal left $\mathscr{G}$-space and a principal right $\mathscr{H}$-space.*
2. *The actions of $\mathscr{G}$ and $\mathscr{H}$ commute.*
3. *The map $r : X \to \mathscr{G}^{(0)}$ induces a homeomorphism between $\mathscr{G}^{(0)}$ and $X/\mathscr{H}$ and the map $s : X \to \mathscr{G}^{(0)}$ induces a homeomorphism between $\mathscr{H}^{(0)}$ and $\mathscr{G}\backslash X$.*

Further, we will say that two locally compact Hausdorff groupoids $\mathscr{G}$ and $\mathscr{H}$ are **Morita equivalent** *if there exists a $(\mathscr{G}, \mathscr{H})$-equivalence.*

One can prove that this is an equivalence relation.

Example 3.3.6 ([43], Example 2.7) The main example of equivalence that we will consider is that arising from a transversal. Suppose that $\mathscr{G}$ is a locally compact

Hausdorff groupoid and that $F \subset \mathscr{G}^{(0)}$ is a closed subset of the unit space that meets every orbit in $\mathscr{G}^{(0)}$. If the restrictions of r and s to $\mathscr{G}_F$ are open, then the space

$$\mathscr{G}_F = \{\gamma \in \mathscr{G} \mid s(\gamma) \in F\}$$

is a $(\mathscr{G}, \mathscr{G}_F^F)$-equivalence.

Example 3.3.7 ([42], Example 5.33.2) Another example that is important to us is the case of isomorphic groupoids. Let $\mathscr{G}$ and $\mathscr{H}$ be locally compact Hausdorff groupoids and let

$$\varphi : \mathscr{H} \to \mathscr{G}$$

be a groupoid isomorphism which is also a homeomorphism. Then $\mathscr{G}$ acts on the left on $\mathscr{G}$ by translation, and $\mathscr{H}$ acts on the right of $\mathscr{G}$ by the formula $\gamma \cdot \eta := \gamma\varphi(\eta)$. With these two actions, $\mathscr{G}$ becomes a $(\mathscr{G}, \mathscr{H})$-equivalence.

3.4 Groupoids Associated with Tilings

We now describe groupoids associated with tilings. Since we have an action of $\mathbb{R}^d$ on Ω, one natural groupoid to consider is the transformation group groupoid $\mathscr{G} = (\Omega, \mathbb{R}^d)$. We use Example 3.3.6 to produce an equivalent groupoid that is easier to deal with.

We start with a primitive substitution system $(\mathcal{P}, \omega)$ which satisfies the conditions of Remark 2.5.8. Let Ω_{punc} be the punctured hull formed from $(\mathcal{P}, \omega)$. Define

$$\mathcal{R}_{\text{punc}} = \mathscr{G}_{\Omega_{\text{punc}}}^{\Omega_{\text{punc}}} = \{(T, T + x) \mid T, T + x \in \Omega_{\text{punc}}\}.$$

It is clear that $\mathcal{R}_{\text{punc}}$ is an equivalence relation. The set $\mathcal{R}_{\text{punc}}$ can be viewed as a subset of $\Omega_{\text{punc}} \times \mathbb{R}^d$, and so we let $\mathcal{R}_{\text{punc}}$ inherit the product topology from $\Omega_{\text{punc}} \times \mathbb{R}^d$.

Lemma 3.4.1 ([1], Proposition 7.2) *Let $T \in \Omega_{\text{punc}}$. Then there is an $\varepsilon > 0$ and a neighbourhood of $T \in U \subset \Omega_{\text{punc}}$ such that map*

$$s : U \times B_\varepsilon(0) \to \Omega$$

$$s(T', x) = T' + x$$

is a homeomorphism onto its image.

Proof: Take $T \in \Omega_{\text{punc}}$ and consider the open neighbourhood $U(\{T(0)\}, T(0))$. Because punctures are in the interior of the tiles, there exists $\varepsilon > 0$ such that $T(B_{2\varepsilon}(0)) = T(0)$. Now

$$s(U(\{T(0)\}, T(0)) \times B_\varepsilon(0)) = \{T' + x \mid T'(0) = T(0), |x| < \varepsilon\}$$

Suppose $T_1 + x_1$ and $T_2 + x_2$ are both in the above set, with $T_i(0) = T(0)$. Then $T_1 = T_2 + x_2 - x_1$. We have that $|x_2 - x_1| < 2\varepsilon$, and both T_1 and T_2 are punctured tilings with $T(0)$ at the origin, so $x_2 - x_1 = 0$ making $T_1 = T_2$. Thus the restriction of r to $U(\{T(0)\}, T(0)) \times B_\varepsilon(0)$ is injective. So restricted to this domain, s is bijective and continuous. We need to show that it is an open map. Suppose we have V open in $U \times B_\varepsilon(0)$. We may assume that

$$V = U(P, T(0)) \times B_\delta(0)$$

for some $\delta > 0$ and some patch P such that $T(0) \in P$. Then

$$s(V) = \{T' + x \mid T' \in U(P, T(0)), |x| < \delta\}.$$

Take any $T' + x \in s(V)$ and find $\delta_0 > 0$ such that $\delta_0 < \frac{1}{4}(\delta - |x|)$ and also such that $P \subset T'(B_{1/\delta_0}(0))$. It is enough to show that the open ball $B^\Omega_{\delta_0}(T' + x) \subset s(V)$. If we take $S \in B^\Omega_{\delta_0}(T'+x)$ then there exists x' with $|x'| < 2\delta_0$ such that $(S+x')(B_{1/\delta_0}(0)) = (T' + x)(B_{1/\delta_0}(0))$. Thus

$$(S + x' - x)(B_{1/\delta_0}(0)) = T'(B_{1/\delta_0}(0) \supseteq P$$

implies that

$$S + x' - x \in U(P, T(0)).$$

We have that

$$|x - x'| \leq |x| + |x'| < |x| + \frac{1}{2}(\delta - |x|) = \frac{1}{2}\delta + \frac{1}{2}|x| < \delta.$$

Hence $S \in s(V)$. ∎

Corollary 3.4.2 *Let $\mathscr{G} = (\Omega, \mathbb{R}^d)$ be the transformation group groupoid associated to a primitive substitution system. Then $\mathscr{G}_{\Omega_{\text{punc}}}$ is a $(\mathscr{G}, \mathcal{R}_{\text{punc}})$-equivalence, and hence $\mathscr{G}$ and $\mathcal{R}_{\text{punc}}$ are Morita equivalent in the sense of Definition 3.3.5.*

Proof: This combines Lemma 3.4.1 with Example 3.3.6, see [1], Proposition 7.2. ∎

For a patch P and tiles $t_1, t_2 \in P$, we define

$$V(P, t_1, t_2) = \{(T, T + x) \in \mathcal{R}_{\text{punc}} \mid T \in U(P, t_1), x = x(t_1) - x(t_2)\}.$$

From the definition we see that if $(T, T') \in V(P, t_1, t_2)$, then $T \in U(P, t_1)$ and $T' \in U(P, t_2)$. Notice again as in the definition of $U(P, t)$, $V(P, t_1, t_2) = V(P+y, t_1+y, t_2+y)$ for any $y \in \mathbb{R}^d$, so that when describing these sets we may use any translational equivalence class of the triple (P, t_1, t_2). It is proved in [33] that these sets are compact open $\mathcal{R}_{\text{punc}}$-sets that form a base for the topology on $\mathcal{R}_{\text{punc}}$. In addition,

$$r(V(P, t_1, t_2)) = U(P, t_1), \qquad s(V(P, t_1, t_2)) = U(P, t_2)$$

and r and s are homeomorphisms when restricted to this domain. This leads to the following.

Theorem 3.4.3 *With the topology inherited from $\Omega_{\text{punc}} \times \mathbb{R}^d$, $\mathcal{R}_{\text{punc}}$ is an étale equivalence relation.*

The proof is in [33]. Though we do not prove this here, we comment on Condition 3 of our definition of étale equivalence relation. Given $(T, T-x) \in \mathcal{R}_{\text{punc}}$, let P be the patch

$$P = T(\{\xi x \mid \xi \in [0,1]\}).$$

In words, P is the set of all tiles in T that intersect the line segment between 0 and x. Then $(T, T-x) \in V(P, T(0), T(x))$, and $V(P, T(0), T(x))$ is the neighbourhood on which r and s are local homeomorphisms.

We now describe an AF subequivalence relation (see Definition 3.2.8) of $\mathcal{R}_{\text{punc}}$. This equivalence relation is constructed, for example, in [40] and [34], though we follow the description and notation in [33].

Notice that if p and p' are two prototiles and x, x' are points in their respective interiors, then the sets $U(\{p\}, p) - x$ and $U(\{p'\}, p) - x'$ are disjoint unless $p = p'$ and $x = x'$. Since ω is a bijection, the sets $\omega^n(U(\{p\}, p)) - \lambda^n x$ and $\omega^n(U(\{p'\}, p')) - \lambda^n x'$ also have this property for any positive integer n.

Let $n \in \mathbb{N}$ and take $p \in \mathcal{P}$. Let $\text{Punc}(n, p)$ be the set of all the punctures in $\omega^n(p)$, i.e.

$$\text{Punc}(n, p) = \{x_t \mid t \in \omega^n(p)\}.$$

For each pair $x, y \in \text{Punc}(n, p)$ we define

$$E_p^n(x, y) = \{(\omega^n(T) - x, \omega^n(T) - y) \mid T \in U(\{p\}, p)\}.$$

Since x and y are both punctures in $\omega^n(p)$, $\omega^n(T) - x$ and $\omega^n(T) - y$ are both in Ω_{punc}. The second is a translate of the first by $x - y$, so $E_p^n(x, y) \subset \mathcal{R}_{\text{punc}}$. We also define, for $x \in \text{Punc}(n, p)$

$$E_p^n(x) = \{\omega^n(T) - x \mid T \in U(\{p\}, p)\} = r\left(E_p^n(x, y)\right).$$

Lemma 3.4.4 *The sets $E_p^n(x, y)$ are clopen in $\mathcal{R}_{\text{punc}}$ for each $n \in \mathbb{N}$, $p \in \mathcal{P}$ and $x, y \in \text{Punc}(n, p)$.*

Proof: Since $\mathcal{R}_{\text{punc}}$ inherits its topology from $\Omega_{\text{punc}} \times \mathbb{R}^d$, we write

$$E_p^n(x,y) = \{(\omega^n(T) - x, x - y) \mid T \in U(\{p\}, p)\}.$$

We prove that $E_p^n(x)$ is clopen. Suppose we have a convergent sequence $(\omega^n(T_i) - x)_{i\in\mathbb{N}}$ converging to $S \in \Omega_{\text{punc}}$. Then

$$\begin{aligned} \omega^n(T_i) - x &\longrightarrow S \\ T_i - \lambda^{-n}x &\longrightarrow \omega^{-n}(S) \\ T_i &\longrightarrow \omega^{-n}(S) + \lambda^{-n}x. \end{aligned}$$

The T_i are all in $U(\{p\}, p)$, which is closed. Hence $\omega^{-n}(S) + \lambda^{-n}x \in U(\{p\}, p)$. Now

$$\omega^n\left(\omega^{-n}(S) + \lambda^{-n}x\right) - x = S.$$

Thus $S \in E_p^n(x)$ and so $E_p^n(x)$ is closed.

To prove that $E_p^n(x)$ is open we use that ω is locally invertible (see Definition 2.4.8). Recall from the discussion after Definition 2.4.8 that local invertibility implies local invertibility of ω^n. Take $\omega^n(T) - x \in E_p^n(x)$, and let $r > 0$ be the radius obtained from the definition of local invertibility of ω^n. Let $t = (\omega^n(T) - x)(0)$ and

$$P = (\omega^n(T) - x)\,(B_r(0)).$$

The tiling $\omega^n(T) - x$ is in the basic open set $U(P, t)$; we claim that this set is contained in $E_p^n(x)$. Let $S \in U(P, t)$. Then $T' = \omega^{-n}(S) + \lambda^{-n}x$ is a tiling such that

$$(\omega^n(T') - x)\,(B_r(0)) = S(B_r(0)) = (\omega^n(T) - x)\,(B_r(0)).$$

Thus by local invertibility we have that $(T - \lambda^{-n}x)(0) = (T' - \lambda^{-n}x)(0)$. Since $\omega^n(T) - x \in E_p^n(x)$, then $(T - \lambda^{-n}x)(0) = p - \lambda^{-n}x$, and so $(T' - \lambda^{-n}x)(0) = p - \lambda^{-n}x$ as well. Thus $T'(0) = p$, giving us that $T' \in U(\{p\}, p)$ and $S = \omega^n(T') - x$, i.e. $S \in E_p^n(x)$. Thus $U(P, t) \subset E_p^n(x)$, and hence $E_p^n(x)$ is clopen.

Now we just find a radius δ around $x-y$ small enough so that

$$E_p^n(x) \times B_\delta(x-y) \cap \mathcal{R}_{\text{punc}} = E_p^n(x) \times \{x-y\}.$$

This is possible because the punctures are in the interiors of tiles. ∎

Let

$$\mathcal{R}_{n,p} = \bigcup_{x,y \in \text{ Punc}(n,p)} E_p^n(x,y).$$

The sets $\mathcal{R}_{n,p}$ are disjoint for different choices of p. They are finite unions of compact open sets, and are hence compact and open. We let

$$\mathcal{R}_n = \bigcup_{p \in \mathcal{P}} \mathcal{R}_{n,p}.$$

It's easy to see that for each n, $\mathcal{R}_n$ is a subgroupoid of $\mathcal{R}_{\text{punc}}$ with unit space equal to Ω_{punc}.

Lemma 3.4.5 *Let $(\mathcal{R}_n)_{n\in\mathbb{N}}$ be as above. Then for $n \geq 1$ we have*

$$\mathcal{R}_n \subset \mathcal{R}_{n+1};$$

that is, the sequence $(\mathcal{R}_n)_{n\in\mathbb{N}}$ is nested.

Proof: Following the notation of [33], for any $p \in \mathcal{P}$ we let

$$I_p := \{(p', x') \mid p' \in \mathcal{P}, p + x' \in \omega(p')\}.$$

Since ω is invertible, if $T \in U(\{p\}, p)$, there is one and only one pair $(p', x') \in I_p$ such that $T \in \omega\left(U(\{p\}, p)\right) - x'$. Hence we can write $U(\{p\}, p)$ as

$$U(\{p\}, p) = \dot{\bigcup_{(p',x') \in I_p}} \left(\omega\left(U(\{p\}, p)\right) - x'\right) \tag{3.4.1}$$

where the dot indicates that the union is of disjoint sets. Now we have

$$E_p^n(x,y) \;=\; \{(\omega^n(T) - x, \omega^n(T) - y) \mid T \in U(\{p\}, p)\}$$

$$
\begin{aligned}
&= \dot{\bigcup_{(p',x')\in I_p}} \{(\omega^n(\omega(T')-x')-x, \omega^n(\omega(T')-x')-y) \mid T' \in U(\{p'\},p')\} \\
&= \dot{\bigcup_{(p',x')\in I_p}} \left\{\left(\omega^{n+1}(T')-\lambda^n x'-x, \omega^{n+1}(T')-\lambda^n x'-y\right) \mid T' \in U(\{p'\},p')\right\} \\
&= \dot{\bigcup_{(p',x')\in I_p}} E_{p'}^{n+1}\left(\lambda^n x'+x, \lambda^n x'+y\right).
\end{aligned}
$$

Thus $E_p^n(x,y)$ can be written as a disjoint union of compact open sets in $\mathcal{R}_{n+1}$, so $\mathcal{R}_n \subset \mathcal{R}_{n+1}$. ∎

Now we can let

$$\mathcal{R}_{AF} := \bigcup_{n\in\mathbb{N}} \mathcal{R}_n.$$

This is an increasing union of compact open subgroupoids of $\mathcal{R}_{\mathrm{punc}}$, with unit space equal to Ω_{punc}, which we know is homeomorphic to a Cantor set. Hence $\mathcal{R}_{AF}$ is an AF Cantor Groupoid in the sense of Definition 3.2.8. The following result originates in [52] and is in [47], Theorem 7.1.

Theorem 3.4.6 *The groupoid $\mathcal{R}_{\mathrm{punc}}$ is an almost AF Cantor Groupoid, with $\mathcal{R}_{AF}$ being the sub-AF groupoid.*

Chapter 4

C*-algebras of a Tiling

4.1 C*-algebras

In this section we recall terms often used in C*-algebra theory. For a good basic reference on C*-algebras, see [17].

A **C*-algebra** is a Banach algebra A with an involution $*$ whose norm satisfies the *C*-condition*: for all $a \in A$ we have

$$\|a^*a\| = \|a\|^2 .$$

A C*-algebra A need not have an identity, but if it does we call it **unital** and denote the identity 1_A. A **projection** is an element p for which $p^* = p = p^2$. Two projections p and q are called **Murray-von Neumann equivalent** (or simply **equivalent**) if there is an element $v \in A$ such that $p = vv^*$ and $q = v^*v$. A projection q is called a **subprojection** of p if $qp = pq = q$, and we write $q \leq p$. If q is equivalent to a subprojection of p then we write $q \precsim p$. A projection in a C*-algebra A is called **finite** if it is not equivalent to a proper subprojection of itself. A unital C*-algebra A is called finite if its identity is finite. If A is a C*-algebra, then the $n \times n$ matrices with entries in A, denoted $\mathbb{M}_n(A)$, is also a C*-algebra with obvious product, involution, and operator norm. We call A **stably finite** if $\mathbb{M}_n(A)$ is finite for all n.

An element $a \in A$ is called **normal** if a commutes with a^*, a is called **self-adjoint** if $a^* = a$ and a is called **unitary** if $a^*a = aa^* = 1_A$. The **spectrum** of an element a of a unital C*-algebra A is the set of all complex numbers z such that $a - z1_A$ is not invertible – if A is not unital the spectrum of $a \in A$ is the set of all complex numbers z such that $a - z1_{\tilde{A}}$ is not invertible in the unitization $\tilde{A}$ of A. An **ideal** in A is a closed sub-C*-algebra $I \subset A$ such that ab and ba are in I whenever $a \in A, b \in I$. We call A **simple** if the only closed ideals of A are $\{0\}$ and A.

The C*-algebra A is said to have **real rank zero** if self-adjoint elements with finite spectrum are dense in the set of all self-adjoint elements of A, and A is said to have **stable rank one** if invertible elements are dense in A.

A **trace** on a C*-algebra A is a positive linear functional $\tau : A \to \mathbb{C}$ such that $\tau(ab) = \tau(ba)$ for all $a, b \in A$. A **tracial state** or **normalized trace** is a trace τ such that $\tau(1_A) = 1$. We denote the set of tracial states on A by $T(A)$. If τ is a tracial state, then it induces a unique tracial state on the C*-algebra $\mathbb{M}_n(A)$ by summing the value of τ along the diagonal and dividing by n, we also denote this tracial state τ. We say that the **order on projections over** A **is determined by traces** if for every $n \in \mathbb{N}$ and projections $p, q \in \mathbb{M}_n(A)$ such that $\tau(p) < \tau(q)$ for every $\tau \in T(A)$ we have $p \precsim q$.

A commutative and unital C*-algebra can be canonically identified with $C(X)$, the continuous complex-valued functions on some compact Hausdorff space X. For us, the primary space of interest is the tiling space Ω.

4.2 The Crossed Product

If A is a C*-algebra, we let $\mathrm{Aut}(A)$ denote the group of $*$-automorphisms on A, equipped with the topology of pointwise norm convergence. This topology gives $\mathrm{Aut}(A)$ the structure of a topological group.

Definition 4.2.1 *Let A be a C^*-algebra, let G be a locally compact group and suppose we have a continuous homomorphism $\alpha : G \to \mathrm{Aut}(A)$. Then the triple (A, G, α) is called a* **C*-dynamical system**.

In this thesis, we mainly deal with 2-dimensional tilings, and the groups that will be acting will all be subgroups of the Euclidean group $E(2)$. In fact, the groups we deal with will either be $\mathbb{R}^2$ or the semidirect product of $\mathbb{R}^2$ by a finite subgroup of $O(2)$. In the following definitions we assume that the group in question is one of these groups, so the definitions simplify somewhat. They of course may be defined in a more general setting (see [73] for a thorough treatment).

Definition 4.2.2 *Let (A, G, α) be a C^*-dynamical system. A* **covariant representation** *of (A, G, α) is a pair (π, U) where $\pi : A \to B(\mathcal{H})$ is a representation of A on a Hilbert space $\mathcal{H}$ and $u : G \to U(\mathcal{H})$ is a unitary representation of G on the same Hilbert space such that*

$$\pi(\alpha_g(a)) = u_g^* \pi(a) u_g.$$

The existence of such pairs may be in doubt, so suppose that $\rho : A \to B(\mathcal{H}_\rho)$ is a representation of A on $\mathcal{H}_\rho$. Then we may define, using the notation of [73], $\mathrm{Ind}_e^G \rho$ to be a pair $(\tilde{\rho}, u)$ of representations on the Hilbert space $L^2(G, \mathcal{H}_\rho) \cong L^2(G) \otimes \mathcal{H}_\rho$ given by

$$\tilde{\rho}(a)h(r) := \rho(\alpha_r^{-1}(a))(h(r)) \qquad u_s h(r) := h(s^{-1}r), \tag{4.2.1}$$

for $h \in L^2(G, \mathcal{H}_\rho)$ and $s, r \in G$. One checks that $\mathrm{Ind}_e^G \rho = (\tilde{\rho}, u)$ is a covariant representation of (A, G, α). Representations of this form are called **regular representations**.

Let (A, G, α) be a C*-dynamical system and consider $C_c(G, A)$, the space of continuous functions with compact support from G to A. We define a product on this space by using Haar measure:

$$f \star g(s) := \int_G f(r) \alpha_r \left(g(r^{-1}s) \right) d\mu(r).$$

We also define an involution by

$$f^*(s) = (f(s^{-1}))^*.$$

These make $C_c(G, A)$ into a $*$-algebra. That $f \star g \in C_c(G, A)$ and $f \star (g \star h) = (f \star g) \star h$ are given in [73], page 48. We have that

$$\|f\|_1 := \int_G \|f(s)\| d\mu(s)$$

is a norm on $C_c(G, A)$ such that $\|f^*\|_1 = \|f\|_1$ and $\|f \star g\|_1 \leq \|f\|_1 \|g\|_1$ for all $f, g \in C_c(G, A)$. If (π, u) is a covariant representation of (A, G, α), we define

$$\pi \rtimes u(f) := \int_G \pi(f(s)) u_s d\mu(s)$$

for $f \in C_c(G, A)$. Thus defined, $\pi \rtimes u$ is a $*$-representation of $C_c(G, A)$, and

$$\|\pi \rtimes u(f)\| \leq \|f\|_1$$

for all $f \in C_c(G, A)$. The representation $\pi \rtimes u$ is called the **integrated form** of the covariant representation of (π, u). We now define

$$\|f\| := \sup\{\|\pi \rtimes u(f)\| \mid (\pi, u) \text{ is a covariant representation of } (A, G, \alpha)\}.$$

One can show (for example [73], Lemma 2.27) that this is a norm on $C_c(G, A)$ which is dominated by $\|\cdot\|_1$ and satisfies the C*-condition. We call it the **universal norm**.

Definition 4.2.3 *Let (A, G, α) be a C*-dynamical system. Then the completion of $C_c(G, A)$ in the universal norm is called the* **crossed product** *of A by G and is denoted $A \rtimes_\alpha G$.*

When the action is implicit we often write $A \rtimes G$. Since the universal norm can be difficult to work with, one can also define the **reduced norm** on $C_c(G, A)$ as follows: let $\mathrm{Ind}_e^G \rho$ be any regular representation of (A, G, α) with ρ faithful. We define

$$\|f\|_{\mathrm{red}} := \|\mathrm{Ind}_e^G \rho(f)\|$$

where $\mathrm{Ind}_e^G\rho(f)$ is understood to be the integrated form of $\mathrm{Ind}_e^G\rho$ on f. One shows that this is independent of the faithful representation chosen. We denote the completion of $C_c(G,A)$ in the reduced norm as $A \rtimes_{\alpha,r} G$ and call this the **reduced crossed product**. The reduced norm is always dominated by the universal norm, and when G is amenable they coincide. We will only be concerned with cases where G is amenable, so we may work with the reduced norm. For a detailed discussion, see [73].

In some special cases the crossed product is easier to describe.

Example 4.2.4 If G is a finite group, then we can realize $A \rtimes_\alpha G$ as the set of all formal sums

$$\sum_{g\in G} a_g\delta_g \qquad a_g \in A$$

where multiplication is determined by the rule $(a\delta_g)(b\delta_h) = a\alpha_g(b)\delta_{gh}$ and involution is $(a\delta_g)^* = \alpha_{g^{-1}}(a^*)\delta_{g^{-1}}$. If $f \in C(G,A)$, we see it as the sum $\sum f(g)\delta_g$. In this case, there is no need to complete. We claim that the reduced norm satisfies

$$\max_{g\in G}\{\|a_g\|_A\} \leq \left\|\sum_{g\in G} a_g\delta_g\right\|_{\mathrm{red}} \leq \sum_{g\in G}\|a_g\|_A. \tag{4.2.2}$$

The right hand term is clearly the $\|\cdot\|_1$ norm, so we check the left inequality. Let ρ be a faithful representation of A on a Hilbert space $\mathcal{H}_\rho$ and consider the covariant pair $\mathrm{Ind}_e^G\rho = (\tilde{\rho}, u)$ from Equation (4.2.1). Let ξ be a norm 1 vector in $\mathcal{H}_\rho$, let $t \in G$ and let $\zeta_t \in L^2(G, \mathcal{H}_\rho)$ be defined by

$$\zeta_t(s) = \begin{cases} \xi & \text{if } s = t \\ 0 & \text{otherwise.} \end{cases}$$

Then ζ_t is a norm 1 vector in $L^2(G,\mathcal{H}_\rho)$. For $f \in C(G,A)$ and $s \in G$ we calculate

$$\begin{aligned} \left(\mathrm{Ind}_e^G\rho(f)\right)\zeta_t(s) &= \left(\sum_{g\in G}\tilde{\rho}(f(g))u_g\right)\zeta_t(s) \\ &= \sum_{g\in G}\tilde{\rho}(f(g))\zeta_t(g^{-1}s) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{g \in G} \rho(\alpha_{g^{-1}s}(f(g)))\zeta_t(g^{-1}s) \\
&= \rho(\alpha_{t^{-1}}(f(st^{-1})))\xi.
\end{aligned}
$$

The last equality is due to the fact that the only nonzero term in the sum occurs when $g^{-1}s = t$, and so $g = st^{-1}$. We have that

$$\| \left(\mathrm{Ind}_e^G \rho(f)\right) \zeta_t\|_2^2 = \sum_{s \in G} \|\rho(\alpha_{t^{-1}}(f(st^{-1})))\xi\|_2^2.$$

Since ζ_t is norm 1 and ρ is faithful we may conclude that

$$\sum_{s \in G} \|\rho(\alpha_{t^{-1}}(f(st^{-1})))\xi\|_2^2 \leq \|f\|_{\mathrm{red}}^2$$

Now taking the supremum over all norm 1 vectors ξ and noticing that ρ and $\alpha_{t^{-1}}$ are isometric allows us to conclude that

$$\sum_{s \in G} \|f(st^{-1})\|_A^2 \leq \|f\|_{\mathrm{red}}^2.$$

Hence we must have that $\|f(g)\|_A^2 \leq \|f\|_{\mathrm{red}}^2$ for all $g \in G$, and so

$$\max_{g \in G}\{\|f(g)\|_A\} \leq \|f\|_{\mathrm{red}}.$$

Example 4.2.5 When A is commutative and unital, i.e. when $A = C(X)$ for some compact Hausdorff space X, one can show (see for example the discussion in [73] on page 53) that the inclusion $C_c(G \times X) \subset C_c(G, C(X))$ is dense in the norm above. Hence when working with crossed products of the form $C(X) \rtimes_\alpha G$ we can work with elements from $C_c(G \times X)$. The formulae for the product and involution on $C_c(G \times X)$ become

$$
\begin{aligned}
f \star g(s,x) &= \int_G f(r,x)g(r^{-1}s, r^{-1}x)d\mu(r) \\
f^*(s,x) &= \overline{f(s^{-1}, s^{-1}x)}.
\end{aligned}
$$

Example 4.2.6 Suppose that G and H are locally compact groups and that $\varphi : H \to \mathrm{Aut}(G)$ is a homomorphism such that the map $H \times G$ to G given by $(h, g) \mapsto \varphi_h(g)$

is continuous. Then we recall that the **semidirect product** $G \rtimes_\varphi H$ is the set $G \times H$ which becomes a group under the operations

$$(g_1, h_1)(g_2, h_2) = (g_1 \varphi_{h_1}(g_2), h_1 h_2), \qquad (g, h)^{-1} = (\varphi_{h^{-1}}(g^{-1}), h^{-1}).$$

When given the product topology from $G \times H$, the semidirect product $G \rtimes_\varphi H$ is locally compact. One also sees that G and H are subgroups of $G \rtimes_\varphi H$. We will need the following proposition to study crossed products associated with tilings

Proposition 4.2.7 *Let A be a C^*-algebra, let G be a finite subgroup of $O(2)$ and suppose that $(A, \mathbb{R}^2 \rtimes_\varphi G, \alpha)$ is a C^*-dynamical system. Then there exists an action*

$$\beta : G \to \mathrm{Aut}(A \rtimes_{\alpha|_{\mathbb{R}^2}} \mathbb{R}^2)$$

such that for all $f \in C_c(\mathbb{R}^2, A)$ we have

$$\beta_g(f)(x) = \alpha_g\left(f(\varphi_g(x))\right).$$

In this case, the natural map

$$\iota : C(G, C_c(\mathbb{R}^2, A)) \to C_c(\mathbb{R}^2 \rtimes_\varphi G, A)$$

extends to an isomorphism from $(A \rtimes_{\alpha|_{\mathbb{R}^2}} \mathbb{R}^2) \rtimes_\beta G$ to $A \rtimes_\alpha (\mathbb{R}^2 \rtimes_\varphi G)$

Proof: This is a special case of [73], Proposition 3.11 (for example). ∎

Let $\mathcal{U}(A)$ denote the group of unitary elements of A. If $\alpha : G \to \mathrm{Aut}(A)$ factors through $\mathcal{U}(A) \to \mathrm{Aut}(A)$ (via $u \mapsto \mathrm{Ad}u$), we say that α is an **inner action**. In this case, we have the following fact.

Lemma 4.2.8 ([73], Lemma 2.73) *Let A be a C^*-algebra and let $\alpha : G \to \mathrm{Aut}(A)$ be an inner action of G on A Then*

$$A \rtimes_\alpha G \cong A \otimes_{\max} C^*(G).$$

4.3 The C*-algebra of a Groupoid

In this section we present the C*-algebra of a locally compact Hausdorff groupoid as defined by Renault in [59]. Muhly's book [42] provides a more direct path to the definition of this C*-algebra, so we follow its development. The groupoids we work with directly in this thesis are mostly r-discrete, so we also state what the definitions reduce to in this case.

Let $\mathscr{G}$ be a locally compact Hausdorff groupoid with Haar system $\{\mu^x\}_{x\in\mathscr{G}^{(0)}}$. Then we may define a product and involution on $C_c(\mathscr{G})$ by integration with respect to the Haar system: if $f_1, f_2 \in C_c(\mathscr{G})$ and $\gamma \in \mathscr{G}$ define

$$f_1 \star f_2(\gamma) = \int f_1(\eta) f_2(\eta^{-1}\gamma)\, d\mu^{r(\gamma)}(\eta) = \int f_1(\gamma\eta) f_2(\eta^{-1})\, d\mu^{s(\gamma)}(\eta),$$

$$f_1^*(\gamma) = \overline{f_1(\gamma^{-1})}.$$

With these operations, and when given the inductive limit topology, $C_c(\mathscr{G})$ becomes a topological $*$-algebra. Recall that the inductive limit topology on $C_c(\mathscr{G})$ is given by saying that a sequence $\{f_n\}$ converges to f in $C_c(\mathscr{G})$ if and only if there exists a compact subset $K \subset \mathscr{G}$ such that the support of f is contained in K, the support of $\{f_n\}$ is eventually in K, and $\{f_n\}$ converges uniformly to f on K. In particular, this is stronger than uniform convergence. A **representation** of $C_c(\mathscr{G})$ is a continuous $*$-homomorphism from $C_c(\mathscr{G})$ with the inductive limit topology into $B(\mathcal{H})$ with the weak operator topology. For $f \in C_c(\mathscr{G})$ define

$$\|f\|_r = \sup_{x\in\mathscr{G}^{(0)}} \left\{ \int |f(\gamma)|\, d\mu^x(\gamma) \right\},$$

$$\|f\|_s = \sup_{x\in\mathscr{G}^{(0)}} \left\{ \int |f(\gamma^{-1})|\, d\mu^x(\gamma) \right\},$$

$$\|f\|_I = \max\{\|f\|_r, \|f\|_s\}.$$

The norm $\|f\|_I$ is a $*$-algebra norm on $C_c(\mathscr{G})$, and the topology it determines is coarser than the inductive limit topology, see [59] Proposition II.1.4. If π is a representation of $C_c(\mathscr{G})$ then $\|\pi(f)\| \leq \|f\|_I$ for any $f \in C_c(\mathscr{G})$. Thus one can show that

$$\|f\| := \sup\{\|\pi(f)\| \mid \pi \text{ is a representation of } C_c(\mathscr{G})\}$$

defines a norm on $C_c(\mathscr{G})$ which satisfies the C*-condition. Hence the completion of $C_c(\mathscr{G})$ in this norm is a C*-algebra, called **the C*-algebra of** $\mathscr{G}$ and denoted $C^*(\mathscr{G})$.

As in the case of the crossed product, there is another norm defined in terms of some explicit representations. Given a Haar system $\{\mu^u\}_{u\in\mathscr{G}^{(0)}}$ with each μ^u supported on $\mathscr{G}^u$ we can define measures μ_u supported on $\mathscr{G}_u$ by composing with the groupoid inverse map: if U is a Borel set then $\mu_u(U) = \mu^u(U^{-1})$. For $f \in C_c(\mathscr{G})$, $x \in \mathscr{G}^{(0)}$ and $\xi \in L^2(\mathscr{G}_x, \mu_x)$ define

$$\lambda_x(f)\xi(\gamma) = \int_{G_x} f(\gamma\eta^{-1})\xi(\eta)\, d\mu_x(\eta).$$

Then each λ_x is a representation of $C_c(\mathscr{G})$. The formula

$$\|f\|_{\text{red}} := \sup_x\{\|\lambda_x(f)\|\}$$

defines a norm on $C_c(\mathscr{G})$ which satisfies the C*-condition. Hence the completion of $C_c(\mathscr{G})$ in this norm is a C*-algebra, called **the reduced C*-algebra of** $\mathscr{G}$ and denoted $C_r^*(\mathscr{G})$. We mention that it is possible to define the C*-algebra of $\mathscr{G}$ when $\mathscr{G}$ is not necessarily Hausdorff, but in this case it is still required that $\mathscr{G}^{(0)}$ be Hausdorff.

In this thesis we deal with two types of groupoids - transformation group groupoids and r-discrete groupoids. As discussed in [42], [59], and elsewhere, the (reduced) C*-algebra of a transformation group groupoid coincides with the (reduced) crossed product discussed last section.

Theorem 4.3.1 *Let X be a locally compact Hausdorff space and let G be a locally compact group acting on X. Let $\mathscr{G} = (X, G)$ be the transformation group groupoid of this action. Then*

$$C^*(\mathscr{G}) \cong C(X) \rtimes G, \text{ and}$$

$$C_r^*(\mathscr{G}) \cong C(X) \rtimes_r G.$$

When $\mathscr{G}$ is r-discrete, its C*-algebra is easier to describe than in the general case. Recall from Section 3.2 that if $\mathscr{G}$ is a locally compact Hausdorff r-discrete groupoid, then if it has a Haar system it essentially consists of counting measures, and it has a Haar system if and only if r and s are local homeomorphisms. Suppose that this is the case. For $f, g \in C_c(\mathscr{G})$ and $\gamma \in \mathscr{G}$, the product $f \star g$ is then given by

$$f \star g(\gamma) := \sum_{\substack{\eta \in \mathscr{G} \\ r(\eta)=s(\gamma)}} f(\gamma\eta)g(\eta^{-1})$$

The following is a special case of the construction of the reduced C*-algebra when $\mathscr{G}$ is r-discrete, and is presented as the definition of the reduced C*-algebra by Phillips [47].

Proposition 4.3.2 ([47], Definition 1.6) *Let $\mathscr{G}$ be a locally compact Hausdorff r-discrete groupoid with counting measures as the Haar system. Let $\mathscr{G}_x = \{\gamma \in \mathscr{G} \mid s(\gamma) = x\}$ and let $C_c(\mathscr{G})$ act on the Hilbert space $l^2(\mathscr{G}_x)$ by*

$$\lambda_x(f)\xi(\gamma) = \sum_{\substack{\eta \in \mathscr{G} \\ s(\eta)=x}} f(\gamma\eta^{-1})\xi(\eta).$$

Then we have

$$\|f\|_{\mathrm{red}} := \sup_x \{\|\lambda_x(f)\|\}.$$

When $\mathscr{G}$ is r-discrete, the formulas for $\|\cdot\|_r$, $\|\cdot\|_s$, and $\|\cdot\|_I$ become

$$\|f\|_r = \sup_{u \in \mathscr{G}^{(0)}} \left\{ \sum_{r(\gamma)=u} |f(\gamma)| \right\}, \tag{4.3.1}$$

$$\|f\|_s = \sup_{u \in \mathscr{G}^{(0)}} \left\{ \sum_{s(\gamma)=u} |f(\gamma)| \right\}, \tag{4.3.2}$$

$$\|f\|_I = \max\{\|f\|_r, \|f\|_s\}. \tag{4.3.3}$$

If $f \in C_c(\mathscr{G})$, then

$$\|f\|_\infty \leq \|f\|_{\mathrm{red}} \leq \|f\|_I,$$

for details see [59], Proposition II.4.2.

Recall from Definition 3.2.7 that a locally compact Hausdorff r-discrete groupoid with counting measures as Haar system is a Cantor groupoid if its unit space is homeomorphic to a Cantor set. If $\mathscr{G}$ is such a groupoid, then Definition 3.2.10 states that $\mathscr{G}$ is an almost AF Cantor groupoid if:

1. There exists an open AF subgroupoid $\mathscr{G}_0 \subset \mathscr{G}$ which contains the unit space such that whenever K is a compact subset of $\mathscr{G} \setminus \mathscr{G}_0$, we have that $s(K)$ is thin in the sense of Definition 3.2.9.

2. For every closed invariant subset $E \subset \mathscr{G}^{(0)}$, and every nonempty relatively open subset $U \subset E$, there is a $\mathscr{G}$-invariant Borel probability measure μ on $\mathscr{G}^{(0)}$ such that $\mu(U) > 0$.

In defining these terms in [47], Phillips notes the following.

Proposition 4.3.3 ([47], Proposition 2.13) *Let $\mathscr{G}$ be a Cantor groupoid. Then $\mathscr{G}$ is an almost AF Cantor groupoid if it satisfies Condition 1 of Definition 3.2.10 and either $C_r^*(\mathscr{G})$ or $C_r^*(\mathscr{G}_0)$ is simple.*

This condition can be checked by using the following.

Proposition 4.3.4 ([59], Proposition II.4.6) *Let $\mathscr{G}$ be an r-discrete locally compact Hausdorff essentially principal groupoid. If the only open invariant subsets of $\mathscr{G}^{(0)}$ are $\mathscr{G}^{(0)}$ and the empty set, then $C_r^*(\mathscr{G})$ is simple.*

4.4 Strong Morita Equivalence

In this section, we describe a well-known notion of equivalence for C*-algebras, that of strong Morita equivalence. Many of the definitions in this section are from the development in [42]. Another excellent reference is [55].

Definition 4.4.1 *Let A be a C^*-algebra and let X be a right module over A. We call X a* **right Hilbert module over** *A if there is a sesquilinear map $\langle \cdot, \cdot \rangle : \mathsf{X} \times \mathsf{X} \to A$, which is conjugate linear in the first variable, such that*

(1) $\langle x, ya \rangle = \langle x, y \rangle a$ *for all $x, y \in \mathsf{X}$ and $a \in A$.*

(2) $\langle x, y \rangle^* = \langle y, x \rangle$ *for all $x, y \in \mathsf{X}$.*

(3) $\langle x, x \rangle \geq 0$ *in A and $\langle x, x \rangle = 0$ only when $x = 0$.*

A **left Hilbert module over** *A is defined similarly, except that $\langle \cdot, \cdot \rangle$ is conjugate linear in the second variable and (1) above is replaced with*

(1a) $\langle ax, y \rangle = a \langle x, y \rangle$ *for all $x, y \in \mathsf{X}$ and $a \in A$.*

If X is a right Hilbert module over A we can define a norm on X via

$$\|x\| = \|\langle x, x \rangle\|_A^{1/2}. \tag{4.4.1}$$

We call this the norm on X coming from A.

Definition 4.4.2 *Let A and B be C^*-algebras. Then we say X is an (A, B)***-equivalence bimodule** *(or an (A, B)***-imprimitivity bimodule***) if the following conditions are satisfied:*

1. *X has A- and B-valued sesquilinear maps making X a left Hilbert module over A and a right Hilbert module over B.*

2. *The sesquilinear maps satisfy the relation*

$$_A\langle x, y \rangle z = x \langle y, z \rangle_B$$

   for all $x, y, z \in \mathsf{X}$.

3. *The following inequalities are satisfied for $a \in A$, $b \in B$, $x \in \mathsf{X}$:*

$$\langle ax, ax \rangle_B \leq \|a\|^2 \langle x, x \rangle_B$$

$$_A\langle xb, xb \rangle \leq \|b\|_A^2 \langle x, x \rangle$$

4. *The linear span of elements of the form $_A\langle x, y \rangle$, for $x, y \in \mathsf{X}$, is dense in A and elements of the form $\langle x, y \rangle_B$ are dense in B (it is sometimes said that X has* **full support** *in both A and B).*

The C-algebras A and B are said to be* **strongly Morita equivalent** *if there exists an (A, B)-equivalence bimodule. In this case, we write $A \sim_m B$.*

Strong Morita equivalence is a weaker notion of equivalence than isomorphism – if two C*-algebras are isomorphic then they are strongly Morita equivalent, see for example [55], Example 3.14. For separable C*-algebras, there is an equivalent formulation of strong Morita equivalence that will be useful.

Definition 4.4.3 *Let $\mathcal{K}$ denote the C*-algebra of compact operators on a separable infinite dimensional Hilbert space. We say that two C*-algebras A and B are* **stably isomorphic** *if $A \otimes \mathcal{K}$ is isomorphic to $B \otimes \mathcal{K}$.*

Theorem 4.4.4 (**Brown-Green-Rieffel**, see for example [55], Theorem 5.55) *Let A and B be two separable C*-algebras. Then A and B are strongly Morita equivalent if and only if they are stably isomorphic.*

The notion of strong Morita equivalence given in Definition 4.4.1 resembles that of Definition 3.3.5, and this similarity is no accident. Suppose, as in Definition 3.3.5 that we have locally compact Hausdorff groupoids $\mathscr{G}$ and $\mathscr{H}$ with Haar systems $\{\mu^u\}$ and $\{\beta^u\}$ and that the space X is a $(\mathscr{G}, \mathscr{H})$-equivalence. As in [42], we let $A_c = C_c(\mathscr{G})$ and $B_c = C_c(\mathscr{H})$ and define a A_c-B_c-bimodule structure on $C_c(X)$. Take $a \in A_c, b \in B_c$

and $\varphi \in C_c(X)$. Then we define

$$a \star \varphi(x) := \int_{\mathscr{G}} a(\gamma)\varphi(\gamma^{-1}x)d\mu^{r(x)}(\gamma) \tag{4.4.2}$$

$$\varphi \star b(x) := \int_{\mathscr{H}} \varphi(x\eta)b(\eta^{-1})d\beta^{s(x)}(\eta). \tag{4.4.3}$$

It can be shown that these do define elements in $C_c(X)$ and that these operations make $C_c(X)$ into a A_c-B_c-bimodule. We can also define $C_c(X)$-valued inner products on A_c and B_c:

$$_A\langle\varphi, \psi\rangle(\gamma) = \int_{\mathscr{H}} \varphi(\gamma x\eta)\overline{\psi(x\eta)}d\beta^{s(x)}(\eta) \tag{4.4.4}$$

$$\langle\varphi, \psi\rangle_B(\eta) = \int_{\mathscr{G}} \overline{\varphi(\gamma^{-1}x)}\psi(\gamma^{-1}x\eta)d\mu^{r(x)}(\gamma). \tag{4.4.5}$$

It is important to note that these are module structures in the traditional sense, i.e., they are not necessarily Hilbert bimodules. However, we may still define norms as in (4.4.1).

The following theorem is from [43] and restated in [42].

Theorem 4.4.5 *If $\mathscr{G}$ and $\mathscr{H}$ are second countable locally compact Hausdorff groupoids and if X is a $(\mathscr{G}, \mathscr{H})$-equivalence, then the completion of $C_c(X)$ with respect to the norm coming from either A_c or B_c is an equivalence bimodule between $C^*(\mathscr{G})$ and $C^*(\mathscr{H})$ with respect to the operations given in* (4.4.2)-(4.4.5)

The authors in [65] prove a similar version of Theorem 4.4.5 for reduced groupoid algebras.

Theorem 4.4.6 ([65], Theorem 13) *If $\mathscr{G}$ and $\mathscr{H}$ are second countable locally compact Hausdorff groupoids which are equivalent in the sense of Definition 3.3.5, then $C_r^*(\mathscr{G})$ and $C_r^*(\mathscr{H})$ are strongly Morita equivalent.*

4.5 AF Algebras

In this section we briefly present a well-studied class of C*-algebras, the AF algebras.

Recall that a finite dimensional C*-algebra A is isomorphic to a direct sum of full matrix algebras, i.e.

$$A = \bigoplus_{i=1}^{k} \mathbb{M}_{n_i}(\mathbb{C}).$$

In particular, a finite dimensional C*-algebra is unital. If

$$B = \bigoplus_{i=1}^{l} \mathbb{M}_{m_i}(\mathbb{C})$$

is another finite dimensional algebra, and $\varphi : A \to B$ is a unital $*$-homomorphism, then φ is determined up to unitary equivalence in B by an $l \times k$ matrix M of non-negative integers such that

$$M \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_k \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_l \end{bmatrix}.$$

The matrix M is called the matrix of **partial multiplicities**. If $M = [M_{ij}]$, then the integer M_{ij} is the multiplicity of the embedding of the summand $\mathbb{M}_{n_j}(\mathbb{C})$ of A into the summand $\mathbb{M}_{m_i}(\mathbb{C})$ of B. For details see [17] Lemma III.2.1.

One way of obtaining the matrix of partial multiplicities is through traces. If τ is a trace on $\mathbb{M}_n(\mathbb{C})$, then it is a positive scalar multiple of the usual matrix trace Tr (this is the sum of the diagonal entries). If A is a finite dimensional algebra written as before,

$$A = \bigoplus_{i=1}^{k} \mathbb{M}_{n_i}(\mathbb{C}).$$

then for each j,

$$\tau_j^A \left((a_i)_{i=1}^k \right) = \mathrm{Tr}(a_j)$$

is a trace on A. Furthermore, every trace on A can be written as a positive linear combination of the τ_j^A since restricting to a summand yields a trace on that summand. Let

$$B = \bigoplus_{i=1}^{l} \mathbb{M}_{m_i}(\mathbb{C})$$

and suppose that $\varphi : A \to B$ is a unital injective homomorphism of C*-algebras. Then for each i between 1 and l, $\tau_i^B \circ \varphi$ is a trace on A. Furthermore, if we denote by q_i the identity on the ith summand in A, $\tau_i^B \circ \varphi(q_s)$ should be the trace of q_s multiplied by the multiplicity of the embedding of the summand $\mathbb{M}_{n_s}(\mathbb{C})$ of A into the summand $\mathbb{M}_{m_i}(\mathbb{C})$ of B. On the other hand, we know that

$$\tau_i^B \circ \varphi = \sum_{j=1}^{k} M_{ij} \tau_j^A \tag{4.5.1}$$

for some positive scalars M_{ij}. Hence,

$$\tau_i^B \circ \varphi(q_s) = \sum_{j=1}^{k} M_{ij} \tau_j^A(q_s) = M_{is} \tau_s^A(q_s) = M_{is} n_s,$$

and so $M = [M_{ij}]$ is the matrix of partial multiplicities of the inclusion. A formula for its entries is given by manipulating the above,

$$M_{ij} = \frac{\tau_i^B \circ \varphi(q_j)}{\tau_j^A(q_j)} \tag{4.5.2}$$

A C*-algebra A is called **approximately finite dimensional** or **AF** if it is the closure of an increasing union of finite dimensional subalgebras A_n. When A is unital, it is required that the A_0 consist only of the scalar multiples of the identity of A. Thus in the unital case, each A_n contains the identity. Given an AF algebra $A = \overline{\cup A_n}$, the inclusion of A_n in A_{n+1} is determined up to unitary equivalence in A_{n+1} by the matrix of partial multiplicities. We may describe this series of inclusions by what is known as a **Bratteli diagram**.

Definition 4.5.1 *A* **Bratteli diagram** *is an infinite directed graph* (E, V)*, where* E *is the set of edges and* V *is the set of vertices, with the following properties:*

1. *The vertex set is a disjoint union finite subsets $V_n \subset V$ for $n \geq 0$,*

2. *the set V_0 consists of one vertex v_0, called the root,*

3. *if $e \in E$ then there exists $n \geq 0$ such that $i(e) \in V_n$ and $t(e) \in V_{n+1}$,*

4. *for $v \in V \setminus V_0$, there exist $e_1, e_2 \in E$ such that $t(e_1) = i(e_2) = v$.*

In the above, $i(e)$ and $t(e)$ denote the initial vertex and terminal vertex of the edge e respectively. We say a Bratteli diagram is **simple** *if for every $v \in V_n$ and $u \in V_{n+1}$ there exists $e \in E$ such that $i(e) = v$ and $t(e) = u$.*

A Bratteli diagram is built from an AF algebra $A = \overline{\cup A_n}$ as follows: the set V_n consists of one vertex for every full matrix summand in A_n. If $M(n)$ is the matrix of partial multiplicities for the inclusion $A_n \subset A_{n+1}$, then we draw $M(n)_{ij}$ edges from the jth vertex in V_n to the ith vertex in V_{n+1}. The requirement that A_0 consist of the scalar multiples of the identity implies that $A_0 \cong \mathbb{C}$, and so V_0 has one vertex as required.

Example 4.5.2 Let $A_n = \mathbb{M}_{2^n}(\mathbb{C})$, and let each inclusion $A_n \subset A_{n+1}$ be determined by

$$a \mapsto \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

Then the matrix of partial multiplicities of each inclusion is the 1×1 matrix $[2]$. The Bratteli diagram of this sequence of finite dimensional algebras is

$$\cdot \rightrightarrows \cdot \rightrightarrows \cdot \rightrightarrows \cdot \rightrightarrows \quad \cdots$$

This algebra is what is known as the **CAR algebra**, see [17], Example III.2.4.

We note the following.

Proposition 4.5.3 ([17], Proposition III.2.7) *Let $A = \overline{\cup A_n}$ and $B = \overline{\cup B_n}$ be two AF algebras. Then if A and B have the same Bratteli diagram, they are isomorphic.*

The act of **telescoping** also results in isomorphic AF algebras. If (V, E) is a Bratteli diagram, we may form another by deleting one of the vertex sets. Pick any $n \in \mathbb{N}$ and let

$$V' = \bigcup_{\substack{i \geq 0 \\ i \neq n}} V_i.$$

Our new edge set E' will consist of all the edges from E which did not have source or range in V_n. We create a new edge in E' for every pair of edges e_1, e_2 with $t(e_1) = i(e_2) \in V_n$. The result (E', V') will be a Bratteli diagram. The incidence matrix between V_{n-1} and V_{n+1} will simply be the product of $M(n)$ and $M(n+1)$. The diagrams (E, V) and (E', V') will have isomorphic AF algebras.

In Definition 3.2.8 we defined an AF Cantor groupoid, and as the terminology suggests there is a connection between this concept and AF algebras. Let (V, E) be a Bratteli diagram. Define

$$X = \{(x_i)_{i \in \mathbb{N}} \mid x_i \in E, s(x_1) = v_0, i(x_{i+1}) = t(x_i)\},$$

the set of all infinite paths in (V, E) which start at the root. If $x \in X$, we define

$$U(x, k) = \{(y_i)_{i \in \mathbb{N}} \mid y_i = x_i, 1 \leq i \leq k\}.$$

This is the set of all infinite paths which look like x up to the kth term. We endow X with the topology generated by sets of this form as x and k vary. If (V, E) is simple, then X with this topology is homeomorphic to the Cantor set. We let

$$\mathcal{R}_k = \{(x, y) \in X \times X \mid x_i = y_i \text{ for all } i \geq k\},$$

$$\mathcal{R} = \bigcup_{n \in \mathbb{N}} \mathcal{R}_n.$$

We see that $\mathcal{R}$ is an equivalence relation on X, and two sequences are equivalent if they are eventually equal. This relation is known as **tail equivalence**. We note that $\mathcal{R}_k \subset \mathcal{R}_{k+1}$ for all $k \in \mathbb{N}$ and that each $\mathcal{R}_k$ contains the diagonal.

Let $x, y \in X$ be such that $t(x_k) = t(y_k)$, i.e., they pass through the same vertex at stage k. Define

$$V(x, y, k) = \{(z, w) \in X \times X \mid z \in U(x, k), w \in U(y, k), z_i = w_i, i > k\}.$$

Then $V(x, y, k) \subset \mathcal{R}_k$. We give $\mathcal{R}$ the topology generated by the $V(x, y, k)$ as x, y, and k vary, keeping in mind it is only defined if $t(x_k) = t(y_k)$. In this topology, $\mathcal{R}_k$ is compact and open in $\mathcal{R}$ for all k. We have that $r(V(x, y, k)) = U(x, k)$, and restricted to this domain r is easily checked to be a homeomorphism. Hence, if (V, E) is simple (possibly after telescoping), the equivalence relation $\mathcal{R}$ is an AF Cantor groupoid. Furthermore, $C^*(\mathcal{R})$ is isomorphic to the AF algebra associated to (V, E). For details on the above construction, see [59], Section III.1.

Example 4.5.4 ([16], Chapter 2 Section 3 and Appendix D) As mentioned in the introduction, Connes associated an AF algebra to a space of Penrose tilings. This was done by constructing a Bratteli diagram (V, E) from the substitution. The diagram we present here is actually obtained from his by telescoping, since it is obtained from a slightly different Penrose substitution. For each $n \in \mathbb{N}$, V_n consists of two vertices,

$$V_n = \{v_{n,1}, v_{n,2}\},$$

one for the small Robinson triangle and one for the large one. Then there is an edge between $v_{n,i}$ and $v_{n+1,j}$ for each instance of tile i in tile j after substituting tile j. We also put an one edge each between the root and $v_{1,1}$ and $v_{1,2}$, see Figure 4.1 (this figure is taken from [35]). The incidence matrix at each stage is $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

There is a finite path of length k starting with $v_{1,i}$ and ending at $v_{k,j}$ for each possible way tile i appears in the substitution applied k times to tile j. Because the Penrose substitution forces its border, each infinite path determines a unique tiling of the plane by Robinson triangles up to orientation. Connes shows that the

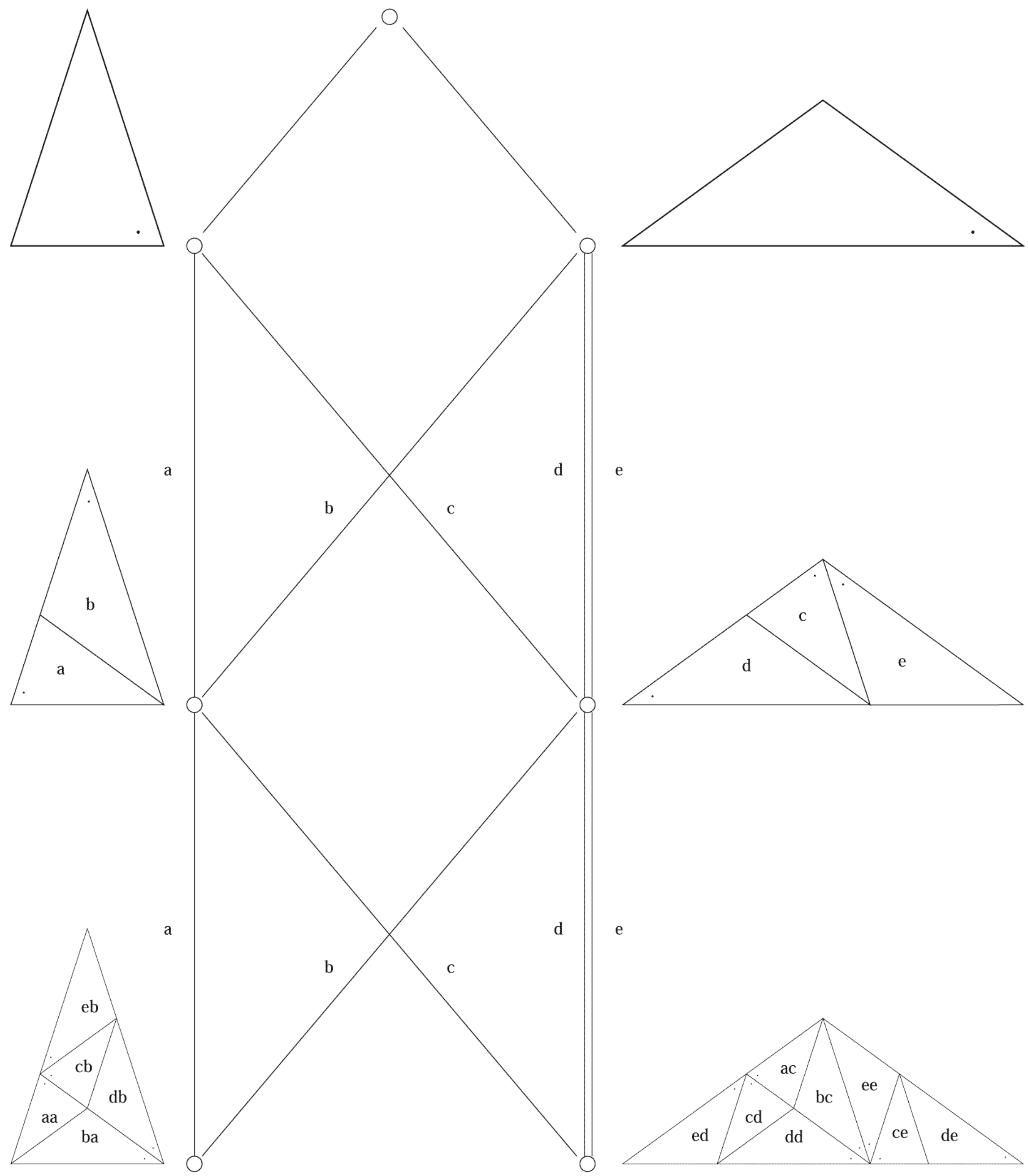


Figure 4.1: Connes' Bratteli diagram from Penrose tilings.

equivalence relation $\mathcal{R}$ on the space X of infinite paths corresponds to equivalence of Penrose tilings by any isometry of the plane.

Example 4.5.4 is an example of a special class of AF algebras which we will be important to us. Suppose that $\{A_i\}_{i=1}^{\infty}$ is an increasing sequence of finite dimensional C*-algebras along with unital inclusions $\phi_i : A_i \hookrightarrow A_{i+1}$ such that A_i has n summands at every stage and the matrix of partial multiplicities associated to ϕ_i is a constant matrix M. Suppose also that the matrix M is primitive, i.e. there exists a k such that the entries of M^k are all strictly positive. Then by [28], Theorem 4.1, the AF algebra $\overline{\cup A_i}$ is simple and has a unique tracial state.

4.6 C*-algebras Associated with Tilings

We have seen in Example 4.5.4 how Connes associated an AF algebra to the Penrose substitution. In further studies on C*-algebras associated to tilings, Kellendonk [35] considers translational equivalence and obtains an algebra that is not AF. In Chapter 5 we link the these two constructions. In this section, we describe the C*-algebras studied by Kellendonk (and later Putnam) in arbitrary dimension. Once again we assume a primitive substitution tiling system $(\mathcal{P}, \omega)$ which satisfies the conditions of Remark 2.5.8.

The first is simply the crossed product $C(\Omega) \rtimes \mathbb{R}^d$ where $\mathbb{R}^d$ acts on Ω by translation. The action of $\mathbb{R}^d$ on Ω is minimal so the crossed product is simple ([73], Corollary 8.22, for example). The group $\mathbb{R}^d$ is amenable so $C(\Omega) \rtimes \mathbb{R}^d \cong C(\Omega) \rtimes_r \mathbb{R}^d$.

The second is the C*-algebra of the r-discrete principal groupoid $\mathcal{R}_{\text{punc}}$. Kellendonk and Putnam denote this algebra as $A_T = C_r^*(\mathcal{R}_{\text{punc}})$. The groupoid $\mathcal{R}_{\text{punc}}$ is an almost AF Cantor groupoid with AF subgroupoid $\mathcal{R}_{AF}$, and so the results of [47] apply. In particular, $C_r^*(\mathcal{R}_{\text{punc}})$ has real rank zero, stable rank one, and the order on its projections is determined by traces. These properties are of importance to questions we will consider in the next chapter, such as whether this algebra or its crossed products are classifiable by their K-theory.

We also have the following lemma which follows directly from the definition of the convolution product. We will omit the convolution product $\star$ from now on for notational convenience.

Lemma 4.6.1 *Let K be a compact open subset of Ω_{punc}. Then the characteristic function on K, $p = \chi_K$ is a projection in $C_c(\mathcal{R}_{\text{punc}})$. Furthermore, if $f \in C_c(\mathcal{R}_{\text{punc}})$, then*

$$(fp)(T,T') = \begin{cases} f(T,T') & T' \in K \\ 0 & T' \notin K \end{cases} \qquad (pf)(T,T') = \begin{cases} f(T,T') & T \in K \\ 0 & T \notin K \end{cases}$$

For a proof, see [47], Lemma 2.7.

We now present a generating set for $C_r^*(\mathcal{R}_{\text{punc}})$ that will be easier to work with than general compactly supported functions. Recall that the topology on $\mathcal{R}_{\text{punc}}$ is generated by sets of the form

$$V(P,t_1,t_2) = \{(T, T+x) \in \mathcal{R}_{\text{punc}} \mid T \in U(P,t_1), x = x(t_1) - x(t_2)\},$$

where P is a patch and t_1, t_2 are tiles in the patch. These sets are clopen, so the functions

$$e(P,t_1,t_2) = \chi_{V(P,t_1,t_2)}$$

are in $C_c(\mathcal{R}_{\text{punc}})$, and so are in $C_r^*(\mathcal{R}_{\text{punc}})$.

Lemma 4.6.2 ([35], Section 2.2) *Let P, P' be patches and let $t_1, t_2, t \in P$ and $t_1', t_2' \in P'$. Assume without loss of generality that $x_{t_2} = 0$ and that $x_{t_1'} = 0$. Then we have the following.*

1. *The product $e(P,t_1,t_2)e(P',t_1',t_2')$ is nonzero precisely when $U(P,t_1) \cap U(P,t_2) \neq \emptyset$ and the patches P and P' agree on the overlap of their supports, i.e., $P \cup P'$ is a patch. In this case the product is $e(P \cup P', t_1, t_2')$.*

2. $e(P, t_1, t_2)^* = e(P, t_2, t_1)$.

3. $e(P, t, t)e(P, t, t) = e(P, t, t)$. *Hence each* $e(P, t, t)$ *is a projection and* $e(P, t_1, t_2)$ *is a partial isometry from* $e(P, t_2, t_2$ *to* $e(P, t_1, t_1)$ *in* $C_c(\mathcal{R}_{\text{punc}})$.

Proof: We verify statement 1. The formula for the product is

$$e(P, t_1, t_2)e(P', t_1', t_2')(T, T') = \sum_{T'' \in [T]} e(P, t_1, t_2)(T, T'')e(P', t_1', t_2')(T'', T'), \quad (4.6.1)$$

where $[T]$ denotes the equivalence class of T in Ω_{punc}. For a given term

$$e(P, t_1, t_2)(T, T'')e(P', t_1', t_2')(T'', T')$$

to be nonzero, we first need $T \in U(P, t_1)$. In this case, there is only one T'' for which $e(P, t_1, t_2)(T, T'')$ is nonzero, and that is

$$T'' = T + x_{t_1} - x_{t_2} = T + x_{t_1}$$

where the second equality is because we assumed that $x_{t_2} = 0$. In this case, we have $T'' \in U(P, t_2)$. For the term $e(P', t_1', t_2')(T'', T')$ to be simultaneously nonzero, we need $T'' \in U(P', t_1')$ and

$$T' = T'' + x_{t_1'} - x_{t_2'} = T'' - x_{t_2'}$$

where again the second equality is because we assumed that $x_{t_1'} = 0$. Rearranging these two equations gives us

$$T' = T + x_{t_1} - x_{t_2'}.$$

Hence the sum in Equation (4.6.1) has at most one nonzero term, and this term is nonzero if and only if $U(P, t_1) \cap U(P, t_2) \neq \emptyset$, $T \in U(P, t_1)$, and $T' = T + x_{t_1} - x_{t_2'}$. In this case, $e(P, t_1, t_2)e(P', t_1', t_2')(T, T') = 1$. This is exactly the definition of $e(P \cup P', t_1, t_2')(T, T')$.

The other two statements are verified in a similar manner, see [71], Section 4.2 for details. ∎

Let

$$\mathcal{E} = \{e(P, t_1, t_2) \mid P \text{ is a patch with } t_1, t_2 \in P\}.$$

Then $\operatorname{span}_{\mathbb{C}} \mathcal{E}$ is a $*$-subalgebra of $C_c(\mathcal{R}_{\text{punc}})$. It has identity $\sum_{p \in \mathcal{P}} e(\{p\}, p, p)$. In [71], Lemma 4.13, Whittaker uses a Stone-Weierstrass argument to show that $\operatorname{span}_{\mathbb{C}} \mathcal{E}$ is dense in $C_c(\mathcal{R}_{\text{punc}})$ with respect to the inductive limit topology. With the following two lemmas we expand this result and show that we may restrict the patches considered to those whose supports have connected interior. Let $\mathcal{E}_c$ denote the set of elements of the form $e(P, t_1, t_2)$ where the support of P has connected interior.

Lemma 4.6.3 *Let P be a patch with $t_1, t_2 \in P$ and $x_{t_1} = 0$. Let $r > 0$ be such that* $\operatorname{supp}(P) \subset B_r(0)$ *and let*

$$Y = \{T(B_r(0)) \mid T \in U(P, t_1)\}.$$

Then Y is a finite set and

$$V(P, t_1, t_2) = \dot{\bigcup_{P' \in Y}} V(P', t_1, t_2)$$

where the union is disjoint.

Proof: That Y is finite follows from finite local complexity. Take $P_1, P_2 \in Y$ with $P_1 \neq P_2$, and suppose that $(T, T + x) \in V(P_1, t_1, t_2) \cap V(P_2, t_1, t_2)$. This implies that $T \in U(P_1, t_1) \cap U(P_2, t_1)$, and hence $P_1, P_2 \in T$. But this means that

$$P_1 = P_1(B_r(0)) = T(B_r(0)) = P_2(B_r(0)) = P_2,$$

a contradiction, and hence the sets $V(P', t_1, t_2)$ are pairwise disjoint.

Let $(T, T + x) \in V(P', t_1, t_2)$ for some $P' \in Y$. Then since $P \subset P'$, we must have that $(T, T + x) \in V(P, t_1, t_2)$. Conversely suppose that $(T, T + x) \in V(P, t_1, t_2)$. Then define $P' = T(B_r(0))$. We have $P' \in Y$ and so $(T, T + x) \in V(P', t_1, t_2)$. ∎

Hence, keeping the notation of Lemma 4.6.2, we have

$$e(P, t_1, t_2) = \sum_{P' \in Y} e(P', t_1, t_2)$$

and so every element in $\mathcal{E}$ can be written as the sum of elements in $\mathcal{E}_c$, whence $\operatorname{span}_{\mathbb{C}} \mathcal{E}_c = \operatorname{span}_{\mathbb{C}} \mathcal{E}$. Thus when working with this dense subalgebra, we can consider only patches whose support has connected interior.[1] This allows us to present a finite generating set of $\operatorname{span}_{\mathbb{C}} \mathcal{E}$. Let

$$\mathcal{E}_2 = \{e(\{t_1, t_2\}, t_1, t_2) \mid \text{the interior of } t_1 \cup t_2 \text{ is connected}\}.$$

This is the set of all two-tile patches whose support has connected interior – one may also think of them as "edge patterns". Because of finite local complexity, this set is finite.

Lemma 4.6.4 *If $e(P, t_1, t_2) \in \mathcal{E}_c$, it is a finite product of elements of $\mathcal{E}_2$.*

Proof: Let $P = \{t_1, t_2, t_3, \ldots, t_n\}$. Assume without loss of generality that $x_{t_1} = 0$. Let $V_P = \{s_1 \cap s_2 \cap \cdots \cap s_{d+1} \mid s_i \in P\}$ be the set of all points in $\operatorname{supp}(P)$ that are $(d+1)$-way intersections of tiles in P. As V_P is contained in the set of vertices of tiles in P, this set is finite. The set $\operatorname{Int}(\operatorname{supp}(P)) \setminus V_P$ is still connected, because we have removed a finite set of points from an open connected subset of $\mathbb{R}^d$. For each $i = 2, 3, \ldots, n$ find a path $p_i : [0, 1] \to \operatorname{Int}(\operatorname{supp}(P)) \setminus V_P$ with $p_i(0) = 0$ and $p_i(1) = x_{t_i}$. Furthermore, we may assume there exist $0 < y_1 < y_2 < \ldots y_{k_i} < 1$ such that if $p_i(y)$ is on the boundary of some tile in P, then $y = y_m$ for some m with $1 \leq m \leq k_i$. For $x \in (y_m, y_{m+1})$, $p_i(x) \in t_{j_m}$ for some tile t_{j_m}. Hence $e(\{t_{j_m}, t_{j_{m+1}}\}, t_{j_m}, t_{j_{m+1}}) \in \mathcal{E}_2$ for all m with $1 \leq m \leq k_i - 1$. We note that $t_{j_1} = t_1$ and $t_{j_{k_i}} = t_i$. Then if we define

$$w_i := e(\{t_{j_1}, t_{j_2}\}, t_{j_1}, t_{j_2}) e(\{t_{j_2}, t_{j_3}\}, t_{j_2}, t_{j_3}) \cdots e(\{t_{j_{k-1}}, t_{j_{k_i}}\}, t_{j_{k_i-1}}, t_{j_{k_i}}).$$

[1] We note that we could have been even more restrictive with the patches considered by insisting that the interiors of the supports of the patches be simply connected. In this case, the product of elements $e(P, t_1, t_2)$ and $e(P', t_1', t_2')$ may not result in another element of this form (since the union of two simply connected sets need not be simply connected), but this problem can be remedied by a similar argument to the one that got us from arbitrary patches to those whose support has connected interior.

Then we see that

$$w_i = e\left(\bigcup_{m=1}^{k_i-1}\{t_{j_m}\}, t_1, t_i\right)$$

and

$$w_i w_i^* = e\left(\bigcup_{m=1}^{k_i-1}\{t_{j_m}\}, t_1, t_1\right).$$

Finally, if we take the product of all of these, we see that the patch obtained must contain each tile in P, so that

$$\prod_{i=1}^{n} w_i w_i^* = e(P, t_1, t_1),$$

and so

$$\left(\prod_{i=1}^{n} w_i w_i^*\right) w_2^* = e(P, t_1, t_1) e\left(\bigcup_{m=1}^{k_2-1}\{t_{j_m}\}, t_1, t_2\right) = e(P, t_1, t_2).$$

∎

Thus, the $*$-algebra $\operatorname{span}_{\mathbb{C}} \mathcal{E}$ is generated by the finite set $\mathcal{E}_2$. Therefore we get:

Corollary 4.6.5 *The C*-algebra $C_r^*(\mathcal{R}_{\text{punc}})$ is generated by the set*

$$\mathcal{E}_2 = \{e(\{t_1, t_2\}, t_1, t_2) \mid \text{the interior of } t_1 \cup t_2 \text{ is connected}\}.$$

In particular, $C_r^*(\mathcal{R}_{\text{punc}})$ is finitely generated.

Putnam and Kellendonk define another C*-algebra associated with a tiling, an AF algebra derived from the substitution. This AF algebra is well-studied - see for example [33], [34] or [52]. For this reason we supply more references than proofs. This AF algebra will be different than the one presented in Example 4.5.4, and later we will see that the AF algebra from Example 4.5.4 can be written as a crossed product of the algebra below.

We follow the description and notation in [33]. Recall from Section 3.4 that for $x, y \in \operatorname{Punc}(n, p)$ we defined

$$E_p^n(x, y) = \{(\omega^n(T) - x, \omega^n(T) - y) \mid T \in U(\{p\}, p)\}.$$

These sets are clopen, and since the vector $x - y$ is fixed, they are compact. We denote by

$$e_p^n(x, y) = \chi_{E_p^n(x,y)},$$

the characteristic function of $E_p^n(x, y)$. These are elements of $C_c(\mathcal{R}_{\text{punc}})$.

Lemma 4.6.6 *If we let*

$$A_{n,p} = \text{span}_{\mathbb{C}} \left\{ e_p^n(x, y) \mid x, y \in \text{ Punc}(n, p) \right\}$$

then $A_{n,p}$ is a $*$-subalgebra of $C_r^*(\mathcal{R}_{\text{punc}})$ isomorphic to the $(m \times m)-$matrices, where $m = \#\text{Punc}(n, p)$. Furthermore, if $p \neq p'$, then $A_{n,p}$ and $A_{n,p'}$ are orthogonal, and hence their direct sum

$$A_n := \bigoplus_{p \in \mathcal{P}} A_{n,p}$$

is also a subalgebra of $C_r^*(\mathcal{R}_{\text{punc}})$.

Proof: We calculate

$$e_p^n(x, y) e_{p'}^n(x', y')(T, T') = \sum_{T'' \in [T]} e_p^n(x, y)(T, T'') e_{p'}^n(x', y')(T'', T').$$

A term $e_p^n(x, y)(T, T'') e_{p'}^n(x', y')(T'', T')$ is only nonzero when

$$(T, T'') \in E_p^n(x, y) \text{ and } (T'', T') \in E_{p'}^n(x', y')$$

which implies

$$T'' = \omega^n(S) - y \text{ for some } S \in U(\{p\}, p),$$

$$T'' = \omega^n(Q) - x' \text{ for some } Q \in U(\{p'\}, p').$$

Applying ω^{-n} to both sides gives us

$$S - \lambda^{-n} y = Q - \lambda^{-n} x'.$$

Now suppose that $p \neq p'$. Since the punctures y and x' are in the interior of p and p' respectively, we get that

$$p - \lambda^{-n} y = (S - \lambda^{-n} y)(0) = (Q - \lambda^{-n} x')(0) = p' - \lambda^{-n} x'.$$

Since prototiles are assumed to not be translates of each other, we get that

$$e_p^n(x, y) e_{p'}^n(x', y') = 0 \qquad \text{if } p \neq p'. \tag{4.6.2}$$

Now if $p = p'$ we get

$$p - \lambda^{-n} y = p - \lambda^{-n} x'$$

which is only true if $y = x'$. Hence

$$e_p^n(x, y) e_{p'}^n(x', y') = 0 \qquad \text{if } p = p' \text{ and } y \neq x'. \tag{4.6.3}$$

In the case that $p = p'$ and $y = x'$, a term $e_p^n(x, y)(T, T'') e_p^n(y, y')(T'', T')$ is only nonzero when

$$T = \omega^n(S) - x \text{ for some } S \in U(\{p\}, p),$$

$$T'' = \omega^n(S) - y,$$

which implies

$$T' = \omega^n(S) - y'.$$

These imply that

$$e_p^n(x, y) e_p^n(y, y')(T, T') = \begin{cases} 1 & (T, T') \in E_p^n(x, y') \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$e_p^n(x, y) e_{p'}^n(x', y') = e_{p'}^n(x, y') \qquad p = p' \text{ and } y = x'. \tag{4.6.4}$$

By labeling the elements of $\mathrm{Punc}(n, p) = \{x_1, \ldots x_m\}$, Equations (4.6.3) and (4.6.4) imply that

$$A_{n,p} = \mathrm{span}_{\mathbb{C}} \left\{ e_p^n(x, y) \mid x, y \in \ \mathrm{Punc}(n, p) \right\} \cong \mathbb{M}_m$$

via

$$e_p^n(x_i, x_j) \mapsto e_{ij}.$$

Equation (4.6.2) implies the orthogonality of $A_{n,p}$ and $A_{n,p'}$ for $p \neq p'$. ∎

In the proof of Lemma 3.4.5 we have the equation

$$E_p^n(x,y) = \dot{\bigcup_{(p',x') \in I_p}} E_{p'}^{n+1}\left(\lambda^n x' + x, \lambda^n x' + y\right).$$

Since the union on the right is disjoint, we have the equation

$$e_p^n(x,y) = \sum_{(p',x') \in I_p} e_{p'}^{n+1}\left(\lambda^n x' + x, \lambda^n x' + y\right).$$

This equation implies that $A_n \subset A_{n+1}$. As discussed in Section 4.5, this inclusion is determined up to unitary equivalence in A_{n+1} by the matrix of partial multiplicities M. Referring to [34], Equation (67) and the discussion on page 24 in [33], the M_{ij} entry in the incidence matrix for our substitution is the number of occurrences of copies of p_j in $\omega(p_i)$. We notice that primitivity of the substitution implies primitivity of the matrix M in the traditional sense, that is M^n has strictly positive entries for some $n \in \mathbb{N}$.

Example 4.6.7 In the case of the Penrose substitution, the entries of M can be read off Figure 2.7. For instance, when one substitutes prototile **1** one obtains one copy of prototile **8** and one copy of prototile **24**. Hence the first column of M will have a 1 in the 8th entry and 24th entry and have 0 in each other entry. For the full matrix, see [68], Section 5.2.

We denote the union

$$AF_\omega = \overline{\bigcup_{n \in \mathbb{N}} A_n}$$

where the closure is with respect to the norm on $C_r^*(\mathcal{R}_{\text{punc}})$. We note that in [33] and [52], Kellendonk and Putnam denote this algebra AF_T for historical reasons. Since each of the generators is a characteristic function of a compact open subset of $\mathcal{R}_{AF}$, we have that AF_ω is a subalgebra of $C^*_r(\mathcal{R}_{AF})$. These algebras are in fact equal, see for example [59], Section III.1 or [52] p. 596.

We now turn our attention to traces on $C^*_r(\mathcal{R}_{\text{punc}})$. In [47] Proposition 2.11, Phillips proves that if $\mathscr{G}$ is an almost AF Cantor groupoid with AF subgroupoid $\mathscr{G}_0$, then the set of tracial states on $C^*_r(\mathscr{G})$ is the same as the set of tracial states on $C^*_r(\mathscr{G}_0)$. In our case, $C^*_r(\mathscr{G}_0) = AF_\omega$ is an AF algebra whose matrix of partial multiplicities, M, is primitive. By [28] Theorem 4.1, such an AF algebra has a unique tracial state. Hence $C^*_r(\mathcal{R}_{\text{punc}})$ has a unique tracial state as well. This trace is well known – for instance see [35], [33] or [52]. We describe the essential properties of this trace and how to calculate it on elements of AF_ω.

Since the matrix M is primitive, by the Perron-Frobenius Theorem M admits left and right eigenvectors whose entries are all positive and whose eigenvalue is positive and strictly larger in modulus than the other eigenvalues of M. For a primitive substitution tiling system $(\mathcal{P}, \omega)$ in $\mathbb{R}^d$ with expansion constant λ, the Perron eigenvalue is λ^d. Furthermore, if $\mathcal{P} = \{p_1, p_2, \dots, p_N\}$ and v_R is the vector whose ith entry is the volume of p_i, then v_R is a right Perron-Frobenius eigenvector of M, see [66], Corollary 2.4[2]. If v_L is the vector whose ith entry is the relative frequency of translates of the prototile p_i in any tiling $T \in \Omega$, then v_L is a left Perron-Frobenius eigenvector of M, see [35], Section 4.

Now, given a basis element $e^n_p(x, y)$, its trace is

$$\tau(e^n_{p_i}(x, y)) = \begin{cases} \lambda^{-2n} v_L(i) & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}. \tag{4.6.5}$$

As before, this is a general fact about AF algebras with constant primitive partial multiplicity matrix. We normalize v_L so that τ is a tracial state, i.e. we need the

[2] Note that the substitution matrix as defined by Solomyak is the transpose of our substitution matrix.

following to hold:

$$\begin{aligned}
1 &= \tau(1) \\
&= \tau\left(\sum_{p_i\in\mathcal{P}} e^0_{p_i}(0,0)\right) \\
&= \sum_{p_i\in\mathcal{P}} \tau(e^0_{p_i}(0,0)) \\
&= \sum_{p_i\in\mathcal{P}} v_L(i).
\end{aligned}$$

We note that if we fix $n\in\mathbb{N}$, we have

$$\sum_{p_i\in\mathcal{P}}\ \sum_{x\in\mathrm{Punc}(n,p_i)} e^n_{p_i}(x,x) = 1.$$

Therefore we also obtain the equation

$$\begin{aligned}
1 &= \tau(1) \\
&= \tau\left(\sum_{p_i\in\mathcal{P}}\ \sum_{x\in\mathrm{Punc}(n,p_i)} e^n_{p_i}(x,x)\right) \\
&= \sum_{p_i\in\mathcal{P}}\ \sum_{x\in\mathrm{Punc}(n,p_i)} \lambda^{-2n} v_L(i) \\
&= \sum_{p_i\in\mathcal{P}} \#\mathrm{Punc}(n,p_i)\lambda^{-2n} v_L(i). \qquad (4.6.6)
\end{aligned}$$

We conclude this chapter by summarizing what is known about the C*-algebras associated to a primitive substitution tiling system. The following facts are all present in the existing literature, but we gather them here for convenience.

Remark 4.6.8 Let $(\mathcal{P},\omega)$ be a primitive substitution tiling system in $\mathbb{R}^d$ satisfying the assumptions of Remark 2.5.8, and let Ω be the associated tiling space. The crossed product $C(\Omega)\rtimes\mathbb{R}^d$ is simple ([73] Corollary 8.22) and strongly Morita equivalent to $C^*_r(\mathcal{R}_{\text{punc}})$ (Corollary 3.4.2 together with Theorem 4.4.5). Strong Morita equivalence preserves ideal structure, so $C^*_r(\mathcal{R}_{\text{punc}})$ is simple as well. The groupoid $\mathcal{R}_{\text{punc}}$ is amenable ([54] Theorem 1.1) and so $C^*(\mathcal{R}_{\text{punc}}) = C^*_r(\mathcal{R}_{\text{punc}})$.

The groupoid $\mathcal{R}_{\text{punc}}$ contains an AF groupoid $\mathcal{R}_{AF}$ relative to which $\mathcal{R}_{\text{punc}}$ has the structure of an almost AF Cantor groupoid (Definition 3.2.10). The C*-algebra $C_r^*(\mathcal{R}_{AF})$ is an AF algebra which we call AF_ω with a unique tracial state given by Equation (4.6.5). By Phillips' results on almost AF Cantor groupoids, $C_r^*(\mathcal{R}_{\text{punc}})$ has real rank zero and stable rank one, and has unique tracial state which when restricted to AF_ω agrees with its unique tracial state ([47] Theorem 4.6, Theorem 5.2 and Proposition 2.11). Furthermore, the order of projections over $C_r^*(\mathcal{R}_{\text{punc}})$ is determined by traces ([52], and [47], Corollary 5.4).

Chapter 5

Finite Group Actions

Many of the tilings we study are acted upon naturally by finite symmetry groups. These groups act on the groupoids and C*-algebras formed from these tilings. We assume for the rest of this work that $(\mathcal{P}, \omega)$ is a primitive substitution tiling system of $\mathbb{R}^2$ satisfying the assumptions in Remark 2.5.8. We restrict to dimension 2 because this case strikes a good balance between richness of examples and complexity (or rather, lack thereof) of the symmetry groups.

5.1 Finite Group Actions on the Tiling Space

We begin by making precise a notion discussed at the end of Section 2.4.

Definition 5.1.1 *Let $(\mathcal{P}, \omega)$ be a substitution tiling system. We say that a subgroup G of $O(2, \mathbb{R})$ is a* **symmetry group for** $(\mathcal{P}, \omega)$ *if for all $p \in \mathcal{P}$ and $g \in G$ we have that:*

1. *The set $gp = \{gx \mid x \in p\}$ is an element of $\mathcal{P}$, and*

2. *$\omega(gp) = g\omega(p)$.*

If G is such a group, then we say that $\mathcal{S}_G \subset \mathcal{P}$ is a set of **standard position** *prototiles for G if $G\mathcal{S}_G = \mathcal{P}$ and $\mathcal{S}_G$ does not properly contain any other such set.*

If G is a symmetry group for $(\mathcal{P}, \omega)$, then Condition 1 says that G acts on the set $\mathcal{P}$ in the traditional sense. This also implies that each $g \in G$ is a bijection of $\mathcal{P}$. Note for Condition 2 that a group element acts on any translate of a prototile via the formula $g(p + x) = gp + gx$. Recall that G acts **freely** on $\mathcal{P}$ if $gp = p$ for some $p \in \mathcal{P}$ if and only if $g = e$. In the case that G acts freely on $\mathcal{P}$, a set $\mathcal{S}_G$ of standard position prototiles has the property that $g\mathcal{S}_G \cap \mathcal{S}_G = \emptyset$ for $g \neq e$. In this work, we only consider tilings which admit finite symmetry groups, and so from now on **we assume that all symmetry groups we consider are finite**.

Example 5.1.2 Consider the Penrose substitution of Example 2.4.10. As mentioned at the end of Section 2.4, a sample symmetry group for $(\mathcal{P}_{\text{Pen}}, \omega)$ is the dihedral group D_{10} generated by r, the counterclockwise rotation by $^{\pi}/_{5}$ and f, the reflection over the x-axis. These elements satisfy the relations

$$r^{10} = f^2 = e, \qquad frf = r^{-1}.$$

In this case, we can take $\mathcal{S}_{D_{10}} = \{\mathbf{1}, \mathbf{21}\}$. Here, D_{10} acts freely on $\mathcal{P}_{\text{Pen}}$, that is, if $gp = p$ for some $g \in D_{10}$ and $p \in \mathcal{P}_{\text{Pen}}$, then $g = e$. We note that for the subgroup $\langle r \rangle$ we have $\mathcal{S}_{\langle r \rangle} = \{\mathbf{1}, \mathbf{11}, \mathbf{21}, \mathbf{31}\}$ and the action of $\langle r \rangle$ also free. In both cases the action commutes with the substitution by definition – it is defined on $\mathcal{S}_G$ and extended by symmetry.

Example 5.1.3 Consider the octagonal substitution of Figure 2.5. Then one symmetry group for the substitution is the dihedral group D_8 generated by r, the counterclockwise rotation by $^{\pi}/_{4}$ and f, the reflection over the x-axis. In this case $\mathcal{S}_{D_8}$ consists of a rhomb and a triangle. Here, the action of D_8 on $\mathcal{P}$ is not free, as the rhomb is fixed by r^4, rf, and $r^5 f$. In Example 2.4.12 we replaced our substitution system with one on which the action of D_8 is free, see Figure 2.6. The tilings obtained from this new substitution system are mutually locally derivable from tilings obtained from the original substitution, yielding homeomorphic tiling spaces by Lemma 2.3.2.

Definition 5.1.4 *Let G be a topological group that acts on a topological space X on the left. We say that X is a G-***space** *if the map from $G \times X$ to X defined by $(g, x) \mapsto gx$ is continuous. We say that a locally compact Hausdorff G-space X is* **proper** *if the map from $G \times X \to X \times X$ defined by $(g, x) \mapsto (gx, x)$ is proper.*

Recall that a continuous map $f : X \to Y$ is proper if $f^{-1}(K)$ is compact in X for every compact $K \subset Y$. We note that if G is finite, then every G-space is a proper G-space. We also note that if X is a G-space then for each $g \in G$ the map $x \mapsto gx$ is a homeomorphism of X.

Lemma 5.1.5 *Suppose that G is a symmetry group for $(\mathcal{P}, \omega)$ and suppose that Ω is the tiling space associated to $(\mathcal{P}, \omega)$. For $T \in \Omega$ and $g \in G$, let*

$$gT = \{gt \mid t \in T\}.$$

Then $gT \in \Omega$.

Proof: If P is a patch in gT, then $g^{-1}P \subset T$ and so appears in $\omega^n(p)$ for some $n \in \mathbb{N}$ and $p \in \mathcal{P}$. But that means $gg^{-1}P = P$ appears in $g\omega^n(p) = \omega^n(gp)$. Every patch of gT appears in $\omega^n(q)$ for some prototile q and some $n \in \mathbb{N}$, so $gT \in \Omega$. ∎

It is clear that this defines an action of G on Ω. We now prove that Ω is a G-space.

Lemma 5.1.6 *Suppose that G is a symmetry group for $(\mathcal{P}, \omega)$ and suppose that Ω is the tiling space associated to $(\mathcal{P}, \omega)$. Then Ω is a G-space.*

Proof: The symmetry group G is finite and Ω is compact; it is enough to show that the map $g : \Omega \to \Omega$ is continuous. Let $0 < \varepsilon < 1$ and suppose that $d(T_1, T_2) < \varepsilon$. Then by definition there exist $x_1, x_2 \in \mathbb{R}^2$ with $|x_1|, |x_2| < \varepsilon$ such that $(T_1 - x_1)(B_{1/\varepsilon}(0)) = (T_2 - x_2)(B_{1/\varepsilon}(0))$. Since $B_{1/\varepsilon}(0)$ is invariant under G,

$$\begin{aligned} g\left((T_1 - x_1)(B_{1/\varepsilon}(0))\right) &= g\left((T_2 - x_2)(B_{1/\varepsilon}(0))\right) \\ (gT_1 - gx_1)(B_{1/\varepsilon}(0)) &= (gT_2 - gx_2)(B_{1/\varepsilon}(0)) \end{aligned}$$

Since $|gx_i| = |x_i|, i = 1, 2$, this shows that $d(gT_1, gT_2) < \varepsilon$ and proves the continuity of g. ∎

We note that the proof above implies that the action of G on Ω is isometric. Since G acts on the prototiles, the action of G on Ω induces a continuous action on Ω_{punc} as well.

5.2 The Orbit Space Ω/G

Definition 5.2.1 *Let X be a G-space, and write $x \sim_G y$ if $gy = x$ for some $g \in G$. Then the* **orbit space** *X/G is defined as the set $X/\sim_G$ with the quotient topology.*

The natural quotient map $\pi : X \to X/G$ is an open map. Indeed, if U is open in X, then $\pi^{-1}(\pi(U)) = \cup_{g\in G} gU$, and each of the gU are open since each g is open. We denote the image of x under π as Gx.

Lemma 5.2.2 *If X is a proper G-space and X is locally compact and Hausdorff, then X/G is locally compact and Hausdorff.*

The goal of this section is to describe the orbit space of the tiling space Ω associated to a primitive substitution tiling system $(\mathcal{P}, \omega)$ by a finite symmetry group G. We will use the description of the tiling space as an inverse limit of CW complexes in Theorem 2.5.3. Since these CW complexes are quotients themselves, we first prove some general facts about the interaction of group actions with quotients by equivalence relations.

Theorem 5.2.3 *Let G be a locally compact Hausdorff group and let X be a locally compact Hausdorff G-space. Let $\mathcal{R}$ be an equivalence relation on X such that for any $g \in G$, $(x, y) \in \mathcal{R}$ if and only if $(gx, gy) \in \mathcal{R}$. Then*

1. *$X/\mathcal{R}$ is a G-space,*
2. *the equivalence relation $\mathcal{R}$ induces an equivalence relation $\mathcal{R}_G$ on X/G such that $(Gx, Gy) \in \mathcal{R}_G$ if and only if there exists $g \in G$ such that $(x, gy) \in \mathcal{R}$, and*
3. *$(X/G)/\mathcal{R}_G \cong (X/\mathcal{R})/G$.*

Proof: 1. Let $Q_\mathcal{R}$ denote the quotient map determined by $\mathcal{R}$, and denote the equivalence class of x in $\mathcal{R}$ by $[x]_\mathcal{R}$. Each $g \in G$ determines a homeomorphism $g : X \to X$, and so $Q_\mathcal{R} \circ g : X \to X/\mathcal{R}$ is a continuous and surjective quotient map. Thus

$$\begin{aligned}
(x, y) \in \mathcal{R} &\Leftrightarrow (gx, gy) \in \mathcal{R} \\
&\Leftrightarrow Q_\mathcal{R}(gx) = Q_\mathcal{R}(gy) \\
&\Leftrightarrow Q_\mathcal{R} \circ g(x) = Q_\mathcal{R} \circ g(y).
\end{aligned}$$

Hence there is a homeomorphism $f_g : X/\mathcal{R} \to X/\mathcal{R}$ such that $f_g \circ Q_\mathcal{R} = Q_\mathcal{R} \circ g$. We have

$$\begin{aligned}
f_g \circ Q_\mathcal{R}(x) &= Q_\mathcal{R} \circ g(x) \\
f_g([x]_\mathcal{R}) &= [gx]_\mathcal{R}.
\end{aligned}$$

That this defines a left G-action on $X/\mathcal{R}$ is obvious. The product map $G \times X/\mathcal{R} \to X/\mathcal{R}$ is continuous because it is the composition of the product map $G \times X \to X$ with $Q_\mathcal{R}$. Hence $X/\mathcal{R}$ is a G-space with $g[x]_\mathcal{R} = [gx]_\mathcal{R}$.

2. We denote the usual quotient space maps by

$$P : X \to X/G,$$

$$x \mapsto Gx,$$

$$P_{\mathcal{R}} : X/\mathcal{R} \to (X/\mathcal{R})/G,$$

$$[x]_{\mathcal{R}} \mapsto G[x]_{\mathcal{R}}.$$

The map $P_{\mathcal{R}} \circ Q_{\mathcal{R}} : X \to (X/\mathcal{R})/G$ is a continuous and surjective quotient map. Suppose we have $x, y \in X$ such that $x = gy$ for some $g \in G$. Then

$$\begin{aligned} P_{\mathcal{R}} \circ Q_{\mathcal{R}}(x) &= P_{\mathcal{R}}([x]_{\mathcal{R}}) \\ &= G[x]_{\mathcal{R}} \\ &= G[gy]_{\mathcal{R}} \\ &= G(g[y]_{\mathcal{R}}) \\ &= G[y]_{\mathcal{R}} \\ &= P_{\mathcal{R}} \circ Q_{\mathcal{R}}(y). \end{aligned}$$

Thus there exists a continuous surjective quotient map $\phi : X/G \to (X/\mathcal{R})/G$ such that $\phi \circ P = P_{\mathcal{R}} \circ Q_{\mathcal{R}}$. For $x \in X$ we have

$$\phi(Gx) = P_{\mathcal{R}} \circ Q_{\mathcal{R}}(x) = G[x]_{\mathcal{R}}.$$

Let $\mathcal{R}_G$ be the equivalence relation on X/G determined by the fibres of ϕ. That is, we say that $(Gx, Gy) \in \mathcal{R}_G$ if and only if $\phi(Gx) = \phi(Gy)$.

3. Let $x, y \in X$. Then $\phi(Gx) = \phi(Gy)$ if and only if $(Gx, Gy) \in \mathcal{R}_G$. Since $\phi : X/G \to (X/\mathcal{R})/G$ is a surjective quotient map, this implies that $(X/G)/\mathcal{R}_G \cong (X/\mathcal{R})/G$. ∎

We also have the following fact about continuous maps on a G-space.

Lemma 5.2.4 *Let G be a locally compact Hausdorff group and let X be a locally compact Hausdorff G-space. Let $f : X \to X$ be continuous and surjective, and suppose that for all $g \in G$ we have that $f(gx) = gf(x)$. Then there exists a continuous surjective map $\tilde{f} : X/G \to X/G$ such that $\tilde{f}(Gx) = Gf(x)$.*

Proof: Once again let $P : X \to X/G$ denote the usual quotient map. The map $P \circ f$ is continuous and surjective. Let $x, y \in X$ such that $x = gy$ for some $g \in G$. Then

$$P \circ f(x) = Gf(x) = G(f(gy)) = G(gf(y)) = Gf(y) = P \circ f(y).$$

Since $P \circ f(x)$ is constant on orbits, there exists a continuous map $\tilde{f} : X/G \to X/G$ such that $\tilde{f} \circ P = P \circ f$. If $x \in X$ then

$$\tilde{f}(Gx) = \tilde{f} \circ P(x) = P \circ f(x) = Gf(x).$$

Surjectivity of f trivially implies surjectivity of $\tilde{f}$. ∎

Let G be a symmetry group for $(\mathcal{P}, \omega)$ which acts freely on $\mathcal{P}$. Recall from Section 2.5 that we consider the disjoint union of the prototiles

$$Y = \{(x, p) \in \mathbb{R}^2 \times \mathcal{P} \mid x \in p\}$$

and define an equivalence relation $\mathcal{R}$ which is the transitive closure of the relation that declares (x, p) and (y, q) to be equivalent if there is a tiling T in Ω such that for some $z_p, z_q \in \mathbb{R}^2$ we have $p + z_p, q + z_q \in T$ and $z_p + x = z_q + y$. The space

$$\Gamma = Y/\mathcal{R}.$$

corresponds to gluing prototiles together along edges that could be next to each other in some tiling. Since G acts on the prototiles, it is trivial that Y is a G-space on setting $g(x, p) = (gx, gp)$. It is also clear that each $g \in G$ is a cellular map (see Definition 2.5.1).

Lemma 5.2.5 *Let G be a symmetry group for $(\mathcal{P},\omega)$ which acts freely on $\mathcal{P}$, and suppose that Y is the disjoint union of the prototiles as above. If $(x,p),(y,q)\in Y$ are $\mathcal{R}$-equivalent, then for all $g\in G$, $(gx,gp),(gy,gq)$ are $\mathcal{R}$-equivalent.*

Proof: It is enough to show that if $x\in p\in\mathcal{P}$, $y\in q\in\mathcal{P}$ such that there is a tiling $T\in\Omega$ and $z_p,z_q\in\mathbb{R}^2$ with $p+z_p,q+z_q\in T$ and $z_p+x=z_q+y$, then (gx,gp) and (gy,gq) are $\mathcal{R}$-equivalent. By Lemma 5.1.5, $gT\in\Omega$. We have that $gp+gz_p,gp+gz_q\in gT$ and $gz_p+gx=gz_q+gy$, and hence (gx,gp) and (gy,gq) are $\mathcal{R}$-equivalent. ∎

Notice that this implies that for all $g\in G$ we have that $(x,y)\in\mathcal{R}$ if and only if $(gx,gy)\in\mathcal{R}$.

Corollary 5.2.6 *Let G be a symmetry group for $(\mathcal{P},\omega)$ which acts freely on $\mathcal{P}$. Then Γ is a G-space, there is an equivalence relation $\mathcal{R}_G$ on Y/G such that $(G(x,p),G(y,q))\in\mathcal{R}_G$ if and only if there exists $g\in G$ such that $((x,p),g(y,q))\in\mathcal{R}$, and*

$$(Y/G)/\mathcal{R}_G\cong\Gamma/G.$$

Proof: This follows from Theorem 5.2.3 with Corollary 5.2.5. ∎

We now describe $(Y/G)/\mathcal{R}_G$. Since Y is the disjoint union of the prototiles and G is a symmetry group which acts freely on $\mathcal{P}$, then Y/G homeomorphic the disjoint union of the prototiles in standard position,

$$Y/G\cong\{(x,p)\mid x\in p,p\in\mathcal{S}_G\}.$$

From Corollary 5.2.6, $(G(x,p),G(y,q))\in\mathcal{R}_G$ if and only if there exists $g\in G$ such that $((x,p),g(y,q))\in\mathcal{R}$. Hence, $\mathcal{R}_G$ is the (transitive) equivalence relation generated after declaring that (x,p) and (y,q) are equivalent if there exists $T\in\Omega$, $z_p,z_q\in\mathbb{R}^2$ and $g\in G$ such that $p+z_p,gq+gz_q\in T$ and $z_p+x=gz_q+gy$. In words, $(Y/G)/\mathcal{R}_G$ is formed by gluing together the standard position prototiles along edges where they could appear next to each other in a tiling in any orientation. From this, it is clear that $(Y/G)/\mathcal{R}_G$ is a CW complex whose 2-cells are the standard position prototiles. Hence by Corollary 5.2.6, Γ/G is homeomorphic to a finite CW complex.

Recall from Section 2.5 that Ω is homeomorphic to the inverse limit of the space Γ under repeated application of the map $\gamma : \Gamma \to \Gamma$ which was defined via the substitution ω. The following tells us that this map descends to the quotient.

Lemma 5.2.7 *Let G be a symmetry group for $(\mathcal{P}, \omega)$ which acts freely on $\mathcal{P}$. Let Γ be the CW complex associated with $(\mathcal{P}, \omega)$, and let $\gamma : \Gamma \to \Gamma$ be defined as in Section 2.5. Then there exists a unique continuous map $\tilde{\gamma} : \Gamma/G \to \Gamma/G$ such that $\tilde{\gamma}(Gx) = G\gamma(x)$.*

Proof: Let $[x, p]_{\mathcal{R}} \in \Gamma$, $g \in G$. Then $\omega(p) = \{q_1 + y_1, q_2 + y_2, \ldots, q_m + y_m\}$ for some $q_j \in \mathcal{P}, y_j \in \mathbb{R}^2$ for $1 \leq j \leq m$. Hence, $\lambda x \in q_i + y_i$ for some i, giving us that $\gamma([x, p]_{\mathcal{R}}) = [\lambda x - y_i, q_i]_{\mathcal{R}}$. Thus $g\gamma([x, p]_{\mathcal{R}}) = [\lambda(gx) - gy_i, gq_i]_{\mathcal{R}}$.

On the other hand, $x \in p$ implies that $gx \in gp$. Since substitution commutes with the action of G, $\omega(gp) = \{gq_1 + gy_1, gq_2 + gy_2, \ldots, gq_m + gy_m\}$. Furthermore, $g\lambda x = \lambda(gx) \in gq_i + gy_i$ implies that $\lambda(gx) - gy_i \in gq_i$. Now the result follows from Lemma 5.2.4. ∎

The following lemma follows directly from the fact that G acts freely on $\mathcal{P}$ and the discussion after Corollary 5.2.6.

Lemma 5.2.8 *Let G be a symmetry group for $(\mathcal{P}, \omega)$ which acts freely on $\mathcal{P}$. Let Γ be the CW complex associated with $(\mathcal{P}, \omega)$. Then the quotient map from Γ to Γ/G given by $x \mapsto Gx$ is a CW map.*

Recall that if X is a compact Hausdorff space and $\gamma : X \to X$ is a continuous surjection, then the inverse limit $\mathscr{X} = \varprojlim(X \xleftarrow{\gamma} X)$ is the subset of $\prod_{n \in \mathbb{N}} X$ consisting of all sequences $(x_n)_{n \in \mathbb{N}}$ such that $\gamma(x_{n+1}) = x_n$ for all $n \in \mathbb{N}$, with the relative topology from the product topology. For an open set $U \subset X$ and $n \in \mathbb{N}$, let $B^{\mathscr{X}}_{U,n}$ denote the set

$$B^{\mathscr{X}}_{U,n} = \{(x_i)_{i\in\mathbb{N}} \mid x_i \in \gamma^{n-i}(U), i = 0, 1, \ldots, n\}.$$

The collection of sets $B^{\mathscr{X}}_{U,n}$ forms a basis for the topology on $\mathscr{X}$. Before we state the next lemma, we recall that if a G is a finite group and X is a compact Hausdorff G-space, then the orbit space X/G is a compact Hausdorff space – this is by Lemma 5.2.2 and the fact that the quotient of a compact space is compact.

Lemma 5.2.9 *Let G be a finite group and let X be a compact Hausdorff G-space. Suppose that $\gamma : X \to X$ is a continuous surjection and that $\gamma(gx) = g\gamma(x)$ for all $g \in G$ and $x \in X$. Let $\tilde{\gamma} : X/G \to X/G$ be the continuous surjection induced by γ from Lemma 5.2.4 such that $\tilde{\gamma}(Gx) = G\gamma(x)$ for all $x \in X$. Then:*

1. *If $\mathscr{X} = \varprojlim \left(X \xleftarrow{\gamma} X\right)$ denotes the inverse limit, then $\mathscr{X}$ is a G-space. If $g \in G$ and $(x_i)_{i\in\mathbb{N}}$ the formula for the action is $g(x_i)_{i\in\mathbb{N}} = (gx_i)_{i\in\mathbb{N}}$.*
2. *The inverse limit $\mathscr{X}_G = \varprojlim \left(X/G \xleftarrow{\tilde{\gamma}} X/G\right)$ is canonically homeomorphic to $\mathscr{X}/G$.*

Proof: 1. This formula clearly defines a left action. Since G is finite, it is enough to show that each g defines a continuous map. If $g \in G$, $U \subset X$ open and $n \in \mathbb{N}$ we have

$$\begin{aligned} gB^{\mathscr{X}}_{U,n} &= \{g(x_i)_{i\in\mathbb{N}} \mid x_i \in \gamma^{n-i}(U), i = 0, 1, \ldots, n\} \\ &= \{(gx_i)_{i\in\mathbb{N}} \mid gx_i \in g\gamma^{n-i}(U), i = 0, 1, \ldots, n\} \\ &= \{(gx_i)_{i\in\mathbb{N}} \mid gx_i \in \gamma^{n-i}(gU), i = 0, 1, \ldots, n\} \\ &= B^{\mathscr{X}}_{gU,n}. \end{aligned}$$

Since g acts by homeomorphisms on X, this is open and so g is continuous.

2. We define

$$f : \mathscr{X} \to \mathscr{X}_G$$

$$(x_i)_{i\in\mathbb{N}} \mapsto (Gx_i)_{i\in\mathbb{N}}.$$

This map is well defined because of part 2 above. To prove that this is surjective, we take $(Gx_i)_{i\in\mathbb{N}} \in \mathscr{X}_G$ and show that there exists $(\overline{x}_i)_{i\in\mathbb{N}} \in \mathscr{X}$ such that $G\overline{x}_i = Gx_i$ for all $i \in \mathbb{N}$. Let $\overline{x}_1 = x_1$ and proceed by induction. We know that

$$\tilde{\gamma}(Gx_2) = G\gamma(x_2) = Gx_1$$

which implies that there exists $g \in G$ such that $g\gamma(x_2) = x_1 = \overline{x}_1$. Since $g\gamma(x_2) = \gamma(gx_2)$, set $\overline{x}_2 = gx_2$. Clearly $Gx_2 = G\overline{x}_2$ so we have so far that

$$(Gx_1, Gx_2, Gx_3, \dots) = (G\overline{x}_1, G\overline{x}_2, Gx_3, \dots).$$

Now suppose we can find $\overline{x}_k, 1 \leq k \leq i$ such that

$$(Gx_1, Gx_2, Gx_3, \dots) = (G\overline{x}_1, G\overline{x}_2, G\overline{x}_3, \dots, G\overline{x}_i, Gx_{i+1}, \dots)$$

such that $\gamma(\overline{x}_{k+1} = \overline{x}_k, 1 \leq k \leq i-1$. Once again we have that

$$\tilde{\gamma}(Gx_{i+1}) = G\gamma(x_{i+1}) = G\overline{x}_i \Rightarrow \exists h \in G \text{ such that } \overline{x}_i = h\gamma(x_{i+1}) = \gamma(hx_{i+1}).$$

Set $\overline{x}_{i+1} = hx_{i+1}$. Then

$$(Gx_1, Gx_2, Gx_3, \dots) = (G\overline{x}_1, G\overline{x}_2, G\overline{x}_3, \dots, G\overline{x}_i, G\overline{x}_{i+1}, Gx_{i+2}, \dots).$$

Hence $f\left((\overline{x}_i)_{i\in\mathbb{N}}\right) = (Gx_i)_{i\in\mathbb{N}} \in \mathscr{X}_G$, so f is surjective.

We now need to show that for $(x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}} \in \mathscr{X}$ we have that $f\left((x_i)_{i\in\mathbb{N}}\right) = f\left((y_i)_{i\in\mathbb{N}}\right)$ if and only if there exists $g \in G$ such that $(x_i)_{i\in\mathbb{N}} = g(y_i)_{i\in\mathbb{N}}$. The "if" direction is obvious, so we need to prove the "only if". Suppose that $(Gx_i)_{i\in\mathbb{N}} = (Gy_i)_{i\in\mathbb{N}}$. Then there exist $g_i \in G$ such that $x_i = g_iy_i$. If $i > j$, then it is easy to see that

$$g_jy_j = x_j = \gamma^{i-j}(x_i) = \gamma^{i-j}(g_iy_i) = g_i\gamma^{i-j}(y_i) = g_iy_j.$$

Since G is a finite group, there is an element $g \in G$ and a subsequence i_n such that $g_{i_n} = g$ for all n. Now if $k \in \mathbb{N}$, we can find n such that $i_n > k$. Hence by the above we have that

$$x_k = g_k y_k = g_{i_n} y_k = g y_k$$

giving us that $x_k = g y_k$ for all $k \in \mathbb{N}$.

Lastly we need to show that f is a continuous quotient map, that is, we need to show that $V \subset \mathscr{X}_G$ open if and only if $f^{-1}(V)$ is open in $\mathscr{X}$. Let $\pi : X \to X/G$ denote the quotient map $\pi(x) = Gx$. Then if $U \subset X/G$ open and $n \in \mathbb{N}$ we have

$$\begin{aligned} B_{U,n}^{\mathscr{X}_G} &= \{(Gx_i)_{i\in\mathbb{N}} \mid Gx_i \in \tilde{\gamma}^{n-i}(U), i = 0, 1, \ldots, n\} \\ f^{-1}\left(B_{U,n}^{\mathscr{X}_G}\right) &= \{(x_i)_{i\in\mathbb{N}} \mid Gx_i \in \tilde{\gamma}^{n-i}(U), i = 0, 1, \ldots, n\} \\ &= \{(x_i)_{i\in\mathbb{N}} \mid x_i \in \pi^{-1}(\tilde{\gamma}^{n-i}(U)), i = 0, 1, \ldots, n\} \\ &= \{(x_i)_{i\in\mathbb{N}} \mid x_i \in \pi^{-1}\gamma^{n-i}(\pi^{-1}(U)), i = 0, 1, \ldots, n\} \\ &= B_{\pi^{-1}(U),n}^{\mathscr{X}}. \end{aligned}$$

Since π is a quotient map, this set is open and so f is continuous. Now, suppose $V \subset \mathscr{X}_G$ with $f^{-1}(V)$ open. If $(Gx_i)_{i\in\mathbb{N}} \in V$, then $(gx_i)_{i\in\mathbb{N}} \in f^{-1}(V)$ for all $g \in G$. Thus we can find $U \in X$ open and $n \in \mathbb{N}$ such that $(x_i)_{\in\mathbb{N}} \in B_{U,n}^{\mathscr{X}} \subset f^{-1}(V)$, and since $gB_{U,n}^{\mathscr{X}} = B_{gU,n}^{\mathscr{X}}$ we may choose U and n so that $gB_{U,n}^{\mathscr{X}} \subset f^{-1}(V)$ for all $g \in G$ as well. We have

$$\begin{aligned} f\left(\bigcup_{g\in G} B_{gU,n}^{\mathscr{X}}\right) &= \bigcup_{g\in G}\{(Gy_i)_{i\in\mathbb{N}} \mid y_i \in \gamma^{n-i}(gU), i = 0, 1, \ldots, n\} \\ &= \left\{(Gy_i)_{i\in\mathbb{N}} \mid y_i \in \gamma^{n-i}\left(\bigcup_{g\in G} gU\right), i = 0, 1, \ldots, n\right\} \\ &= \left\{(Gy_i)_{i\in\mathbb{N}} \mid y_i \in \gamma^{n-i}\left(\pi^{-1}(\pi(U))\right), i = 0, 1, \ldots, n\right\} \\ &= \left\{(Gy_i)_{i\in\mathbb{N}} \mid y_i \in \pi^{-1}\left(\gamma^{n-i}(\pi(U))\right), i = 0, 1, \ldots, n\right\} \\ &= \{(Gy_i)_{i\in\mathbb{N}} \mid Gy_i \in \gamma^{n-i}(\pi(U)), i = 0, 1, \ldots, n\} \\ &= B_{\pi(U),n}^{\mathscr{X}} \end{aligned}$$

and $(Gx_i)_{i\in\mathbb{N}} \in B^{\mathscr{X}}_{\pi(U),n} \subset V$. Hence f is a quotient map and we obtain that $\mathscr{X}_G \cong \mathscr{X}/G$. ∎

Corollary 5.2.10 *Suppose that G is a symmetry group for $(\mathcal{P},\omega)$ and suppose that Γ is the CW complex associated to $(\mathcal{P},\omega)$. Suppose also that G acts freely on $\mathcal{P}$. Then if Ω/G is the orbit space of Ω under the action of G, we have*

$$\Omega/G \cong \lim_{\leftarrow} \left(\Gamma/G \xleftarrow{\tilde{\gamma}} \Gamma/G\right).$$

Corollary 5.2.10 allows us to identify the quotient space Ω/G as (homeomorphic to) the inverse limit of CW complexes. In [45] Ormes, Radin and Sadun calculated the cohomology of Ω/G for the Penrose tiling when G is the rotation group $\mathbb{Z}_{10}$. Below, we do the calculation in detail for the octagonal tiling.

Example 5.2.11 Octagonal tiling, $G = \mathbb{Z}_8$.

The cohomology of the orbit space can be calculated by taking the orbit space of the CW complex and then taking its direct limit. Referring to Figure 2.9, there are 32 prototiles in the CW complex for the octagonal, so the orbit space has 4 different 2-cells. We take $\mathcal{S}_{\mathbb{Z}_8} = \{\mathbf{17},\mathbf{25},\mathbf{1},\mathbf{5}\}$, and rename these tiles $\mathbf{A}=\mathbf{17}, \mathbf{B}=\mathbf{25}, \mathbf{C}=\mathbf{1}$, and $\mathbf{D}=\mathbf{5}$. By identifying each of the standard position prototiles with their orbits under $\mathbb{Z}_8$, we see that the all the edges in the set $\{\mathtt{9},\mathtt{10},\ldots,\mathtt{16}\}$ become one edge in the orbit space, call it a. Similarly, the edges in the set $\{\mathtt{1},\mathtt{2},\ldots,\mathtt{8}\}$ are renamed b, the edges in the set $\{\mathtt{17},\mathtt{18},\ldots,\mathtt{24}\}$ are renamed c and the edges in the set $\{\mathtt{25},\mathtt{26},\ldots,\mathtt{32}\}$ are renamed d. Let $E=\{a,b,c,d\}$ and $F=\{\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\}$. There are five vertices before modding out by $\mathbb{Z}_8$, and 4 of them get identified. Call them α,β and set $V=\{\alpha,\beta\}$. The cochain complex for the integer cohomology for $\Gamma/\mathbb{Z}_8$ is isomorphic to (see for example [29], Theorem 3.5)

$$0 \to C(V,\mathbb{Z}) \to C(E,\mathbb{Z}) \to C(F,\mathbb{Z}) \to 0$$

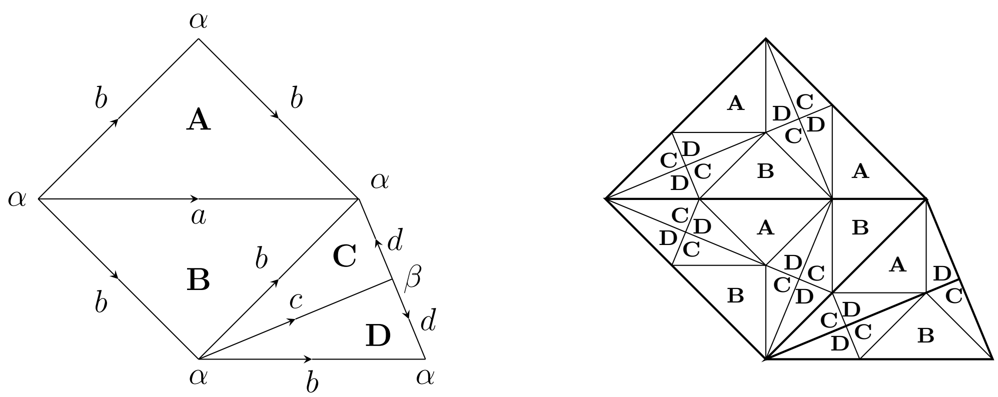


Figure 5.1: $\Gamma/\mathbb{Z}_8$ for the octagonal tiling with substitution rule.

which becomes

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\partial_0} \mathbb{Z}^4 \xrightarrow{\partial_1} \mathbb{Z}^4 \longrightarrow 0.$$

One obtains the matrices ∂_0 and ∂_1 from Figure 5.1 after giving each element of F a clockwise orientation:

$$\partial_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \partial_1 = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix},$$

where, for instance, the first row of ∂_1 indicates that the boundary of the tile **A** has two copies of edge b and one copy of edge a. The -1 indicates that the boundary of **A** contains one copy of a and arrow on the a runs against the clockwise orientation of **A**. The $+2$ indicates that the boundary of **A** contains two copies of b and arrows of both copies run with the clockwise orientation of **A**. We have

$$\ker \partial_0 = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle \qquad \mathrm{Im}\partial_0 = \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\rangle$$

$$\ker \partial_1 = \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle \qquad \mathrm{Im}\partial_1 = \left\langle \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\rangle.$$

This gives us

$$H^0(\Gamma/\mathbb{Z}_8, \mathbb{Z}) = \ker \partial_0 \cong \mathbb{Z}$$

$$H^1(\Gamma/\mathbb{Z}_8, \mathbb{Z}) = \ker \partial_1/\mathrm{Im}\partial_0 \cong \mathbb{Z}$$

$$H^2(\Gamma/\mathbb{Z}_8, \mathbb{Z}) = \mathbb{Z}^4/\mathrm{Im}\partial_1 \cong \mathbb{Z}^2.$$

The space $\Omega/\mathbb{Z}_8$ is homeomorphic to the inverse limit of $\Gamma/\mathbb{Z}_8$ by the map $\tilde{\gamma}$ obtained from the substitution. The map $\tilde{\gamma}$ is cellular. The group $H^i(\Gamma/\mathbb{Z}_8, \mathbb{Z})$ is a subgroup of $\mathbb{Z}^{n(i)}$, where $n(i)$ is the number of i-cells in the CW complex $\Gamma/\mathbb{Z}_8$, and the cohomology groups $H^i(\Omega/\mathbb{Z}_8, \mathbb{Z})$ are obtained by taking the stationary direct limit of the groups $H^i(\Gamma/\mathbb{Z}_8, \mathbb{Z})$ under the $(n(i) \times n(i))$ matrix A_i whose (k, j) entry is the number of copies of the kth i-cell obtained when one performs the substitution on the jth i-cell (for details on this see [1]). These substitution matrices can be read off Figure 5.1:

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 4 & 4 & 1 & 2 \\ 4 & 4 & 2 & 1 \end{bmatrix},$$

where, for instance, the first column of A_2 indicates that substituting the tile **A** results in two copies of **A**, one copy of **B**, and four copies each of **C** and **D**. All of these have determinant either ± 1, so they are invertible over the integers. One calculates these inverse limits to be

$$H^0(\Omega/\mathbb{Z}_8, \mathbb{Z}) = \lim \left(H^0(\Gamma/\mathbb{Z}_8, \mathbb{Z}) \xrightarrow{A_0} H^0(\Gamma/\mathbb{Z}_8, \mathbb{Z}) \right) \cong \mathbb{Z}$$

$$H^1(\Omega/\mathbb{Z}_8, \mathbb{Z}) = \lim \left(H^1(\Gamma/\mathbb{Z}_8, \mathbb{Z}) \xrightarrow{A_1} H^1(\Gamma/\mathbb{Z}_8, \mathbb{Z}) \right) \cong \mathbb{Z}$$

$$H^2(\Omega/\mathbb{Z}_8, \mathbb{Z}) = \lim \left(H^2(\Gamma/\mathbb{Z}_8, \mathbb{Z}) \xrightarrow{A_2} H^2(\Gamma/\mathbb{Z}_8, \mathbb{Z}) \right) \cong \mathbb{Z}^2.$$

Example 5.2.12 As stated above, the cohomology of the orbit space $\Omega/\mathbb{Z}_{10}$ was calculated in [45] to be

$$H^0(\Omega/\mathbb{Z}_{10}, \mathbb{Z}) \cong \mathbb{Z}$$

$$H^1(\Omega/\mathbb{Z}_{10}, \mathbb{Z}) \cong \mathbb{Z}$$

$$H^2(\Omega/\mathbb{Z}_{10}, \mathbb{Z}) \cong \mathbb{Z}^2.$$

Before leaving this section, we note that in [5] the authors provide a framework for computing the cohomology of the orbit space for a more general class of tilings.

5.3 Fixed Points of Ω Under the Group Action

Once again we let $(\mathcal{P}, \omega)$ be a primitive substitution system and let G be a finite symmetry group which acts freely on $\mathcal{P}$. We describe in this section what the fixed points of Ω are under the action of G. Descriptions of tilings of this type were mentioned in [34] among other places, but since our calculations depend on the exact nature of such tilings we derive conditions necessary and sufficient for a tiling to be fixed by a given group element. Since G is always a finite subgroup of $O(2, \mathbb{R})$, it consists of rotations and/or reflections.

Suppose $g \in G$ is a rotation and suppose that $gT = T$. Recall that Ω is homeomorphic to Ω_0, the inverse limit of the spaces Γ under the map γ. Suppose that $T = (x_1, x_2, \dots)$ when viewed as an element of Ω_0. If $gT = T$ then $gx_i = x_i$ for all i. Since G acts freely on the prototiles, we must have that x_i is either a vertex or on an edge in the CW complex Γ (that is, x_i is a 0-cell or in a 1-cell). If x_i is in an edge and $gx_i = x_i$, then g must be rotation by π and x_i must be in the middle of an edge. If we create a new vertex at x_i, then this does not change the cohomology of the CW complex or that of its direct limit, nor does it change the fact that the quotient map from Γ to Γ/G is a CW map. Hence if we create a vertex in the middle of every edge whose middle is fixed by g, we may assume that $gT = T$ implies that x_i is a vertex for all i.

Since the action of G commutes with the substitution, $gT = T$ implies that $(\omega^n(T)(0))_{n\in\mathbb{Z}}$ is a bi-infinite sequence of vertex patterns, each of which is fixed by g. By finite local complexity there are only finitely many such patterns, and hence this bi-infinite sequence must be periodic. To see this, consider $(\omega^n(P)(0))_{n\in\mathbb{Z}\setminus\mathbb{N}}$. Since the number of vertex patterns is finite one of them, say Q, must repeat infinitely often in $(\omega^n(P)(0))_{n\in\mathbb{Z}\setminus\mathbb{N}}$. Furthermore, the gaps between the Qs must be constant, because the ith term in the sequence is determined by the $(i-1)$th term. This implies periodicity of $(\omega^n(P)(0))_{n\in\mathbb{Z}}$ and hence periodicity of $(\omega^n(P)(0))_{n\in\mathbb{N}}$. Conversely, if one finds a vertex pattern P such that $gP = P$ and $\omega^n(P)(0) = P$, then we can build a tiling via

$$T = \bigcup_{i=1}^{\infty} \omega^{in}(P)$$

and in this case, $gT = T$. Thus, tilings for which $gT = T$ are in one-to-one correspondence with vertex patterns P for which $gP = P$ and $\omega^n(P)(0) = P$.

Finding tilings for which $gT = T$ then comes down to writing down all the vertex patterns fixed by g and iteratively substituting each of them and seeing if the pattern at the origin repeats. Also notice that this implies that the set of tilings fixed by G is a finite (and hence isolated) set; this will become important for calculations in the next chapter. We record the above discussion in the following lemma.

Lemma 5.3.1 *Suppose that g is a rotation. Let*

$$\mathcal{P}_g^* = \{P \mid P = T'(0) \text{ for some } T' \in \Omega, gP = P, \text{ and } \omega^n(P)(0) = P \text{ for some } n \in \mathbb{N}\}$$

Then the set of tilings $T \in \Omega$ for which $T = gT$ is in one-to-one correspondence with $\mathcal{P}_g^*$. Furthermore, there are only a finite number of such tilings.

Example 5.3.2 Consider the Penrose tiling with r denoting rotation counterclockwise by $^{\pi}/_{5}$, see Example 5.1.2 and Figure 2.7. By repeatedly applying the substitution, one finds that the only vertex patterns which are fixed under some rotation are the two "star" patterns in the bottom left of Figure 2.7 which we call P_1 and P_3, as well as the patches in the centers of $\omega(P_1)$ and $\omega(P_3)$ which we call P_2 and P_4. These are fixed by $\langle r^2 \rangle \subset D_{10}$. The two decagons in the top left of Figure 2.7 are fixed by some rotation, but do not appear in tilings in Ω. This is seen by noticing that the smallest edges are only shared by the small triangles, and no edge between small triangles is ever deleted under ω^{-1}. One finds that

$$\omega(P_1)(0) = P_2,$$

$$\omega(P_2)(0) = P_3,$$

$$\omega(P_3)(0) = P_4,$$

$$\omega(P_4)(0) = P_1.$$

Hence all four of these patches give rise to tilings fixed under the subgroup $\langle r^2 \rangle$, defined by

$$T_i = \bigcup_{k=0}^{\infty} \omega^{4k}(P_i).$$

We also note that $rT_1 = T_3$, $rT_2 = T_4$.

Example 5.3.3 Consider the octagonal tiling whose cell complex is given in Figure 2.9, and let r denote rotation counterclockwise by $^{\pi}/_{4}$. After writing down all vertex patterns one finds that there are only five vertex patterns which are fixed under rotations. The patch in the middle of Figure 2.9 has stabilizer $\mathbb{Z}_8$; we call this patch P_1. The patch $P_2 = \{\mathbf{1}, \mathbf{5}, \mathbf{9}, \mathbf{13}\}$ has stabilizer $\langle r^4 \rangle \cong \mathbb{Z}_2$. If we let $rP_2 = P_3$, $rP_3 = P_4$ and $rP_4 = P_5$, then these have stabilizer $\langle r^4 \rangle \cong \mathbb{Z}_2$ as well. Also,

$$\omega(P_1) = P_1,$$

$$\omega^2(P_i) = P_i, \quad i = 2, 3, 4, 5.$$

Hence we get five tilings fixed under rotations,

$$T_1 = \bigcup_{k=1}^{\infty} \omega^k(P_1),$$

$$T_i = \bigcup_{k=1}^{\infty} \omega^{2k}(P_i), \quad i = 2, 3, 4, 5.$$

We again note that $T_5 = rT_4 = r^2T_3 = r^3T_2$.

When g is a reflection, it is a little more difficult to describe the tilings which are fixed by g.

Definition 5.3.4 *Let G be a finite symmetry group of $(\mathcal{P}, \omega)$ and let $g \in G$ be a reflection. Let Γ be the CW complex associated with $(\mathcal{P}, \omega)$, and let E be the set of 1-cells in this complex. Let ℓ_g be the axis of symmetry of g. We say an edge $e \in E$ parallel to ℓ_g is* **substitution symmetric** *if whenever we have $T = (x_1, x_2, \dots)$ and $x_1 \in \mathrm{Int}(e)$, then $\omega(T)(0) = g\omega(T)(0)$. Let $SS(g)$ be the set of all substitution symmetric edges which are parallel to ℓ_g.*

In other words, an edge is substitution symmetric if whenever you substitute any pattern containing it, the tiles which intersect the substituted edge's interior are fixed by reflection over the edge. Notice that the group element g is fixed once you specify the edge since there is only one line which it parallel to it. Since the substitution commutes with g, substituting a substitution symmetric edge results in a union of substitution symmetric edges. We define

$$\Gamma_g = \{x \in \Gamma \mid x \in e, e \in SS(g)\}.$$

If $gT = T$ with $T = (x_1, x_2, \dots)$, again we have that $gx_i = x_i$ for all i. Thus x_i has to be contained in an edge parallel to ℓ_g. The converse is not true in general. For example, consider the collaring procedure from Remark 2.5.4 where each prototile

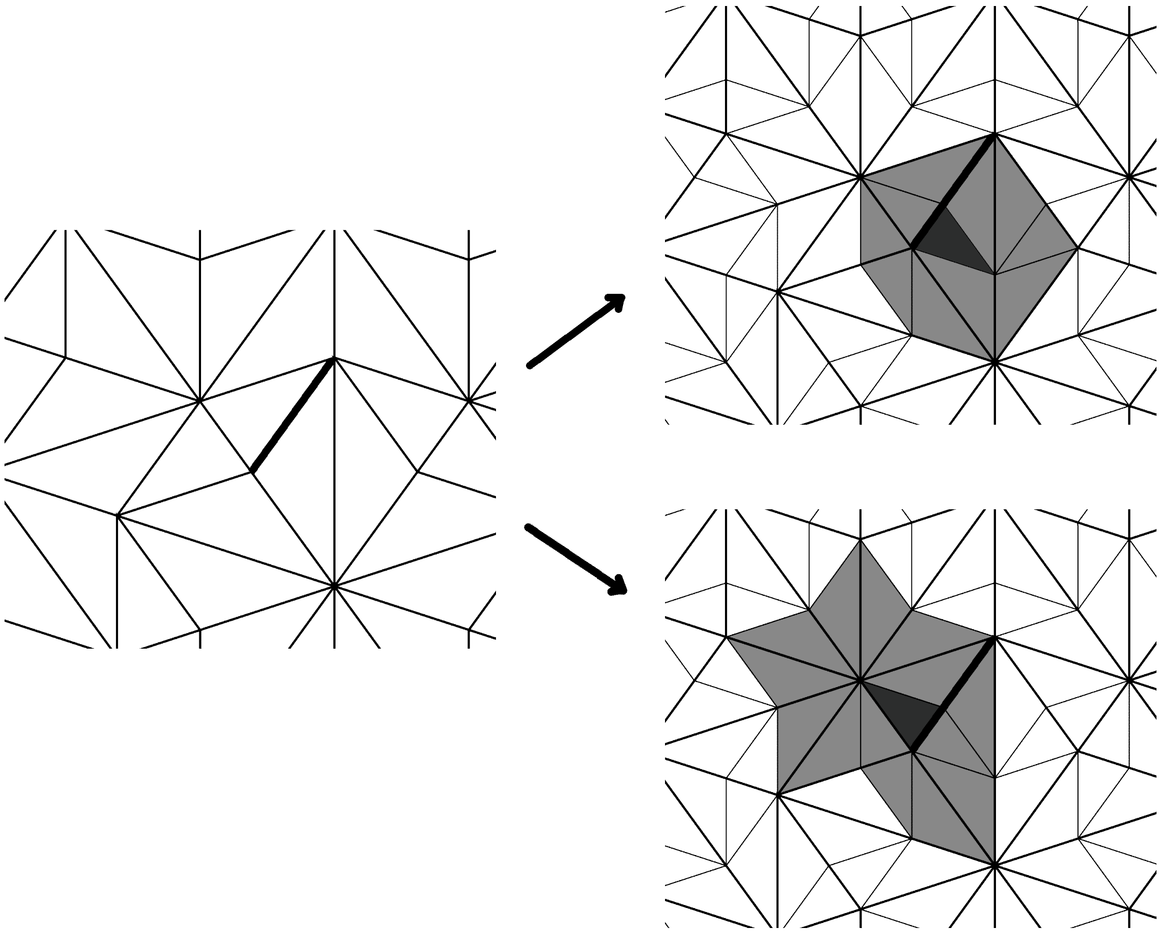


Figure 5.2: After collaring, edges may fail to be substitution symmetric.

was replaced by finitely many labeled copies with the label depending on the possible configurations of tiles surrounding them in tilings. As Figure 5.2 shows, collaring Penrose tiles results in a substitution for which not all edges are substitution symmetric.

Lemma 5.3.5 *Suppose that we have $T = (x_i)_{i\in\mathbb{N}}$ such that x_i is in a substitution symmetric edge for all i. Then there exists a reflection g such that x_i is in a substitution symmetric edge in the direction ℓ_g for all i and $gT = T$.*

Proof: Since there are only a finite number of reflections possible, there exists a strictly increasing sequence of positive integers $(i_n)_{n\in\mathbb{N}}$ and a group element g such that $x_{i_n} \in \Gamma_g$ for all $n \in \mathbb{N}$. In addition, if $x_{j+1} \in \Gamma_g$, $\gamma(x_{j+1}) = x_j \in \Gamma_g$ as well. Together, these imply that $x_i \in \Gamma_g$ for all g.

We now want to show that $gT = T$, or equivalently that $x_i = gx_i$ for all i. If $x_{i+1} \in \Gamma_g$ then $\gamma(x_{i+1}) = x_i$ and whenever we have a tiling $S = (x_i, y_1, y_2, \dots)$, we must have that $\omega(S)(0) = g\omega(S)(0)$. This means precisely that $gx_i = x_i$. Since i was arbitrary, $gx_i = x_i$ for all i. ∎

We state but do not use the following result.

Lemma 5.3.6 *The elements of $SS(g)$ are the prototiles for a one-dimensional substitution tiling system. The tiling space of this system is denoted Ω_g. This is a closed subspace of Ω. In addition,*

$$\Omega_g = \lim_{\leftarrow} \left(\Gamma_g \xleftarrow{\gamma} \Gamma_g \right).$$

Proof: The second statement is by definition; Ω_g is the subspace of the inverse limit such that each element in the sequence is in $SS(g)$. ∎

5.4 The Semidirect Product Groupoid

If G is a symmetry group for substitution tiling system $(\mathcal{P}, \omega)$, then G acts on the groupoids associated to $(\mathcal{P}, \omega)$.

Lemma 5.4.1 *Let G be a symmetry group for $(\mathcal{P}, \omega)$, let Ω be the tiling space associated to $(\mathcal{P}, \omega)$ and let $(\Omega, \mathbb{R}^2)$ denote the transformation group groupoid associated to the translation action on Ω. Then $\alpha : G \to \mathrm{Aut}(\Omega, \mathbb{R}^2)$ defined for $g \in G$, $(T, x) \in (\Omega, \mathbb{R}^2)$, by*

$$\alpha_g(T, x) = (gT, gx)$$

is a homomorphism. Furthermore, each α_g is continuous.

Proof: If $g, h \in G$, then $\alpha_{gh}(T, x) = (ghT, ghx) = \alpha_g(hT, hx) = \alpha_g \circ \alpha_h(T, x)$. If $g \in G$, $(T, x) \in (\Omega, \mathbb{R}^2)$, then $\alpha_g(T, x) = (gT, gx)$. This is continuous in each of the coordinates hence continuous. It is also easy to see that this has continuous inverse given by $\alpha_{g^{-1}}$.

If $(T, x), (T + x, y) \in (\Omega, \mathbb{R}^2)$, then

$$\begin{aligned}
\alpha_g((T,x)(T+x,y)) &= \alpha_g(T, x+y) \\
&= (gT, gx + gy) \\
&= (gT, gx)(gT + gx, gy) \\
&= \alpha_g(T,x)\alpha_g(T+x,y) \\
\alpha_g((T,x)^{-1}) &= \alpha_g(T+x, -x) \\
&= (gT + gx, -gx) \\
&= (\alpha_g(T,x))^{-1}.
\end{aligned}$$

∎

Since G acts on the prototiles, the action above restricts to an action on $\mathcal{R}_{\text{punc}}$. That is, $\alpha_g|_{\mathcal{R}_{\text{punc}}}$ is a continuous automorphism of $\mathcal{R}_{\text{punc}}$, and $g \mapsto \alpha_g|_{\mathcal{R}_{\text{punc}}}$ defines a group homomorphism from G to $\text{Aut}(\mathcal{R}_{\text{punc}})$. From now on, we will simply write $g\gamma$ in place of $\alpha_g(\gamma)$ for $\gamma \in \mathcal{R}_{\text{punc}}$ or $\gamma \in (\Omega, \mathbb{R}^2)$.

If G is a symmetry group for $(\mathcal{P}, \omega)$, then G acts on $\mathbb{R}^2$ and so we can form the semidirect product group $\mathbb{R}^2 \rtimes G$. As a set, $\mathbb{R}^2 \rtimes G$ is $\mathbb{R}^2 \times G$ and it becomes a group under the binary operation $(x, g)(y, h) = (x + gy, gh)$ and inverse $(x, g)^{-1} = (-g^{-1}x, g^{-1})$. Since $(x, g) = (x, e)(0, g)$, $\mathbb{R}^2 \rtimes G$ acts on the left on Ω via the formula

$$(x, g)T = gT + x,$$

and Lemma 5.1.6 together with the discussion above imply that it acts by homeomorphisms on Ω. Hence we can form the transformation group groupoid $\mathscr{G} = (\Omega, \mathbb{R}^2 \rtimes G)$.

The unit space of $\mathscr{G}$ is Ω. The space $\Omega_{\text{punc}} \subset \Omega$ is closed and intersects every orbit in G. Hence to use Example 3.3.6 we must show the following.

Lemma 5.4.2 *Restricted to $\mathscr{G}_{\Omega_{\text{punc}}} = \{\gamma \in \mathscr{G} \mid s(\gamma) \in \Omega_{\text{punc}}\}$, the maps r and s are both open.*

Proof: Suppose that $V \subset \mathscr{G}_{\Omega_{\text{punc}}}$ is open. Then there exists W open in $\mathscr{G}$ such that $V = W \cap \mathscr{G}_{\Omega_{\text{punc}}}$. Let $T \in r(V)$. Then we can find $g \in \mathbb{R}^2 \rtimes G$ such that $(g^{-1}T, g) \in V$. Since W is open, we can find $U \subset \Omega, H \subset \mathbb{R}^2 \rtimes G$ both open such that

$$g^{-1}T \in U, \qquad g \in H, \qquad U \times H \subset W.$$

Now we have

$$\begin{aligned} r\left((U \times H) \cap \mathscr{G}_{\Omega_{\text{punc}}}\right) &= \left\{T' \in \bigcup_{h \in H} hU \mid T' \in \Omega_{\text{punc}}\right\} \\ &= \bigcup_{h \in H} hU \cap \Omega_{\text{punc}} \end{aligned}$$

This set is open in the relative topology, is contained in $r(V)$ and contains T. Therefore the restriction of r is open.

To prove s is open, again take $V = W \cap \mathscr{G}_{\Omega_{\text{punc}}}$ open. Let $T \in s(V)$ and find $(x, g) \in \mathbb{R}^2 \rtimes G$ such that $(T, (x, g)) \in V$. In particular $gT + x \in \Omega_{\text{punc}}$. Find $\varepsilon > 0$ such that

$$B^{\Omega}_{\varepsilon}(T) \times B_{\frac{\varepsilon}{2}}(x, g) \subset W.$$

By taking ε smaller if necessary, we may assume that $|x| < \frac{1}{2\varepsilon}$, $B_{\frac{\varepsilon}{2}}(x, g) = \{(x', g) \mid |x - x'| < \frac{\varepsilon}{2}\}$ and $S(B_{2\varepsilon}(0)) = S(0)$ for all $S \in \Omega_{\text{punc}}$.
We claim that $B^{\Omega}_{\varepsilon/4}(T) \subset s(V)$. Let $T' \in B^{\Omega}_{\varepsilon/4}(T)$. Then by definition there exists x' with $|x'| < \frac{\varepsilon}{2}$ such that

$$(T' + x')(B_{\frac{4}{\varepsilon}}(0)) = T(B_{\frac{4}{\varepsilon}}(0)).$$

Since $|x| < \frac{4}{\varepsilon}$ and $(x, g)T$ is punctured, $(x, g)(T' + x')$ must be as well. But

$$(x, g)(T' + x') = gT' + gx' + x = (gx' + x, g)T'.$$

We have that $T' \in B_\varepsilon^\Omega(T)$ and $(gx'+x, g) \in B_{\varepsilon/2}(x,g)$ and $(gx'+x,g)T' \in \Omega_{\mathrm{punc}}$. Thus $T' \in s(V)$, hence s is an open map. ∎

We now prove that $(\Omega, \mathbb{R}^2 \rtimes G)$ and $(\Omega, \mathbb{R}^2) \rtimes G$ are naturally isomorphic.

Lemma 5.4.3 *If $\mathscr{H}$ is the semidirect product groupoid $(\Omega, \mathbb{R}^2) \rtimes G$, then the map $\varphi : \mathscr{G} \to \mathscr{H}$ defined by*

$$\varphi((T,(x,g))) = \left((T, g^{-1}x), g^{-1}\right), \quad \text{for } (T,(x,g)) \in \mathscr{G},$$

is a topological groupoid isomorphism.

Proof: Let us first prove that φ is a groupoid homomorphism. If $\gamma, \eta \in \mathscr{G}$ form a composable pair, then there exist $g \in G$, $x \in \mathbb{R}^2$ and $T \in \Omega$ such that

$$\gamma = (T,(x,g)), \qquad \eta = (gT + x, (y,h)), \qquad \gamma\eta = (T, (hx+y, hg)).$$

Then

$$\begin{aligned}\varphi(\gamma\eta) &= \left((T, (hg)^{-1}(hx+y)), (hg)^{-1}\right)\\ &= \left((T, g^{-1}x + g^{-1}h^{-1}y), g^{-1}h^{-1}\right)\end{aligned}$$

We calculate

$$\begin{aligned}\varphi(\gamma) &= \left((T, g^{-1}x), g^{-1}\right)\\ \varphi(\eta) &= \left((gT + x, h^{-1}y), h^{-1}\right)\\ &= \left((g(T + g^{-1}x), h^{-1}y), h^{-1}\right).\end{aligned}$$

We have that $r(\varphi(\gamma)) = gT + x = s(\varphi(\eta))$ so $(\varphi(\gamma), \varphi(\eta))$ is composable in $\mathscr{H}$. By the definition of the product in the semidirect product groupoid we have

$$\begin{aligned}\varphi(\gamma)\varphi(\eta) &= \left((T, g^{-1}x), g^{-1}\right)\left((g(T+g^{-1}x), h^{-1}y), h^{-1}\right)\\ &= \left((T, g^{-1}x), g^{-1}\right)\left((g^{-1}\cdot(T+g^{-1}x), g^{-1}h^{-1}y), h^{-1}\right)\end{aligned}$$

$$\begin{aligned}
&= \big(((T, g^{-1}x)(T + g^{-1}x), g^{-1}h^{-1}y), g^{-1}h^{-1}\big) \\
&= \big((T, g^{-1}x + g^{-1}h^{-1}y), g^{-1}h^{-1}\big) \\
&= \varphi(\gamma\eta).
\end{aligned}$$

The map from $\Omega \times \mathbb{R}^2 \times G \to \Omega \times \mathbb{R}^2 \times G$ given by $(T, x, g) \mapsto (T, g^{-1}x, g^{-1})$ is clearly a bijection (in fact as a set map it is its own inverse). Under this map, an open set $U \times V \times W$ gets mapped to $U \times W^{-1}V \times W^{-1}$. The sets U and W^{-1} are open and

$$W^{-1}V = \bigcup_{g \in W} g^{-1}V$$

is a union of open sets, hence open. Thus φ is a homeomorphism that is also a groupoid homomorphism, hence it is a topological groupoid isomorphism. ∎

The above lemma gives us the following.

Corollary 5.4.4 *If $\mathscr{G} = (\Omega, \mathbb{R}^2 \rtimes G)$, then $\mathscr{G}_{\Omega_{\mathrm{punc}}}^{\Omega_{\mathrm{punc}}} \cong \mathcal{R}_{\mathrm{punc}} \rtimes G$.*

Theorem 5.4.5 *Let G be a symmetry group for $(\mathcal{P}, \omega)$, and let Ω be the tiling space associated to $(\mathcal{P}, \omega)$. Then the groupoids $\mathscr{G} = (\Omega, \mathbb{R}^2 \rtimes G)$ and $\mathcal{R}_{\mathrm{punc}} \rtimes G$ are equivalent in the sense of Definition 3.3.5. The space $\mathscr{G}_{\Omega_{\mathrm{punc}}}$ is the $(\mathscr{G}, \mathcal{R}_{\mathrm{punc}} \rtimes G)$-equivalence.*

Proof: This combines Example 3.3.6 with the Corollary 5.4.4. ∎

By Proposition 3.2.5, $\mathcal{R}_{\mathrm{punc}} \rtimes G$ is r-discrete and admits a Haar system. We now show that if G acts freely on $\mathcal{P}$ then $\mathcal{R}_{\mathrm{punc}} \rtimes G$ is almost AF in the sense of Definition 3.2.10.

Lemma 5.4.6 *Suppose that G is a finite symmetry group for $(\mathcal{P}, \omega)$ and that G acts freely on $\mathcal{P}$. Let $\mathcal{R}_{AF}$ be that AF Cantor groupoid associated to $(\mathcal{P}, \omega)$. Then $\mathcal{R}_{AF} \rtimes G$ is an AF Cantor groupoid in the sense of Definition 3.2.8.*

Proof: It is enough to show that $\mathcal{R}_{AF} \rtimes G$ is an increasing union of compact open principal subgroupoids each with unit space Ω_{punc}. For $N \in \mathbb{N}$ consider $\mathcal{R}_N \rtimes G$. Let

$((T_1, T_1+x_1), g_1)$ and $((T_2, T_2+x_2), g_2)$ be elements of $\mathcal{R}_N \rtimes G$ and suppose they have the same range and source. As for $i = 1, 2$,

$$r((T_i, T_i + x_i), g_i) = (r(T_i, T_i + x_i), e) = ((T_i, T_i), e),$$

then $T_1 = T_2$. Let $T = T_1 = T_2$, making our two elements $((T, T + x_1), g_1)$ and $((T, T + x_2), g_2)$. Since for $i = 1, 2$,

$$s((T, T + x_i), g_i) = (s(T, T + x_i) \cdot g_i, e) = (g_i^{-1}(T + x_i), g_i^{-1}(T + x_i), e),$$

then $g_1^{-1}(T + x_1) = g_2^{-1}(T + x_2)$, or

$$T + x_1 = g_1 g_2^{-1}(T + x_2)$$

The pairs $(T, T + x_1)$ and $(T, T + x_2)$ are both in $\mathcal{R}_N$. This means that $\omega^{-N}(T + x_1)$ and $\omega^{-N}(T+x_1)$ are both tilings with the same tile around the origin, only translated. That is to say that $\omega^{-N}(T + x_1)(0) = t$ and $\omega^{-N}(T + x_2)(0) = t + \lambda^{-N}(x_1 - x_2)$. But the above then implies that $t = g_1 g_2^{-1}(t + \lambda^{-N}(x_1 - x_2))$ which is only possible if $x_2 = x_1$ and $g_1 = g_2$ since G acts freely on the prototiles. Thus each $\mathcal{R}_N \rtimes G$ is principal. It is easy to see that

$$\mathcal{R}_{AF} \rtimes G = \bigcup_{N \in \mathbb{N}} \mathcal{R}_N \rtimes G$$

and so $\mathcal{R}_{AF} \rtimes G$ is an increasing union of compact principal groupoids. Since $\mathcal{R}_N \rtimes G$ inherits the product topology from $\mathcal{R}_N \times G$ and $\mathcal{R}_N$ is open in $\mathcal{R}_{N+1}$, we must have that $\mathcal{R}_N \rtimes G$ is open in $\mathcal{R}_{N+1} \rtimes G$. This completes the proof. ∎

The following lemma helps us simplify verification of Definition 3.2.10.

Lemma 5.4.7 *Suppose that G is a finite symmetry group for $(\mathcal{P}, \omega)$. Then the only open invariant subsets of Ω_{punc} with respect to the groupoid $\mathcal{R}_{\text{punc}} \rtimes G$ are $\emptyset$ and Ω_{punc}. Hence, $C_r^*(\mathcal{R}_{\text{punc}} \rtimes G)$ is simple.*

Proof: Every $\mathcal{R}_{\text{punc}} \rtimes G$-orbit in Ω_{punc} is the union of $\mathcal{R}_{\text{punc}}$-orbits, and each $\mathcal{R}_{\text{punc}}$-orbit is dense in Ω_{punc}. Hence every $\mathcal{R}_{\text{punc}} \rtimes G$-orbit is dense. That $C_r^*(\mathcal{R}_{\text{punc}} \rtimes G)$ is simple follows from Lemma 4.3.4. ∎

We are now in a position to prove the following important theorem.

Theorem 5.4.8 *Suppose that G is a finite symmetry group for $(\mathcal{P}, \omega)$ and that G acts freely on $\mathcal{P}$. Then $\mathcal{R}_{\text{punc}} \rtimes G$ is an almost AF Cantor groupoid with respect to the sub-AF groupoid $\mathcal{R}_{AF} \rtimes G$.*

Proof: By Lemma 5.4.7, $C_r^*(\mathcal{R}_{\text{punc}} \rtimes G)$ is simple. Thus by Proposition 4.3.3 we only need to check Condition 1 of Definition 3.2.10.

Consider the sets

$$L_r = \{(T, T-x) \in \mathcal{R}_{\text{punc}} \setminus \mathcal{R}_{AF} \mid |x| \leq r\}$$

$$M_r = \{((T, T-x), g) \in (\mathcal{R}_{\text{punc}} \rtimes G \setminus \mathcal{R}_{AF} \rtimes G) \mid |x| \leq r\}$$

Then $M_r = L_r \times G$. In [51] it is shown that L_r is compact and that any compact set in $\mathcal{R}_{\text{punc}} \setminus \mathcal{R}_{AF}$ is contained in L_r for some suitable r. Notice that $r(M_r) = r(L_r)$. Suppose that $K \subset (\mathcal{R}_{\text{punc}} \rtimes G \setminus \mathcal{R}_{AF} \rtimes G)$ is compact. Then

$$K = \bigcup_{g \in G} K_g \quad \text{where} \quad K_g = K \cap (\mathcal{R}_{\text{punc}} \setminus \mathcal{R}_{AF}) \times \{g\}.$$

Each of the K_g is compact because $(\mathcal{R}_{\text{punc}} \setminus \mathcal{R}_{AF}) \times \{g\}$ is closed. If $\pi_1 : \mathcal{R}_{\text{punc}} \rtimes G \to \mathcal{R}_{\text{punc}}$ is the usual projection, then $\pi_1(K_g)$ is compact, and hence included in L_{r_g} for some r_g. Let $r = \max\{r_g\}$ and consider M_r. We have

$$K = \bigcup_{g \in G} K_g \subset \bigcup_{g \in G} (L_{r_g} \times \{g\}) \subset \bigcup_{g \in G} (L_r \times \{g\}) = M_r$$

giving us that $K \subset M_r$ and thus $r(K) \subset r(M_r) = r(L_r)$. Since $r(L_r)$ is thin, $r(K)$ must also be thin in the sense of Definition 3.2.9. Thus Condition 1 of Definition 3.2.10 is satisfied and we have that $\mathcal{R}_{\text{punc}} \rtimes G$ is an almost AF Cantor groupoid. ∎

The above theorem yields the following, which we will use in Section 5.7.

Corollary 5.4.9 *Suppose that G is a finite symmetry group for $(\mathcal{P}, \omega)$ acting freely on $\mathcal{P}$. Then the C^*-algebra $C_r^*(\mathcal{R}_{\text{punc}} \rtimes G)$ has real rank zero, stable rank one, and order on its projections is determined by traces.*

Proof: This follows from the above theorem together with [47] Theorem 4.6, Theorem 5.2, Corollary 5.4 and Proposition 2.11. ∎

5.5 The Crossed Product $C_r^*(\mathcal{R}_{\text{punc}}) \rtimes G$

Finite symmetry groups act on the C*-algebras associated to the tilings. In this section, we prove that the groupoid C*-algebra $C_r^*(\mathcal{R}_{\text{punc}} \rtimes G)$ is isomorphic to the crossed product C*-algebra $C_r^*(\mathcal{R}_{\text{punc}}) \rtimes G$. Then in Theorem 5.5.2 we compile results from the previous section and come up with a list of C*-algebras strongly Morita equivalent to $C_r^*(\mathcal{R}_{\text{punc}} \rtimes G)$.

The first proposition of this section is certainly well-known, but for the sake of completeness we include the proof.

Proposition 5.5.1 *Let $\mathscr{G}$ be a locally compact Hausdorff r-discrete groupoid which admits a Haar system, let G be a finite group, and let $\alpha : G \to \mathrm{Aut}(\mathscr{G})$ be a homomorphism. Recall from Proposition 3.2.5 that $\mathscr{G} \rtimes_\alpha G$ is r-discrete and admits Haar system. Then:*

1. *α induces an action $\beta : G \to \mathrm{Aut}(C_r^*(\mathscr{G}))$ such that*

$$\beta_g(f)(\gamma) = f(\alpha_g^{-1}(\gamma)), \qquad f \in C_c(\mathscr{G}), \gamma \in \mathscr{G}. \tag{5.5.1}$$

2. *There exists a $*$-isomorphism*

$$\Phi : C_r^*(\mathscr{G}) \rtimes_\beta G \to C_r^*(\mathscr{G} \rtimes_\alpha G)$$

such that

$$\Phi(f\delta_h)(\gamma, g) = \begin{cases} f(\gamma) & \text{if } g = h \\ 0 & \text{otherwise,} \end{cases} \qquad f \in C_c(\mathscr{G}), \gamma \in \mathscr{G}, g, h \in G. \tag{5.5.2}$$

Proof: Let X be the unit space of $\mathscr{G}$, that is $X = \mathscr{G}^{(0)}$. We recall from the definition of the semidirect product groupoid (Definition 3.1.8) that the unit space of $\mathscr{G} \rtimes_\alpha G$ is also X and that $r(\gamma, g) = r(\gamma)$ for all $\gamma \in \mathscr{G}$ and $g \in G$. For $\gamma \in \mathscr{G}$ and $g \in G$ we will also use the notation $\gamma \cdot g := \alpha_{g^{-1}}(\gamma)$ introduced in Definition 3.1.8 and note that it defines a right action.

1. We first prove that (5.5.1) defines a $*$-automorphism of the $*$-algebra $C_c(\mathscr{G})$. Let $f_1, f_2 \in C_c(\mathscr{G})$, $g \in G$ and $\gamma \in \mathscr{G}$. Then

$$\begin{aligned}
\beta_g(f_1 \star f_2)(\gamma) &= f_1 \star f_2(\gamma \cdot g) \\
&= \sum_{\substack{\eta \in \mathscr{G} \\ r(\eta) = s(\gamma \cdot g)}} f_1((\gamma \cdot g)\eta) f_2(\eta^{-1}) \\
&= \sum_{\substack{\eta' \in \mathscr{G} \\ r(\eta') = s(\gamma)}} f_1((\gamma\eta') \cdot g) f_2(\eta'^{-1} \cdot g) \quad (\text{letting } \eta' = \eta \cdot g^{-1}) \\
&= \sum_{\substack{\eta' \in \mathscr{G} \\ r(\eta') = s(\gamma)}} \beta_g(f_1)(\gamma\eta')\beta_g(f_2)(\eta'^{-1}) \\
&= \beta_g(f_1) \star \beta_g(f_2)(\gamma).
\end{aligned}$$

$$\begin{aligned}
\beta_g(f_1^*)(\gamma) &= f_1^*(\gamma \cdot g) \\
&= \overline{f_1\left((\gamma \cdot g)^{-1}\right)} \\
&= \overline{f_1\left(\gamma^{-1} \cdot g\right)} \\
&= \overline{\beta_g(f_1)\left(\gamma^{-1}\right)} \\
&= \beta_g(f_1)(\gamma)^*.
\end{aligned}$$

For each $g \in G$, each β_g is clearly linear and bijective, and so is a bijective $*$-algebra homomorphism. To prove continuity, we first let $g \in G$, $x \in X$ and $\xi \in l^2(\mathscr{G}_x)$. Then

$$\begin{aligned}\|\xi\|_2^2 &= \sum_{\eta \in \mathscr{G}_x} |\xi(\eta)|^2 \\ &= \sum_{\alpha_g(\eta) \in \mathscr{G}_{\alpha_g(x)}} |\xi(\eta)|^2 \\ &= \sum_{\gamma \in \mathscr{G}_{\alpha_g(x)}} |\xi(\gamma \cdot g)|^2\end{aligned}$$

Thus we may define $\beta'_g : l^2(\mathscr{G}_x) \to l^2(\mathscr{G}_{\alpha_g(x)})$ by

$$\beta'_g(\xi)(\gamma) = \xi(\gamma \cdot g), \qquad \gamma \in \mathscr{G}_{\alpha_g(x)},$$

and $\beta'_g \in B\left(l^2(\mathscr{G}_x), l^2(\mathscr{G}_{\alpha_g(x)})\right)$ with $\|\beta'_g\| = 1$. We are now in a position to prove that each β_g is continuous. Let $x \in X$ and consider the representation

$$\lambda_x(f) : C_c(\mathscr{G}) \to B(l^2(\mathscr{G}_x))$$

$$\lambda_x(f)\xi(\gamma) = \sum_{\eta \in \mathscr{G}_x} f(\gamma\eta^{-1})\xi(\eta)$$

from Proposition 4.3.2. For $f \in C_c(\mathscr{G})$, $x \in X$, $\xi \in l^2(\mathscr{G}_x)$, $\gamma \in \mathscr{G}_x$ and $g \in G$ we have

$$\begin{aligned}\lambda_x(\beta_g(f))\xi(\gamma) &= \sum_{\eta \in \mathscr{G}_x} \beta_g(f)(\gamma\eta^{-1})\xi(\eta) \\ &= \sum_{\eta \in \mathscr{G}_x} f((\gamma\eta^{-1}) \cdot g)\xi(\eta \cdot gg^{-1}) \\ &= \sum_{\eta \in \mathscr{G}_x} f((\gamma \cdot g)(\eta^{-1} \cdot g))\beta'_{g^{-1}}(\xi)(\eta \cdot g) \\ &= \sum_{\eta' \in \mathscr{G}_{x \cdot g}} f((\gamma \cdot g)\eta'^{-1})\beta'_{g^{-1}}(\xi)(\eta') \quad \text{(letting } \eta' = \eta \cdot g) \\ &= \lambda_{x \cdot g}(f)\beta'_{g^{-1}}(\xi)(\gamma \cdot g) \\ &= \beta'_g(\lambda_{x \cdot g}(f)\beta'_{g^{-1}}(\xi))(\gamma).\end{aligned}$$

If $\|\xi\|_2 = 1$, we have

$$\|\lambda_x(\beta_g(f))\xi\|_2 = \|\beta'_g(\lambda_{x \cdot g}(f)\beta'_{g^{-1}})(\xi)\|_2$$

$$
\begin{aligned}
&\leq \|\beta'_g(\lambda_{x\cdot g}(f)\beta'_{g^{-1}})\| \\
&= \|\lambda_{x\cdot g}(f)\beta'_{g^{-1}}\| \\
&\leq \|\lambda_{x\cdot g}(f)\| \\
&\leq \|f\|_{\text{red}}.
\end{aligned}
$$

Now, taking the supremum of the left side over $\xi \in l^2(\mathscr{G}_x)$ of norm 1 and $x \in \mathscr{G}$ gives us that $\|\beta_g(f)\|_{\text{red}} \leq \|f\|_{\text{red}}$. Hence β_g is continuous for each g and extends to a $*$-automorphism of $C_r^*(\mathscr{G})$. That β is a homomorphism is obtained by calculating

$$\beta_{gh}(f)(\gamma) = f(\gamma \cdot gh) = \beta_h(f)(\gamma \cdot g) = \beta_g(\beta_h(f))(\gamma) = \beta_g \circ \beta_h(f)(\gamma)$$

for all $\gamma \in \mathscr{G}$, $g, h \in G$, and $f \in C_c(\mathscr{G})$. That β is continuous is immediate due to G being discrete.

2. The $*$-algebra

$$C_c(\mathscr{G})G = \left\{ \sum_{g\in G} f_g\delta_g \mid f_g \in C_c(\mathscr{G}) \right\}$$

(see Example 4.2.4) is a dense subalgebra of $C_r^*(\mathscr{G}) \rtimes_\beta G$. The map Φ is defined on elements $f\delta_g$ and extended by linearity to $C_c(\mathscr{G})G$. We check that Φ as defined in (5.5.2) is a $*$-automorphism of the $*$-algebra $C_c(\mathscr{G})G$; if $f_1, f_2 \in C_c(\mathscr{G})$, $g, g_1, g_2 \in G$ and $\gamma \in \mathscr{G}$, then we have

$$
\begin{aligned}
\Phi(f_1\delta_{g_1} f_2\delta_{g_2})(\gamma, g) &= \Phi(f_1\beta_{g_1}(f_2)\delta_{g_1g_2})(\gamma, g) \\
&= \begin{cases} f_1\beta_{g_1}(f_2)(\gamma) & \text{if } g = g_1g_2 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \displaystyle\sum_{\substack{\eta\in\mathscr{G} \\ r(\eta)=s(\gamma)}} f_1(\gamma\eta)\beta_{g_1}(f_2)(\eta^{-1}) & \text{if } g = g_1g_2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

$$= \begin{cases} \sum_{\substack{\eta \in \mathscr{G} \\ r(\eta)=s(\gamma)}} f_1(\gamma\eta) f_2(\eta^{-1} \cdot g_1) & \text{if } g = g_1 g_2 \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$\Phi(f_1\delta_{g_1}) \star \Phi(f_2\delta_{g_2})(\gamma, g) = \sum_{\substack{(\eta,h) \in \mathscr{G} \rtimes G \\ r(\eta,h)=s(\gamma,g)}} \Phi(f_1\delta_{g_1})((\gamma, g), (\eta, h)) \Phi(f_2\delta_{g_2})((\eta, h)^{-1})$$

Now, we have

$$r(\eta, h) = r(\eta), \quad s(\gamma, g) = s(\gamma) \cdot g, \quad (\eta, h)^{-1} = (\eta^{-1} \cdot h, h^{-1}).$$

Hence, $\Phi(f_1\delta_{g_1})((\gamma, g), (\eta, h))$ is only nonzero if $g_1 = gh$ and $\Phi(f_2\delta_{g_2})((\eta, h)^{-1})$ is nonzero if $g_2 = h^{-1}$. These together imply that $g = g_1g_2$. We also have

$$(\gamma, g)(\eta, h) = (\gamma, g)((\eta \cdot g^{-1}) \cdot g, g_2^{-1}) = (\gamma(\eta \cdot g^{-1}), gg_2^{-1}) = (\gamma(\eta \cdot g^{-1}), g_1).$$

In this case we have that

$$\Phi(f_1\delta_{g_1})(\gamma(\eta \cdot g^{-1}), g_1) = f_1(\gamma(\eta \cdot g^{-1}))$$

and

$$\Phi(f_2\delta_{g_2})(\eta^{-1} \cdot g_2^{-1}, g_2) = f_2(\eta^{-1} \cdot g_2^{-1}).$$

Hence we obtain

$$\Phi(f_1\delta_{g_1}) \star \Phi(f_2\delta_{g_2})(\gamma, g) = \begin{cases} \sum_{\substack{\eta \in \mathscr{G} \\ r(\eta)=s(\gamma)\cdot g}} f_1(\gamma(\eta \cdot g^{-1})) f_2(\eta^{-1} \cdot g_2^{-1}) & \text{if } g = g_1 g_2 \\ 0 & \text{otherwise.} \end{cases}$$

If we let $\nu = \eta \cdot g^{-1}$, then $r(\nu) = s(\gamma)$ and straightforward computation shows that $\eta^{-1} \cdot g_2^{-1} = \nu \cdot g_1$. Thus,

$$\Phi(f_1\delta_{g_1}) \star \Phi(f_2\delta_{g_2})(\gamma, g) = \begin{cases} \sum_{\substack{\nu \in \mathscr{G} \\ r(\nu)=s(\gamma)}} f_1(\gamma\nu) f_2(\nu \cdot g_1) & \text{if } g = g_1 g_2 \\ 0 & \text{otherwise} \end{cases}$$

$$= \quad \Phi(f_1\delta_{g_1}f_2\delta_{g_2})(\gamma, g).$$

Thus Φ respects the product. Now let $f \in C_c(\mathscr{G})$, $\gamma \in \mathscr{G}$ and $g, h \in G$. Then

$$\begin{aligned}
\Phi((f\delta_h)^*)(\gamma, g) &= \Phi(\beta_{h^{-1}}(f^*)\delta_{h^{-1}})(\gamma, g) \\
&= \begin{cases} \beta_{h^{-1}}(f^*)(\gamma) & \text{if } g = h^{-1} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} f^*(\gamma \cdot h^{-1}) & \text{if } g = h^{-1} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \overline{f(\gamma^{-1} \cdot h^{-1})} & \text{if } g = h^{-1} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \overline{f(\gamma^{-1} \cdot g)} & \text{if } g = h^{-1} \\ 0 & \text{otherwise} \end{cases} \\
&= \overline{\Phi(f\delta_h)(\gamma^{-1} \cdot g, g^{-1})} \\
&= \overline{\Phi(f\delta_h)\left((\gamma, g)^{-1}\right)} \\
&= \Phi\left((f\delta_h)^*\right)(\gamma, g).
\end{aligned}$$

The map Φ is clearly injective. Let $f \in C_c(\mathscr{G} \rtimes_\alpha G)$, and suppose that K is a compact subset of $\mathscr{G} \rtimes_\alpha G$ such that $\mathrm{supp}(f) \subset K$. Since the set $\mathscr{G} \times \{g\}$ is closed in $\mathscr{G} \rtimes_\alpha G$ for all $g \in G$, the complex-valued function

$$\mathfrak{e}_g(f) : \mathscr{G} \to \mathbb{C}$$

$$\gamma \mapsto f(\gamma, g)$$

is continuous and supported on $K \cap \mathscr{G} \times \{g\}$, and hence is an element of $C_c(\mathscr{G})$ for each $g \in G$. We see that if $(\gamma, h) \in \mathscr{G} \rtimes_\alpha G$ then

$$\Phi\left(\sum_{g \in G} \mathfrak{e}_g(f)\delta_g\right)(\gamma, h) = \mathfrak{e}_h(f)(\gamma) = f(\gamma, h)$$

and so Φ is surjective. We also note that

$$\mathfrak{e}_g\left(\Phi\left(\sum_{h\in G} f_h\delta_g\right)\right) = f_g. \tag{5.5.3}$$

It remains to show that Φ is continuous. The norm on $C_c(\mathscr{G} \rtimes_\alpha G)$ is determined by the representations

$$\pi_x : C_c(\mathscr{G} \rtimes_\alpha G) \to B(l^2(\mathscr{G} \rtimes_\alpha G_x))$$

$$\pi_x(f)\xi(\gamma, g) = \sum_{(\eta,h)\in\mathscr{G}\rtimes_\alpha G_x} f((\gamma, g)(\eta, h)^{-1})\xi(\eta, h)$$

for $x \in X$, $\xi \in l^2(\mathscr{G} \rtimes_\alpha G_x)$ and $(\gamma, g) \in \mathscr{G} \rtimes_\alpha G_x$. If $\xi \in l^2(\mathscr{G} \rtimes_\alpha G_x)$ then the function $\xi_g(\gamma) := \xi(\gamma \cdot g^{-1}, g)$ is an element of $l^2(\mathscr{G}_x)$ and $\|\xi_h\|_2 \leq \|\xi\|_2$ for all $h \in G$. Straightforward computation also shows that

$$(\gamma, g)(\eta, h)^{-1} = (\gamma(\eta^{-1} \cdot hg^{-1}), gh^{-1}).$$

Thus, for $k \in C_c(\mathscr{G} \rtimes_\alpha G)$

$$\begin{aligned}
\pi_x(k)\xi(\gamma, g) &= \sum_{(\eta,h)\in\mathscr{G}\rtimes_\alpha G_x} k(\gamma(\eta^{-1} \cdot hg^{-1}), gh^{-1})\xi(\eta, h) \\
&= \sum_{h\in G}\left(\sum_{\substack{\eta\in\mathscr{G} \\ s(\eta)=x\cdot h^{-1}}} k_{gh^{-1}}(\gamma(\eta^{-1} \cdot hg^{-1}))\xi_h(\eta \cdot h)\right) \\
&= \sum_{h\in G}\left(\sum_{\substack{\eta\in\mathscr{G} \\ s(\eta)=x\cdot h^{-1}}} \beta_{g^{-1}}(k_{gh^{-1}})((\gamma \cdot g)(\eta^{-1} \cdot h))\xi_h(\eta \cdot h)\right) \\
&= \sum_{h\in G}\left(\sum_{\substack{\nu\in\mathscr{G} \\ s(\nu)=x}} \beta_{g^{-1}}(k_{gh^{-1}})((\gamma \cdot g)\nu^{-1})\xi_h(\nu)\right) \qquad (\nu = \eta \cdot h) \\
&= \sum_{h\in G} \lambda_x\left(\beta_{g^{-1}}(k_{gh^{-1}})\right)\xi_h(\gamma \cdot g).
\end{aligned}$$

So if we take an arbitrary element

$$\sum_{g\in G} f_g\delta_g \in C_c(\mathscr{G})G,$$

and define

$$f := \Phi\left(\sum_{g\in G} f_g \delta_g\right)$$

then Equation (5.5.3) tells us that $\mathfrak{e}_g(f) = f_g$, that is to say $f_g(\gamma) = f(\gamma, g)$ for all $\gamma \in \mathscr{G}$ and $g \in G$. So we let $x \in X$, $\xi \in l^2(\mathscr{G} \rtimes_\alpha G_x)$ with $\|\xi\|_2 \leq 1$, and calculate

$$\begin{aligned}
\|\pi_x(f)\xi\|_2^2 &= \sum_{(\gamma,g)\in\mathscr{G}\rtimes_\alpha G_x} |\pi_x(f)\xi(\gamma,g)|^2 \\
&= \sum_{(\gamma,g)\in\mathscr{G}\rtimes_\alpha G_x} \left|\sum_{h\in G} \lambda_x\left(\beta_{g^{-1}}(f_{gh^{-1}})\right)\xi_h(\gamma\cdot g)\right|^2 \\
&\leq \sum_{(\gamma,g)\in\mathscr{G}\rtimes_\alpha G_x} \sum_{h\in G} \left|\lambda_x\left(\beta_{g^{-1}}(f_{gh^{-1}})\right)\xi_h(\gamma\cdot g)\right|^2 \\
&= \sum_{g,h\in G}\left(\sum_{s(\gamma)=\gamma\cdot g^{-1}} \left|\lambda_x\left(\beta_{g^{-1}}(f_{gh^{-1}})\right)\xi_h(\gamma\cdot g)\right|^2\right) \\
&= \sum_{g,h\in G} \left\|\lambda_x\left(\beta_{g^{-1}}(f_{gh^{-1}})\right)\xi_h\right\|_2^2 \\
&\leq \sum_{g,h\in G} \left\|\lambda_x\left(\beta_{g^{-1}}(f_{gh^{-1}})\right)\right\|^2 \\
&\leq \sum_{g,h\in G} \|\beta_{g^{-1}}(f_{gh^{-1}})\|_{\text{red}}^2 \\
&= \sum_{g,h\in G} \|f_{gh^{-1}}\|_{\text{red}}^2 \\
&\leq (\#G)^2 \max_{g\in G}\{\|f_g\|_{\text{red}}^2\} \\
&\leq (\#G)^2 \left\|\sum_{g\in G} f_g\delta_g\right\|_{\text{red}}^2,
\end{aligned}$$

where the last inequality is by Equation (4.2.2). Taking the supremum of the left hand side over norm one vectors and $x \in X$ yields

$$\left\|\Phi\left(\sum_{g\in G} f_g\delta_g\right)\right\|_{\text{red}}^2 \leq (\#G)^2 \left\|\sum_{g\in G} f_g\delta_g\right\|^2$$

and so Φ is continuous. Hence it extends to a $*$-isomorphism

$$\Phi : C_r^*(\mathscr{G}) \rtimes_\beta G \to C_r^*(\mathscr{G} \rtimes_\alpha G)$$

as required. ∎

We are now ready to prove the following.

Theorem 5.5.2 *Let $(\mathcal{P}, \omega)$ be a substitution tiling system satisfying the assumptions of Remark 2.5.8 and let G be a finite symmetry group for $(\mathcal{P}, \omega)$. Let Ω and Ω_{punc} be the topological spaces associated with $(\mathcal{P}, \omega)$, and let $\mathcal{R}_{\text{punc}}$ denote translational equivalence on Ω_{punc}. We denote, as in Lemma 5.4.3*

$$\mathscr{G} = (\Omega, \mathbb{R}^2 \rtimes G), \quad \mathscr{H} = (\Omega, \mathbb{R}^2) \rtimes G.$$

Then the following C^*-algebras strongly Morita equivalent:

(1) $(C(\Omega) \rtimes \mathbb{R}^2) \rtimes G$,
(2) $C(\Omega) \rtimes (\mathbb{R}^2 \rtimes G)$,
(3) $C_r^*(\mathscr{G})$,
(4) $C_r^*(\mathscr{H})$,
(5) $C_r^*(\mathcal{R}_{\text{punc}}) \rtimes G$,
(6) $C_r^*(\mathcal{R}_{\text{punc}} \rtimes G)$, *and*
(7) $C_r^*(\mathscr{G}_{\Omega_{\text{punc}}}^{\Omega_{\text{punc}}})$.

Proof:

- (1)$\sim_m$(2) – this follows from Proposition 4.2.6; in fact, these C*-algebras are isomorphic.

- (2)$\sim_m$(3) – $C(\Omega) \rtimes (\mathbb{R}^2 \rtimes G) \cong C^*(\mathscr{G})$ by [42], Example 2.34 and $C^*(\mathscr{G}) \cong C_r^*(\mathscr{G})$ by [59] Proposition II.3.2, Definition II.3.6 and Example 3.10.

- (3)$\sim_m$(4) – this follows from Lemma 5.4.3 and Example 3.3.7.

- (3)$\sim_m$(7) – by Lemma 5.4.2, the groupoids $\mathscr{G}_{\Omega_{\text{punc}}}^{\Omega_{\text{punc}}}$ and $\mathscr{G}$ are equivalent in the sense of Definition 3.3.5. Hence by [65], Theorem 13 we have that $C_r^*(\mathscr{G}_{\Omega_{\text{punc}}}^{\Omega_{\text{punc}}})$ and $C_r^*(\mathscr{G})$ are strongly Morita equivalent.

- (5)$\sim_m$(6) – this follows from Proposition 5.5.1 and Theorem 3.4.3; in fact, these C*-algebras are isomorphic.

- (6)$\sim_m$(7) – this is due to Corollary 5.4.4 and Example 3.3.7.

∎

Since strong Morita equivalence preserves ideal structure, by Lemma 5.4.7 each of the above C*-algebras is simple.

5.6 The Crossed Product $AF_\omega \rtimes G$

In this section, we examine the crossed product of the AF algebra AF_ω by a finite symmetry group G and explicitly compute incidence matrices for our examples.

Our first definition describes the crossed product actions we encounter when studying the crossed product of AF_ω by a finite symmetry group G.

Definition 5.6.1 *Let $n \geq 1$ and $k \geq 2$ be integers and let A be the finite dimensional algebra*

$$A = \bigoplus_{i=1}^{n} \mathbb{M}_k = C(I) \otimes \mathbb{M}_k$$

where $I = \{1, 2, \ldots, n\}$. Let G be a finite group and let $\alpha : G \to \mathrm{Aut}(A)$ be a homomorphism. Then we say α **transitively permutes the summands of** *A if the restriction of α on $C(I)$ acts by transitively permuting I.*

If A is as above and G transitively permutes the summands of A, then the restriction of α to the $Z(A)$ (the center of A) acts transitively on $Z(A) = C(I)$. For $i \in I$, let $q_i = \chi_{\{i\}} \otimes 1_k$ where 1_k is the $k \times k$ identity matrix, and let G_i denote the stabilizer subgroup of q_i. The element q_i is the identity on the ith summand. The set of cosets G/G_1 has n elements, $g_1 = e, g_2, \ldots g_n$ such that $\sum \alpha_{g_i}(q_1) = 1_A$.

Proposition 5.6.2 *Let $n \geq 1$ and $k \geq 2$ be integers and let A be the finite dimensional algebra*

$$A = \bigoplus_{i=1}^{n} \mathbb{M}_k = C(I) \otimes \mathbb{M}_k$$

where $I = \{1, 2, \ldots, n\}$. Let G be a finite group and let $\alpha : G \to \mathrm{Aut}(A)$ be an action which transitively permutes the summands of A. Then there exists a $*$-isomorphism

$$\Phi : \mathbb{M}_n \otimes (q_1 A \rtimes G_1) \to A \rtimes G$$

such that

$$\begin{aligned} e_{ij} \otimes q_1 a \delta_g &\mapsto (1\delta_{g_i})(q_1 a \delta_g)(1\delta_{g_j^{-1}}) \\ &= q_i \alpha_{g_i}(a) \delta_{g_i g g_j^{-1}}, \end{aligned}$$

where the g_i are as above.

Proof: We first note that since G acts transitively on I, for $i \in I$ we have $G \cdot i = I$. By counting elements, this implies that $\#G = \#G_i \cdot n$. We also note that both $\mathbb{M}_n \otimes (q_1 A \rtimes G_1)$ and $A \rtimes G$ are finite dimensional as complex vector spaces. We have

$$\dim(A \rtimes G) = \dim(A) \cdot \#G = (k^2 n)(n \cdot \#G_1) = k^2 n^2 \#G_1,$$

$$\dim(\mathbb{M}_n \otimes (q_1 A \rtimes G_1)) = n^2 \cdot k^2 \cdot \#G_1.$$

Hence, it is enough to show that Φ as defined is a $*$-homomorphism and either surjective or injective – we will show surjective. The map Φ defined on elementary tensors is extended by linearity. If the product of $e_{ij} \otimes q_1 a \delta_g$ and $e_{lk} \otimes q_1 b \delta_h$ is nonzero, then $j = l$. So we compute

$$\begin{aligned} \Phi\left((e_{ij} \otimes q_1 a \delta_g)(e_{jk} \otimes q_1 b \delta_h)\right) &= \Phi(e_{ik} \otimes (q_1 a \delta_g)(q_1 b \delta_h)) \\ &= (1\delta_{g_i})(q_1 a \delta_g)(q_1 b \delta_h)(1\delta_{g_k^{-1}}) \\ &= (1\delta_{g_i})(q_1 a \delta_g)(1\delta_{g_j^{-1}})(1\delta_{g_j})(q_1 b \delta_h)(1\delta_{g_k^{-1}}) \end{aligned}$$

$$= \Phi(e_{ij}\otimes q_1 a\delta_g)\Phi(e_{jk}\otimes q_1 b\delta_h)$$

$$\begin{aligned}
\Phi((e_{ij}\otimes q_1 a\delta_g)^*) &= \Phi(e_{ji}\otimes \delta_g^* a^* q_1^*)\\
&= (1\delta_{g_j})\delta_g^* a^* q_1^*(1\delta_{g_i^{-1}})\\
&= (1\delta_{g_j^{-1}})^*\delta_g^* a^* q_1^*(1\delta_{g_i})^*\\
&= ((1\delta_{g_i})q_1 a\delta_g(1\delta_{g_j^{-1}}))^*\\
&= (\Phi(e_{ij}\otimes q_1 a\delta_g))^*.
\end{aligned}$$

To see that Φ is surjective, we notice that for every $1\le i\le n$, the sets $\{g_iG_1g_j^{-1}\}_{j=1}^L$ form a partition of G. Thus for $h\in G$ and $1\le i\le n$ there is exactly one $1\le j\le n$ such that $g_i^{-1}hg_j\in G_1$. Then for $b\in A$, $h\in G$ we have

$$\begin{aligned}
\Phi\left(\sum_{\substack{1\le i,j\le n\\ g_i^{-1}hg_j\in G_1}} e_{ij}\otimes q_1\alpha_{g_i^{-1}}(b)\delta_{g_i^{-1}hg_j}\right) &= \sum_{\substack{1\le i,j\le n\\ g_i^{-1}hg_j\in G_1}} \Phi\left(e_{ij}\otimes q_1\alpha_{g_i^{-1}}(b)\delta_{g_i^{-1}hg_j}\right)\\
&= \sum_{\substack{1\le i,j\le n\\ g_i^{-1}hg_j\in G_1}} q_i\alpha_{g_i}\left(\alpha_{g_i^{-1}}(b)\right)\delta_h\\
&= \left(\sum_{\substack{1\le i,j\le n\\ g_i^{-1}hg_j\in G_1}} q_i b\right)\delta_h\\
&= b\delta_h.
\end{aligned}$$

The last equality is because every $1\le i\le n$ is represented in the sum exactly once. ∎

We obtain the following in the case that q_1 (and hence q_i) have trivial stabilizer.

Corollary 5.6.3 *Suppose that A and G are as in Definition 5.6.1 and suppose in addition that G permutes the summands* freely. *Then*

$$A\rtimes G\cong \mathbb{M}_{\#G}\otimes(q_1A)\cong \mathbb{M}_{\#G\cdot k}.$$

If $g \in G_1$, then $\alpha_g(q_1) = q_1$. Since all automorphisms of simple finite dimensional algebras are inner, there exists a unitary u_g such that $\alpha_g(a) = u_g a u_{g^{-1}}$ for all $a \in A_1 = q_1 A$.

Lemma 5.6.4 *Suppose that A and G are as in Definition 5.6.1. Then there exists a $*$-isomorphism $\Psi : q_1 A \rtimes G_1 \to q_1 A \otimes C^*(G_1)$ such that $a\delta_g \mapsto au_g \otimes \delta_g$.*

Proof: This follows from Lemma 4.2.8 and its proof in [73], Lemma 2.73. ∎

Let AF_ω be the AF subalgebra of $C_r^*(\mathcal{R}_{\text{punc}})$ presented in Section 4.6. Recall that

$$AF_\omega = \overline{\bigcup_{n\in\mathbb{N}} A_n},$$

where

$$A_n = \bigoplus_{p\in\mathcal{P}} A_{n,p},$$

with

$$\begin{aligned} A_{n,p} &= \operatorname{span}_{\mathbb{C}} \{e_p^n(x,y) \mid x, y \in \operatorname{Punc}(n,p)\} \\ &\cong \mathbb{M}_{\#\operatorname{Punc}(n,p)}. \end{aligned}$$

Let G be a finite symmetry group for $(\mathcal{P}, \omega)$, and as before let β denote the action induced by G. For $g \in G$ we have

$$\begin{aligned} \beta_g(e_p^n(x,y))(T,T') &= e_p^n(x,y)(g^{-1}T, g^{-1}T') \\ &= \begin{cases} 1 & (g^{-1}T, g^{-1}T') \in E_p^n(x,y) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & (g^{-1}T, g^{-1}T') = (\omega^n(S) - x, \omega^n(S) - y), \\ & \quad S \in U(\{p\}, p) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} 1 & (T,T') = (\omega^n(gS) - gx, \omega^n(gS) - gy), \\ & \quad gS \in U(\{gp\}, gp) \\ 0 & \text{otherwise} \end{cases} \\
&= e^n_{gp}(gx, gy)(T, T').
\end{aligned}
$$

By definition of $A_{n,p}$, we then have $\beta_g(A_{n,p}) = A_{n,gp}$. Let $\mathcal{S}_G$ be a set of standard position prototiles for G and assume that G acts freely on $\mathcal{P}$. Because $\mathcal{P} = G\mathcal{S}_G$, we have

$$A_n = \bigoplus_{p \in \mathcal{S}_G} (\oplus_{g \in G} A_{n,gp}) \cong \bigoplus_{p \in \mathcal{S}_G} \left(C(Gp) \otimes \mathbb{M}_{\#\mathrm{Punc}(n,p)} \right),$$

where Gp is the finite set $\{gp \mid g \in G\}$. Let $B_{n,p} = \oplus_{g \in G} A_{n,gp} \cong C(Gp) \otimes \mathbb{M}_{\#\mathrm{Punc}(n,p)}$. The action β acts freely and transitively on the set Gp, that is, β restricted to each of the $B_{n,p}$ transitively permutes the summands of $B_{n,p}$ in the sense of Definition 5.6.1. Hence we have

$$
\begin{aligned}
A_n \rtimes_\beta G &= \left(\bigoplus_{p \in \mathcal{S}_G} B_{n,p} \right) \rtimes_\beta G \\
&= \bigoplus_{p \in \mathcal{S}_G} (B_{n,p} \rtimes_\beta G) \\
&\cong \bigoplus_{p \in \mathcal{S}_G} \mathbb{M}_{\#G \cdot \mathrm{Punc}(n,p)}.
\end{aligned}
$$

We are now ready to describe the crossed product $AF_\omega \rtimes G$, which by Proposition 5.5.1 is isomorphic to $C^*_r(\mathcal{R}_{AF} \rtimes G)$.

Theorem 5.6.5 *Let G be a symmetry group for $(\mathcal{P}, \omega)$. Then*

$$AF_\omega \rtimes G \cong \overline{\bigcup_{n \in \mathbb{N}} A_n \rtimes G}.$$

If G acts freely on $\mathcal{P}$, the number of summands in the finite dimensional algebras $A_n \rtimes G$ is the number of elements of $\mathcal{S}_G$, and if M is the incidence matrix for the unital inclusion $A_n \rtimes G \subset A_{n+1} \rtimes G$, then M_{ij} is the number of images of p_j under the action of $\mathbb{R}^2 \rtimes G$ in $\omega(p_i)$.

Proof: We denote the inclusion of A_n in A_{n+1} by ι. Then there is an inclusion

$$I : A_n \rtimes G \hookrightarrow A_{n+1} \rtimes G$$

$$a\delta_g \mapsto \iota(a)\delta_g.$$

The unit for each of these algebras is $1\delta_e$, and since ι is unital I is as well. Since each of the $A_n \rtimes G$ is a sub C*-algebra of $AF_\omega \rtimes G$ and they are nested, their union is a sub $*$-algebra of $AF_\omega \rtimes G$. Hence the closure is a sub C*-algebra of $AF_\omega \rtimes G$. To get the other inclusion, let $\varepsilon > 0$ and take

$$\sum_{g\in G} a_g\delta_g \in AF_\omega \rtimes G \qquad a_g \in AF_\omega.$$

For each $g \in G$, find $b_g \in \cup A_n$ such that $\|b_g - a_g\| < \frac{\varepsilon}{\#G}$. Since G is finite we may take each of the b_g to be in the same A_n. So $\sum_{g\in G} b_g\delta_g \in A_n \rtimes G$, and

$$\begin{aligned}
\left\|\sum_{g\in G} a_g\delta_g - \sum_{g\in G} b_g\delta_g\right\| &= \left\|\sum_{g\in G}(a_g - b_g)\delta_g\right\| \\
&\leq \sum_{g\in G} \|a_g - b_g\| \\
&< \sum_{g\in G} \frac{\varepsilon}{\#G} \\
&= \varepsilon.
\end{aligned}$$

Hence $\sum_{g\in G} a_g\delta_g \in \overline{\bigcup_{n\in\mathbb{N}} A_n \rtimes G}$, and we have proved the first statement. The second statement is by the discussion directly above the theorem.

We now find the incidence matrix of the inclusions. To do this, we use Equation (4.5.2). Let $q_{n,p}$ denote the identity of $A_{n,p}$. Then the identity of the pth summand of $A_n \rtimes G$ is

$$\sum_{g\in G} q_{n,gp}\delta_e.$$

The trace on the pth summand is

$$\tau_p^{A_n \rtimes G}(a\delta_g) = \begin{cases} \mathrm{Tr}\left(a \sum_{h\in G} q_{n,hp}\right) & \text{if } g = e \\ 0 & \text{otherwise.} \end{cases}$$

And we have

$$\begin{aligned} I\left(\sum_{g\in G} q_{n,gp}\delta_e\right) &= \iota\left(\sum_{g\in G} q_{n,gp}\right)\delta_e \\ &= \sum_{g\in G} \iota\left(q_{n,gp}\right)\delta_e \end{aligned}$$

Thus for $p_i, p_j \in \mathcal{S}_G$ we have

$$\begin{aligned} \tau_{p_i}^{A_{n+1}\rtimes G} \circ I\left(\sum_{g\in G} q_{n,gp_j}\delta_e\right) &= \tau_{p_i}^{A_{n+1}\rtimes G}\left(\sum_{g\in G} \iota(q_{n,gp})\delta_e\right) \\ &= \mathrm{Tr}\left(\left(\sum_{h\in G} q_{n+1,hp_i}\right)\left(\sum_{g\in G}\iota(q_{n,gp_j})\right)\right) \\ &= \sum_{h\in G}\sum_{g\in G}\mathrm{Tr}\left(q_{n+1,hp_i}\iota(q_{n,gp_j})\right) \end{aligned}$$

The term $\mathrm{Tr}\left(q_{n+1,hp_i}\iota(q_{n,gp_j})\right)$ is the number of translates of gp_j in $\omega(hp_i)$, by the discussion in Section 4.5. Hence

$$\begin{aligned} \tau_{p_i}^{A_{n+1}\rtimes G} \circ I\left(\sum_{g\in G} q_{n,gp_j}\delta_e\right) &= \sum_{h\in G}\sum_{g\in G} \#\mathrm{Punc}(n,p_j)\begin{pmatrix}\#\text{ of translates of}\\ gp_j \text{ in } \omega(hp_i)\end{pmatrix} \\ &= \#\mathrm{Punc}(n,p_j)\sum_{h\in G}\sum_{g\in G}\begin{pmatrix}\#\text{ of translates of}\\ h^{-1}gp_j \text{ in } \omega(p_i)\end{pmatrix} \end{aligned}$$

For fixed h, $\sum_{g\in G}$ (# of translates of $h^{-1}gp_j$ in $\omega(p_i)$) is the number of images of p_j under the action of $\mathbb{R}^2 \rtimes G$ in $\omega(p_i)$. Hence

$$\begin{aligned} \tau_{p_i}^{A_{n+1}\rtimes G} \circ I\left(\sum_{g\in G} q_{n,gp_j}\delta_e\right) &= \#\mathrm{Punc}(n,p_j)\sum_{h\in G}\begin{pmatrix}\#\text{ of images of } p_j \text{ under the}\\ \text{action of } \mathbb{R}^2\rtimes G \text{ in } \omega(p_i)\end{pmatrix} \\ &= \#G\#\mathrm{Punc}(n,p_j)\begin{pmatrix}\#\text{ of images of } p_j \text{ under the}\\ \text{action of } \mathbb{R}^2\rtimes G \text{ in } \omega(p_i)\end{pmatrix} \end{aligned}$$

On the other hand,

$$\tau_{p_j}^{A_n\rtimes G}\left(\sum_{g\in G} q_{n,gp_j}\delta_e\right) = \#G\#\mathrm{Punc}(n,p_j)$$

because $\sum_{g\in G} q_{n,gp_j}\delta_e$ is the identity on the p_jth summand, $\tau_{p_j}^{A_n\rtimes G}$ is the matrix trace restricted to the p_jth summand, and the size of the p_jth summand is $\#G\#\mathrm{Punc}(n,p_j)$. Hence

$$\frac{\tau_{p_i}^{A_{n+1}\rtimes G}\circ I\left(\sum_{g\in G} q_{n,gp_j}\delta_e\right)}{\tau_{p_j}^{A_n\rtimes G}\left(\sum_{g\in G} q_{n,gp_j}\delta_e\right)} = \left(\begin{array}{c}\#\text{ of images of } p_j \text{ under the}\\ \text{action of } \mathbb{R}^2\rtimes G \text{ in } \omega(p_i)\end{array}\right).$$

Thus by Equation (4.5.2), the incidence matrix of the inclusion is as given in the statement of the theorem. ∎

Notice that the incidence matrix does not depend on n. In fact, it is the same for each inclusion, just as it is for AF_ω. Primitivity of the substitution implies primitivity of the incidence matrix for $AF_\omega \rtimes G$.

Proposition 5.6.6 *Let G be a finite symmetry group for $(\mathcal{P},\omega)$ such that G acts freely on the prototiles. Then both AF algebra $AF_\omega \rtimes G$ and $C_r^*(\mathcal{R}_{\mathrm{punc}})\rtimes G$ have unique tracial states.*

Proof: As stated before, that $AF_\omega \rtimes G$ has a unique tracial state is a general fact about AF algebra with constant primitive substitution matrix, again see [28], Theorem 4.1. As mentioned in Section 4.6, that the tracial states of $C_r^*(\mathcal{R}_{\mathrm{punc}})\rtimes G$ and $AF_\omega \rtimes G$ coincide is a general fact about almost AF Cantor groupoids, see [47], Proposition 2.11. ∎

Example 5.6.7 Penrose tiling, $G = D_{10}$.

Referring to Figure 2.4, we set $\mathcal{S}_{D_{10}} = \{\mathbf{1},\mathbf{21}\}$. Then $\omega(\mathbf{1})$ contains one image each of $\mathbf{1}$ and $\mathbf{21}$ under the action of $\mathbb{R}^2\rtimes D_{10}$, and $\omega(\mathbf{21})$ contains one image of $\mathbf{1}$ and two images of $\mathbf{21}$ under the action of $\mathbb{R}^2\rtimes D_{10}$. Hence the incidence matrix for

$AF_\omega \rtimes D_{10}$ is

$$\beta = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Example 5.6.8 Octagonal tiling, $G = D_8$.

Here we see that the structure of $AF_\omega \rtimes G$ depends on the exact form of the substitution. Recall that we replaced the substitution in Figure 2.5 by that in Figure 2.6 to break symmetry and allow D_8 to act freely on $\mathcal{P}$. This resulted in homeomorphic CW complexes and hulls, but does not result in isomorphic $AF_\omega \rtimes G$.

We first consider the substitution given in Figure 2.6. Here the action is free so we can read the incidence matrix off the substitution. Let p_1 be the small right angle triangle and let p_2 be the isosceles triangle. Then $\omega(p_1)$ contains three images of p_1 and one image of p_2 under the action of $\mathbb{R}^2 \rtimes D_8$, and $\omega(p_2)$ contains eight images of p_1 and three images of p_2 under the action of $\mathbb{R}^2 \rtimes D_8$. Hence the incidence matrix for $AF_\omega \rtimes D_8$ is

$$M = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix}.$$

In the case of the substitution in Figure 2.5, D_8 still acts on AF_ω and this action is locally representable, so $AF_\omega \rtimes D_8$ is an AF algebra. For consistency with Proposition 5.5.1 we denote this action β. We cannot use Theorem 5.6.5, but Lemma 5.6.4 still applies. The calculation in this case is quite long, and involves keeping track of the values of traces composed with the inclusion.

The algebras A_n each have 20 summands, 16 corresponding to triangles, and 4 to rhombs. The summands corresponding to the triangles are permuted transitively and freely by D_8, while the 4 rhomb summands are permuted transitively but not freely. Thus, if we let p_1 denote the rhomb in standard position and p_2 denote the triangle in standard position we have

$$A_n = \left(\bigoplus_{i=1}^{4} \mathbb{M}_{\#\mathrm{Punc}(n,p_1)} \right) \oplus \left(\bigoplus_{i=1}^{16} \mathbb{M}_{\#\mathrm{Punc}(n,p_2)} \right). \tag{5.6.1}$$

Let q_1 be the identity on the first summand; this corresponds to the rhomb in standard position. Its stabilizer subgroup is $G_1 = \{e, r^4, f, r^4 f\}$, where r is rotation by $^{\pi}/_{4}$ and f is the reflection over the line through the origin which makes an angle of $^{\pi}/_{8}$ with the x-axis. We choose for our elements of our cosets $g_1 = e, g_2 = r, g_3 = r^2$ and $g_4 = r^3$. In this case we get that $A_n \rtimes D_8$ decomposes into a direct sum

$$\begin{aligned} A_n \rtimes D_8 &\cong (\mathbb{M}_4 \otimes q_1 A_n \rtimes G_1) \oplus \mathbb{M}_{16 \cdot \#\mathrm{Punc}(n,p_2)} \\ &\cong (\mathbb{M}_4 \otimes q_1 A_n \otimes C^*(G_1)) \oplus \mathbb{M}_{16 \cdot \#\mathrm{Punc}(n,p_2)} \end{aligned}$$

In this case G_1 is a finite abelian group with 4 characters

$$\chi_1 \begin{bmatrix} e \\ f \\ r^4 \\ r^4 f \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \chi_2 \begin{bmatrix} e \\ f \\ r^4 \\ r^4 f \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\chi_3 \begin{bmatrix} e \\ f \\ r^4 \\ r^4 f \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \qquad \chi_4 \begin{bmatrix} e \\ f \\ r^4 \\ r^4 f \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix},$$

and $C^*(G_1)$ is isomorphic to $\mathbb{C}^4$ via the isomorphism

$$\delta_g \mapsto (\chi_1(g), \chi_2(g), \chi_3(g), \chi_4(g)) .$$

Thus

$$\begin{gathered} q_1 A_n \rtimes G_1 \cong q_1 A_n \otimes G_1 \cong \bigoplus_{i=1}^{4} \mathbb{M}_{\#\mathrm{Punc}(n,p_1)} \\ a\delta_g \mapsto au_g \otimes \delta_g \mapsto (au_g\chi_1(g), au_g\chi_2(g), au_g\chi_3(g), au_g\chi_4(g)) . \end{gathered}$$

Then the matrix trace on each summand, denoted $\rho_i^{(n)}$ for $i = 1, 2, 3, 4$ when composed with this isomorphism is given by

$$\rho_i^{(n)}(a\delta_g) = \mathrm{Tr}(au_g)\chi_i(g).$$

We have

$$A_n \rtimes D_8 \cong (\mathbb{M}_4 \otimes q_1 A_n \otimes C^*(G_1)) \oplus \mathbb{M}_{16\cdot\#\mathrm{Punc}(n,p_2)}$$

If a is in the first four summands in Equation (5.6.1), then under this isomorphism

$$a\delta_g \mapsto \left(\sum_{1\le i,j\le 4} e_{ij} \otimes \left(q_1 \beta_{g_i^{-1}g}(q_j)\beta_{g_i^{-1}}(a)u_{g_i^{-1}gg_j}\right) \otimes \delta_{g_i^{-1}gg_j}, 0\right)$$

The traces on the first four summands, denoted τ_k for $k = 1,2,3,4$ when composed with this isomorphism are given by

$$\begin{aligned} \tau_k^{(n)}(a\delta_g) &= \sum_{i=1}^{4} \rho_i^{(0)} \left(q_1\beta_{g_i^{-1}g}(q_i)\beta_{g_i^{-1}}(a)u_{g_i^{-1}gg_i}\delta_{g_i^{-1}gg_i}\right) \\ &= \sum_{i=1}^{4} \mathrm{Tr}\left(q_1\beta_{g_i^{-1}g}(q_i)\beta_{g_i^{-1}}(a)u_{g_i^{-1}gg_i}\right)\chi_k(g_i^{-1}gg_i) \end{aligned}$$

The product inside the Tr is nonzero if and only if $g_i^{-1}gg_i \in G_1$, so the χ_k make sense here. Let $Q_1^{(n)}$ and $Q_2^{(n)}$ be the identities on the first and second group of summands respectively in Equation (5.6.1). Referencing Corollary 5.6.3, the trace on the last summand is given by

$$\tau_5^{(n)}(a\delta_g) = \begin{cases} \mathrm{Tr}(Q_2^{(n)}a) & \text{if } g = e \\ 0 & \text{otherwise.} \end{cases}$$

The inclusion

$$I : A_n \hookrightarrow A_{n+1}$$

gives rise to an inclusion $A_n \rtimes D_8 \hookrightarrow A_{n+1} \rtimes D_8$ which we also call I:

$$I(a\delta_g) = I(a)\delta_g.$$

We obtain the incidence matrix of this inclusion by using the fact that traces on the algebra $A_{n+1} \rtimes D_8$ give rise to traces on $A_n \rtimes D_8$, and these must be positive linear combinations of the traces on the matrix summands of $A_n \rtimes D_8$. In other words,

$$\tau_k^{(n+1)} \circ I = c_{k1}\tau_1^{(n)} + c_{k2}\tau_2^{(n)} + c_{k3}\tau_3^{(n)} + c_{k4}\tau_4^{(n)} + c_{k5}\tau_5^{(n)}. \tag{5.6.2}$$

By Equation (4.5.1), the matrix $[c_{ij}]$ will be the matrix of partial multiplicities. We can solve for the constants c_{ij} by applying both sides of (5.6.2) to various elements of $A_n \rtimes D_8$. For instance

$$\begin{aligned}
\tau_5^{(n+1)} \circ I(Q_2^{(n)}\delta_e) &= \tau_5^{(n+1)}(I(Q_2^{(n)})\delta_e) \\
&= \mathrm{Tr}(Q_2^{(n+1)} I(Q_2^{(n)})) \\
&= 3 \cdot 16 \cdot \#\mathrm{Punc}(n, p_2).
\end{aligned}$$

We get the third line in the above equation because there are 3 triangles in the substitution of each triangles, and there are 16 triangles total. On the other hand, if we apply the right hand side of Equation (5.6.2) to $Q_2^{(n)}\delta_e$, the only nonzero term will come from $c_{55}\tau_5^{(n)}$, and we get

$$c_{55}\tau_5^{(n)}(Q_2^{(n)}\delta_e) = c_{55}\mathrm{Tr}(Q_2^{(n)}) = c_{55}16 \cdot \#\mathrm{Punc}(n, p_2),$$

giving us that $c_{55} = 3$. We calculate the traces on the right hand side of (5.6.2) on $Q_1^{(n)}\delta_e$, $Q_1^{(n)}\delta_f$, $Q_2^{(n)}\delta_{r^4}$ and $q_1^{(n)}\delta_f$. For these calculations, we keep in mind that $\beta_g(Q_i^{(n)}) = Q_i^{(n)}$ for $i = 1, 2$ and that $q_1 u_g = u_g$ for all $g \in G_1$. For $k = 1, 2, 3, 4$,

$$\begin{aligned}
\tau_k^{(n)}(Q_1^{(n)}\delta_e) &= \sum_{i=1}^{4} \mathrm{Tr}\left(q_1 \beta_{g_i^{-1}e}(q_i)\beta_{g_i^{-1}}(Q_1^{(n)})u_{g_i^{-1}eg_i}\right)\chi_k(g_i^{-1}eg_i) \\
&= \sum_{i=1}^{4} \mathrm{Tr}\left(q_1 \beta_{g_i^{-1}}(Q_1^{(n)})u_e\right) \\
&= \sum_{i=1}^{4} \mathrm{Tr}\,(q_1) \\
&= 4 \cdot \#\mathrm{Punc}(n, p_1).
\end{aligned}$$

$$\begin{aligned}
\tau_k^{(n)}(Q_1^{(n)}\delta_f) &= \sum_{i=1}^{4} \mathrm{Tr}\left(q_1 \beta_{g_i^{-1}f}(q_i)\beta_{g_i^{-1}}(Q_1^{(n)})u_{g_i^{-1}fg_i}\right)\chi_k(g_i^{-1}eg_i) \\
&= \mathrm{Tr}\left(q_1 \beta_f(q_1) Q_1^{(n)} u_f\right)\chi_k(f) + \\
&\quad \cdots + \mathrm{Tr}\left(q_1 \beta_{r^{-1}f}(q_2) Q_1^{(n)} u_{r^{-1}fr}\right)\chi_k(r^{-1}fr) +
\end{aligned}$$

$$\begin{aligned}
&\cdots + \mathrm{Tr}\left(q_1\beta_{r^{-2}f}(q_3)Q_1^{(n)}u_{r^{-2}fr^2}\right)\chi_k(r^{-2}fr^2) + \\
&\cdots + \mathrm{Tr}\left(q_1\beta_{r^{-3}f}(q_4)(Q_1^{(n)}u_{r^{-3}fr^3}\right)\chi_k(r^{-3}fr^3) \\
=\;& \mathrm{Tr}(q_1u_f)\chi_k(f) + 0 + \mathrm{Tr}(q_1u_{r^4f})\chi_k(r^4f) + 0 \\
=\;& \mathrm{Tr}(u_f)\chi_k(f) + \mathrm{Tr}(u_{r^4f})\chi_k(r^4f).
\end{aligned}$$

$$\begin{aligned}
\tau_k^{(n)}(Q_1^{(n)}\delta_{r^4}) &= \sum_{i=1}^{4}\mathrm{Tr}\left(q_1\beta_{g_i^{-1}r^4}(q_i)\beta_{g_i^{-1}}(Q_1^{(n)})u_{g_i^{-1}r^4g_i}\right)\chi_k(g_i^{-1}r^4g_i) \\
&= \mathrm{Tr}\left(q_1\beta_{r^4}(q_1)Q_1^{(n)}u_{r^4}\right)\chi_k(r^4) + \mathrm{Tr}\left(q_1\beta_{r^3}(q_2)Q_1^{(n)}u_{r^4}\right)\chi_k(r^4) + \\
&\quad \cdots + \mathrm{Tr}\left(q_1\beta_{r^2}(q_3)Q_1^{(n)}u_{r^4}\right)\chi_k(r^4) + \\
&\quad \cdots + \mathrm{Tr}\left(q_1\beta_{r}(q_4)(Q_1^{(n)}u_{r^4}\right)\chi_k(r^4) \\
&= 4\mathrm{Tr}(u_{r^4})\chi_k(r^4).
\end{aligned}$$

To find $\tau_k^{(n)}(q_1\delta_f)$, we notice that the only summand which will be nonzero will be the one corresponding to $g_1 = e$; for others $\beta_{g_i}(q_1)$ will be orthogonal to q_1. Hence

$$\begin{aligned}
\tau_k^{(n)}(q_1\delta_f) &= \mathrm{Tr}(q_1u_f)\chi_k(f) \\
&= \mathrm{Tr}(u_f)\chi_k(f)
\end{aligned}$$

If we apply $\tau_5^{(n+1)} \circ I$ to each of these, we will get a system of 4 linear equations in the 4 unknowns c_{5k}, $k = 1, 2, 3, 4$. The map $\tau_5^{(n+1)} \circ I$ will be 0 on all of them except $Q_1^{(n)}\delta_e$, on which we get

$$\begin{aligned}
\tau_5^{(n+1)} \circ I(Q_1^{(n)}\delta_e) &= \mathrm{Tr}\left(Q_2^{(n+1)}I(Q_1^{(n)})\right) \\
&= 2 \cdot 16 \cdot \#\mathrm{Punc}(n, p_1).
\end{aligned}$$

We get this because there are 2 rhombs in each substituted triangle, and there are 16 triangles total. For convenience of the calculation, we let

$$b_1 = \mathrm{Tr}(u_f),$$

$$b_2 = \mathrm{Tr}(u_{r^4f}),$$

$$b_3 = \#\mathrm{Punc}(n, p_1),$$

and we also note that $\mathrm{Tr}(u_{r^4}) = 1$ because r^4 fixes exactly one tile in the rhomb. From the above we get the system of equations

$$\begin{bmatrix} 4b_3 & 4b_3 & 4b_3 & 4b_3 \\ (b_1+b_2) & (-b_1-b_2) & (-b_1+b_2) & (b_1-b_2) \\ 4b_2 & 4b_2 & -4b_2 & -4b_2 \\ b_1 & -b_1 & -b_1 & b_1 \end{bmatrix} \begin{bmatrix} c_{51} \\ c_{52} \\ c_{53} \\ c_{54} \end{bmatrix} = \begin{bmatrix} 32b_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which has solution $c_{5k} = 2$ for $k = 1, 2, 3, 4$.

Now recall that the right hand side of Equation (5.6.2) applied to $Q_2\delta_e$ is $16c_{k5}\#\mathrm{Punc}(n, p_2)$ no matter what k is. For $k \neq 5$, the left hand side is

$$\begin{aligned} \tau_k^{(n+1)} \circ I(Q_2^{(n)}\delta_e) &= \tau_k^{(n+1)}(I(Q_2^{(n)})\delta_e) \\ &= \sum_{i=1}^{4} \mathrm{Tr}(q_1 I(Q_2^{(n)})) \\ &= 4 \cdot 4 \cdot \#\mathrm{Punc}(n, p_2). \end{aligned}$$

The second line is due to the fact that $\beta_g(I(Q_2^{(n)})) = I(Q_2^{(n)})$ for any $g \in G$ and the third line is due to there being 4 equal summands each corresponding to there being 4 triangles present when one substitutes the rhomb in standard position. Hence $c_{k5} = 1$ for $k = 1, 2, 3, 4$. One calculates

$$\begin{aligned} \tau_k^{(n+1)} \circ I(Q_1^{(n)}\delta_e) &= 12b_3 \\ \tau_k^{(n+1)} \circ I(Q_1^{(n)}\delta_{r^4}) &= 4\chi_k(r^4) \\ \tau_k^{(n+1)} \circ I(Q_1^{(n)}\delta_f) &= (2b_1 + b_2)\chi_k(f) + b_1\chi_k(r^4 f) \\ \tau_k^{(n+1)} \circ I(q_1^{(n)}\delta_f) &= b_1(2\chi_k(f) + \chi_i(r^4 f)) \end{aligned}$$

We omit the details, but they are along the same lines as the calculations done so far. The corresponding linear systems can be solved to give the matrix of partial multiplicities M for the inclusion $A_n \rtimes D_8 \hookrightarrow A_{n+1} \rtimes D_8$ to be

$$M = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 3 \end{bmatrix}.$$

5.7 The Rokhlin Property and Tracial Rokhlin Property

In Proposition 5.6.2, we considered the crossed product of a finite dimensional C*-algebra A by a finite group G in the case where the identity of A could be decomposed as a sum of projections which were permuted by the elements of G. In Corollary 5.6.3, we saw that the crossed product had a particularly nice form when G permuted the projections freely.

In this section we present weaker versions of the above conditions for actions of finite groups on unital C*-algebras which are still strong enough to imply some results about the crossed products. We also discuss to what extent these properties are satisfied for actions of finite symmetry groups on tiling C*-algebras.

Definition 5.7.1 ([48], Definition 2.1) *Let A be a unital C*-algebra, and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the* **Rokhlin property** *if for every finite set $\mathcal{F} \subset A$ and every $\varepsilon > 0$ there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:*

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ *for all* $g, h \in G$.
2. $\|e_g f - f e_g\| < \varepsilon$ *for all* $g \in G$ *and all* $f \in \mathcal{F}$.
3. $\sum_{g\in G} e_g = 1$.

We call the $(e_g)_{g\in G}$ *a* **family of Rokhlin projections** *for* $\alpha, \mathcal{F}$ *and* ε.

As we can see, this is a weakening of the conditions in Corollary 5.6.3. The below is a further weakening of this property.

Definition 5.7.2 ([48], Definition 3.1) *Let A be an infinite dimensional unital C*-algebra, and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the* **tracial Rokhlin property** *if for every finite set $\mathcal{F} \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:*

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ *for all* $g, h \in G$.
2. $\|e_g f - f e_g\| < \varepsilon$ *for all* $g \in G$ *and all* $f \in \mathcal{F}$.
3. *If* $e = \sum_{g\in G} e_g$, *then* $1 - e$ *is Murray-von Neumann equivalent to a projection in* $\overline{xAx}$.
4. *Taking* e *as above,* $\|exe\| > 1 - \varepsilon$.

This definition simplifies somewhat when A is finite, in this case the 4th condition is not needed – see Lemma 1.12 of [50]. It is also clear that the Rokhlin property is strictly stronger than the tracial Rokhlin property provided A is infinite dimensional – if $e = 1$ the last two conditions in Definition 5.7.2 are automatically satisfied. We will see in Lemma 5.7.5 why the "tracial" modifier is used in the name of this property.

For A and B C*-algebras with $B \subset A$, $a \in A$ and $\varepsilon > 0$, we write $a \in_\varepsilon B$ if $\inf\{\|b - a\| \mid b \in B\} < \varepsilon$. We state a definition of Lin.

Definition 5.7.3 ([38], Definition 2.1) *Let A be a unital simple C*-algebra. Then we say A has* **tracial rank zero** *if for any $\varepsilon > 0$, any finite set $\mathcal{F} \subset A$, and any positive $x \neq 0$ there exists a finite dimensional C*-algebra $F \subset A$ with $p = 1_F$ such that*

1. $\|pf - fp\| < \varepsilon$ *for all* $f \in \mathcal{F}$,

2. $pfp \in_\varepsilon F$ *for all* $f \in \mathcal{F}$, *and*

3. $1 - p$ *is Murray-von Neumann equivalent to a projection in* $\overline{xAx}$.

The tracial rank zero property was first defined by Lin in [38]. Tracial rank can be seen as a noncommutative analogue of topological dimension, see [39]. In further work on tracial rank zero algebras, Lin proves in [39] the following, which was stated as below by Brown in [14]:

Theorem 5.7.4 ([14], Theorem 4.5.1) *Let A be a simple C^*-algebra. Then A has tracial rank zero if and only if A has real rank zero, stable rank one, the order of projections on A is determined by traces, and for every finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$ there exists a finite dimensional subalgebra $F \subset A$ with $p = 1_F$ such that:*

1. $\|pf - fp\| < \varepsilon$ *for all* $f \in \mathcal{F}$,

2. $pfp \in_\varepsilon F$ *for all* $f \in \mathcal{F}$, *and*

3. $\tau(e) > 1 - \varepsilon$ *for all* $\tau \in T(A)$.

That an action has the tracial Rokhlin property is somewhat easier to verify in the presence of tracial rank zero, as the following lemma shows.

Lemma 5.7.5 ([49], Lemma 1.8) *Let A be a separable infinite dimensional simple unital C^*-algebra with tracial rank zero. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Suppose that for every finite set $\mathcal{F} \subset A$ and each $\varepsilon > 0$ that there is for each $g \in G$ a positive element $a_g \in A$ with $0 \leq a_g \leq 1$ such that*

1. $a_g a_h = 0$ *for each* $g, h \in G$ *with* $g \neq h$,

2. $\|\alpha_g(a_h) - a_{gh}\| < \varepsilon$ *for all* $g, h \in G$,

3. *$\|a_g f - f a_g\| < \varepsilon$ for all $g \in G$ and $f \in \mathcal{F}$, and*

4. *$\tau\left(1 - \sum_{g\in G} a_g\right) < \varepsilon$ for every normalized trace τ on A.*

Then α has the tracial Rokhlin property.

In this section we prove two results. The first is that if G is a symmetry group of the primitive substitution tiling system $(\mathcal{P}, \omega)$ and G acts freely on $\mathcal{P}$, then the action of G on AF_ω has the Rokhlin property. This follows from the following more general result:

Proposition 5.7.6 *Suppose that $A = \overline{\cup A_n}$ is a unital AF algebra and that $\alpha : G \to Aut(A)$ is an action of a finite group G on A. Suppose that for each $n \in \mathbb{N}$ there exists a positive integer $k(n)$ such that*

$$A_n = \bigoplus_{i\in I(n)} \left(\bigoplus_{g\in G} \mathbb{M}_{k(n)} \right)$$

and that the restriction of α freely and transitively permutes the summands of $\bigoplus_{g\in G} \mathbb{M}_{k(n)}$. Then α has the Rokhlin property.

Proof: For A_n written as above, let $q_{i,g}$ denote the identity of the summand corresponding to $i \in I(n)$ and $g \in G$. Since G acts freely and transitively on $\bigoplus_{g\in G} \mathbb{M}_{k(n)}$ we lose no generality by supposing that $gq_{i,e} = q_{i,g}$. Let $\mathcal{F} \subset A$ be a finite set, and let $\varepsilon > 0$. There exists $n \in \mathbb{N}$ such that for each $f \in \mathcal{F}$ there exists $a_f \in A_n$ such that $\|a_f - f\| < \frac{\varepsilon}{2}$. The identity of A_n is the identity of A, and can be expressed as the sum of the projections

$$1_{A_n} = 1_A = \sum_{\substack{i\in I(n)\\ g\in G}} q_{i,g}.$$

We let

$$e_g = \sum_{i\in I(n)} q_{i,g},$$

then e_g is the sum mutually orthogonal central projections, and hence are themselves central projections. Thus they commute with every element of A_n. Hence, for any $g \in G$ and $f \in \mathcal{F}$ we have

$$\begin{aligned}
\|e_g f - f e_g\| &= \|e_g f - e_g a_f + e_g a_f - f e_g\| \\
&= \|e_g f - e_g a_f + a_f e_g - f e_g\| \\
&\leq \|e_g f - e_g a_f\| + \|a_f e_g - f e_g\| \\
&\leq \|e_g\|\|f - a_f\| + \|a_f - f\|\|e_g\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Furthermore, it is also easy to see that $g e_h = e_{gh}$, and so Condition 1 of Definition 5.7.1 is satisfied. Condition 3 of Definition 5.7.1 also follows directly from the definition of the $q_{i,g}$. ∎

The following theorem follows as a corollary of Proposition 5.7.6.

Theorem 5.7.7 *Let G be a symmetry group for $(\mathcal{P}, \omega)$, and suppose that G acts freely on $\mathcal{P}$. Then the action of G on AF_ω has the Rokhlin property.*

Proof: In the proof of the above theorem, we take $I(n)$ to be $\mathcal{S}_G$ for every n. For $p \in \mathcal{S}_G$ we take $q_{s,g} = \sum_{x \in \mathrm{Punc}(n,p)} e^n_{gp}(gx, gx)$. ∎

The second result of this section is that **if** $C^*_r(\mathcal{R}_{\mathrm{punc}})$ has tracial rank zero, then the action of G on $C^*_r(\mathcal{R}_{\mathrm{punc}})$ has the tracial Rokhlin property. Phillips conjectures that if $\mathscr{G}$ is an almost AF Cantor groupoid, then $C^*_r(\mathscr{G})$ has tracial rank zero ([47], Question 8.1), though this is currently unresolved. The following result may therefore be vacuous, though we believe it provides insight into how one might tackle Phillips' conjecture. In the interests of readability, for the remainder of this section we will write $\|a\|$ for the reduced norm of a for any $a \in C^*_r(\mathcal{R}_{\mathrm{punc}})$.

First, we need a lemma of Putnam. In what follows, $\partial(X)$ denotes the topological boundary of the space X.

Lemma 5.7.8 ([52], Lemma 2.3) *Let $(\mathcal{P}, \omega)$ be a substitution tiling system satisfying the conditions of Remark 2.5.8. For $p \in \mathcal{P}$, $n \in \mathbb{N}$, and $x \in Punc(n, p)$, we define $D(x)$ to be*

$$D(x) = \sup\{\|x - y\| \mid y \in \partial(\mathrm{supp}(\omega^n(p)))\}.$$

Then the quotient

$$\frac{\#\{x \in \mathrm{Punc}(n, p) \mid D(x) < R\}}{\#\mathrm{Punc}(n, p)}$$

converges to 0 as n goes to infinity.

Intuitively, the numerator scales with the perimeter of a tile while the denominator scales with the area, and so the limit being 0 is expected. This allows us to build positive elements which approximately commute with the generating set

$$\mathcal{E}_2 = \{e(\{t_1, t_2\}, t_1, t_2) \mid \text{the interior of } t_1 \cup t_2 \text{ is connected}\}$$

of $C_r^*(\mathcal{R}_{\mathrm{punc}})$ and have large trace.

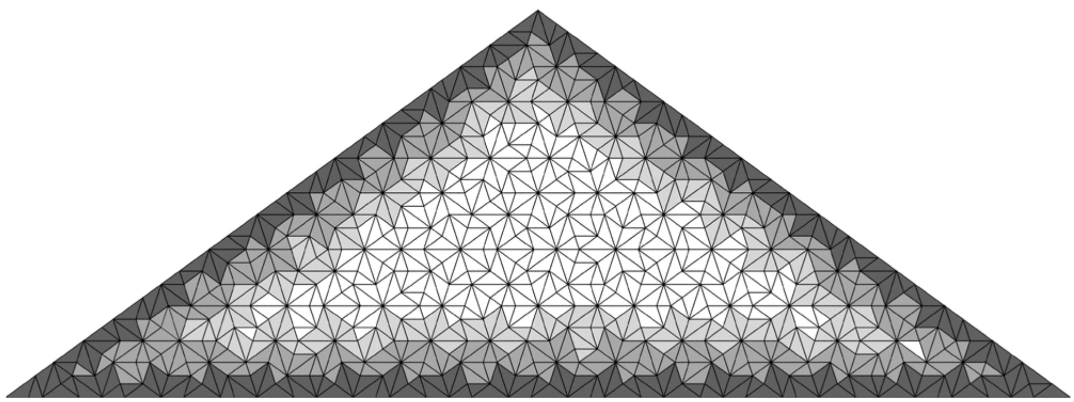
Lemma 5.7.9 *For any $\varepsilon > 0$ we can find a positive element $a_g \in C_r^*(\mathcal{R}_{\mathrm{punc}})$ for each $g \in G$ with $0 \leq a_g \leq 1$ such that*

1. *$a_g a_h = 0$ for each $g, h \in G$ with $g \neq h$,*
2. *$\|\alpha_g(a_h) - a_{gh}\| < \varepsilon$ for all $g, h \in G$,*
3. *$\|a_g f - f a_g\| < \varepsilon$ for all $g \in G$ and $f \in \mathcal{E}_2$, and*
4. *$\tau\left(1 - \sum_{g \in G} a_g\right) < \varepsilon$, where τ is the unique trace on $C_r^*(\mathcal{R}_{\mathrm{punc}})$ described at the end of Section 4.6.*

Proof: For $x \in \mathrm{Punc}(n,p)$ let $t(x)$ be the tile such that $x \in t(x)$, and if X is a set of punctures denote the set of tiles with elements of X as punctures by $t(X)$. We define, for $k \geq 0$, collections of punctures $\rho^k(n,p)$ as follows:

$$\begin{aligned}
\rho^0(n,p) &= \{x \in \mathrm{Punc}(n,p) \mid t(x) \cap \partial(\mathrm{supp}(\omega^n(p))) \neq \emptyset\} \\
\rho^k(n,p) &= \left\{x \in \mathrm{Punc}(n,p) \mid t(x) \cap \left(\partial\left(\mathrm{supp}\left(\omega^n(p) \setminus \cup_{i=0}^{k-1} t(\rho^i(n,p))\right)\right)\right) \neq \emptyset\right\}
\end{aligned}$$

Here, $\partial(A)$ denotes the topological boundary of the set $A \subset \mathbb{R}^2$. Loosely speaking, $\rho^0(n,p)$ is the set of punctures around the edge of the patch $\omega^n(p)$, $\rho^1(n,p)$ is the set of punctures around the edge of the patch $\omega^n(p) \setminus t(\rho^0(n,p))$ and so on.



In this picture, the punctures of the darkest tiles are $\rho^0(n,p)$, the punctures of the next darkest are $\rho^1(n,p)$ and the punctures of the lightest gray are $\rho^2(n,p)$. We notice that there exists $k' \in \mathbb{N}$ such that for all $k > k'$ we have that $\rho^k(n,p)$ is empty.

Let $\varepsilon > 0$ and find $N \in \mathbb{N}$ such that $N > \frac{2}{\varepsilon}$. Let d be the maximum diameter among the prototiles, and let $R > 0$ be such that $R > 2Nd$. By Lemma 5.7.8, there exists $s \in \mathbb{N}$ such that

$$\frac{\#\{x \in \mathrm{Punc}(s,p) \mid D(x) < R\}}{\#\mathrm{Punc}(s,p)} < \varepsilon. \tag{5.7.1}$$

Since punctures can be no more than $2d$ apart by the triangle inequality, for all i with $0 \leq i \leq N$ and $x \in \rho^i(s,p)$ we must have that $D(x) < R$. Let b_k be numbers such that $b_0 = 0$, $0 < b_j - b_{j-1} < \frac{\varepsilon}{2}$, and $b_k = 1$ for all $k > N$. We may find these numbers because $N \cdot \frac{\varepsilon}{2} > 1$. For the identity element $e \in G$, let

$$a_e = \sum_{k=0}^{\infty} \left(b_k \sum_{\substack{p \in \mathcal{S}_G \\ x \in \rho^k(s,p)}} e_p^s(x,x) \right),$$

and more generally

$$a_g = g a_e = \sum_{k=0}^{\infty} \left(b_k \sum_{\substack{p \in \mathcal{S}_G \\ x \in \rho^k(s,p)}} e_{gp}^s(gx,gx) \right).$$

Notice that these sums are finite because $\rho^k(s,p)$ are eventually empty, and that each a_g is an element of the finite dimensional algebra A_s. In addition, each a_g is a real-valued function in $C(\Omega_{\mathrm{punc}}) \subset C_r^*(\mathcal{R}_{\mathrm{punc}})$ which takes values between 0 and 1. Hence both a_g and $1 - a_g$ have positive square roots, and so $0 \le a_g \le 1$ for all $g \in G$.

Let us pause for a moment to give an intuitive description of the a_g. If we think of them as functions on $\mathrm{Punc}(s,p)$, they take the value 0 on the punctures around the boundary of $\omega^s(p)$, they take the value 1 on most of the punctures in the interior, and the values increase gradually from 0 to 1 as we move from the boundary towards the middle. The values that the a_g take on punctures whose tiles share an edge always differ by less than ε. Furthermore, they only take values less than 1 in a relatively small band of punctures near the boundary.

Since a_g is supported on the diagonal of $\mathcal{R}_{\mathrm{punc}}$ for all $g \in G$, we have that for any given $T \in \Omega_{\mathrm{punc}}$, $a_g(T,T')$ is only possibly nonzero if $T = T'$, and so by Equations (4.3.1) and (4.3.2)

$$\|a_g\|_r = \sup_{T \in \Omega_{\mathrm{punc}}} \left\{ \sum_{T' \in [T]} |a_g(T,T')| \right\} = \sup_{T \in \Omega_{\mathrm{punc}}} |a_g(T,T)| = 1,$$

$$\|a_g\|_s = \sup_{T \in \Omega_{\mathrm{punc}}} \left\{ \sum_{T' \in [T]} |a_g(T',T)| \right\} = \sup_{T \in \Omega_{\mathrm{punc}}} |a_g(T,T)| = 1.$$

Since $\|a_g\|_I$ is the max of these two norms and $\|a_g\|_I$ dominates the reduced norm, we have $\|a_g\| \leq 1$. The a_g elements satisfy Condition 1 trivially, and from the previous section we see that $ga_h = a_{gh}$.

To prove Condition 3, recall that $e^s_p(x,x)$ is the characteristic function of the set $E^s_p(x,x)$, which is a compact open subset of the unit space. Hence we may use Lemma 4.6.1 to calculate the products of the a_g with elements of $\mathcal{E}_2$. Take

$$q = e(\{t_1, t_2\}, t_1, t_2) \in \mathcal{E}_2,$$

and calculate

$$(a_g q)(T, T') = \sum_{k=0}^{\infty} \left(b_k \sum_{\substack{p \in \mathcal{S}_G \\ x \in \rho^k(s,p)}} e^s_{gp}(gx, gx) q(T, T') \right).$$

We know from Lemma 4.6.1 that

$$\begin{aligned} e^s_{gp}(gx,gx)q(T,T') &= \begin{cases} q(T,T') & \text{if } T \in E^s_{gp}(gx,gx) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } (T,T') \in V(\{t_1,t_2\},t_1,t_2) \text{ and} \\ & T \in E^s_{gp}(gx,gx) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Given $T \in \Omega_{\text{punc}}$, there exist unique $g \in G$, $p \in \mathcal{P}$ and $x \in \text{Punc}(s,p)$ such that $T \in E^s_{gp}(gx, gx)$. This puncture x must be an element of $\rho^k(s,p)$ for some k. Then in this case we have

$$(a_g q)(T, T') = \begin{cases} b_k & \text{if } (T,T') \in V(\{t_1,t_2\},t_1,t_2) \\ 0 & \text{otherwise.} \end{cases}$$

Now we calculate qa_g:

$$(qa_g)(T, T') = \sum_{k=0}^{\infty} \left(b_k \sum_{\substack{p \in \mathcal{S}_G \\ x \in \rho^k(s,p)}} q e^s_{gp}(gx, gx)(T, T') \right),$$

and similar to above

$$qe^s_{gp'}(gy,gy)(T,T') = \begin{cases} q(T,T') & \text{if } T' \in E^s_{gp'}(gy,gy), \\ 0 & T' \notin E^s_{gp'}(gy,gy) \end{cases}$$
$$= \begin{cases} 1 & \text{if } (T,T') \in V(\{t_1,t_2\},t_1,t_2) \text{ and} \\ & T' \in E^s_{gp}(gy,gy) \\ 0 & \text{otherwise.} \end{cases}$$

As above, given $T' \in \Omega_{\text{punc}}$ there exist unique $g \in G$, $p \in \mathcal{P}$ and $y \in \text{Punc}(s,p)$ such that $T \in E^s_{gp}(gy,gy)$. This puncture y must be an element of $\rho^m(s,p)$ for some k. Then in this case we have

$$(qa_g)(T,T') = \begin{cases} b_m & \text{if } (T,T') \in V(\{t_1,t_2\},t_1,t_2) \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we may calculate the difference

$$(a_gq - qa_g)(T,T') = \begin{cases} b_k - b_m & (T,T') \in V(\{t_1,t_2\},t_1,t_2), \\ & T \in E^s_{gp}(gx,gx), x \in \rho^k(s,p), \text{ and} \\ & T' \in E^s_{gp'}(gy,gy), y \in \rho^m(s,p') \\ 0 & \text{otherwise.} \end{cases}$$

If we are in the first case and $p \neq p'$, then k and m must both be zero. Indeed, if $p \neq p'$, then $\{t_1,t_2\}$ is a two-tile pattern whose edge lies along the boundary of gp and gp', and hence $t(x)$ and $t(y)$ intersect the boundaries of $\omega^s(p)$ and $\omega^s(p')$ respectively. Thus $x \in \rho^0(s,p)$ and $y \in \rho^0(s,p')$. In the case where $p = p'$, the conditions in the first case above imply that the patch $\{t(gx), t(gy)\}$ is a translate of $\{t_1,t_2\}$. Hence the difference between k and m is at most 1, and by the definition of the b_i this implies that $|b_k - b_m| < \frac{\varepsilon}{2}$. Furthermore, if $T \in \Omega_{\text{punc}}$, there is at most one T' for which $(a_gq - qa_g)(T,T')$ is nonzero, namely $T' = T + x_{t_1} - x_{t_2}$ if T happens to be in $U(\{t_1,t_2\},t_1)$. Hence

$$\|a_gq - qa_g\|_r = \sup_{T\in\Omega_{\text{punc}}} \left\{ \sum_{T'\in[T]} |(a_gq - qa_g)(T,T')| \right\} \leq \frac{\varepsilon}{2}$$

$$\|a_g q - q a_g\|_s = \sup_{T' \in \Omega_{\text{punc}}} \left\{ \sum_{T \in [T']} |(a_g q - q a_g)(T, T')| \right\} \leq \frac{\varepsilon}{2}$$

$$\|a_g q - q a_g\| \leq \max\{\|a_g q - q a_g\|_r, \|a_g q - q a_g\|_s\} \leq \frac{\varepsilon}{2} < \varepsilon.$$

Hence Condition 3 is satisfied. To prove Condition 4 we use Equation (5.7.1). The function $1 - \sum_g a_g$ is nonnegative and is only nonzero on elements (T, T) such that

$$(T, T) \in \bigcup_{p \in \mathcal{P}} \bigcup_{i=0}^{N-1} \bigcup_{x \in \rho^i(s,p)} E_p^s(x, x).$$

Notice that the above union is a disjoint union. Hence for every $T \in \Omega_{\text{punc}}$ we have that

$$\left(1 - \sum_g a_g\right)(T, T) \leq \sum_{p \in \mathcal{P}} \sum_{i=0}^{N-1} \sum_{x \in \rho^i(s,p)} e_p^s(x, x)(T, T).$$

Recall from Section 4.6 that $C_r^*(\mathcal{R}_{\text{punc}})$ has a unique trace τ such that for all $n \in \mathbb{N}$, $p \in \mathcal{P}$ and $x \in \text{Punc}(n, p)$ we have $\tau(e_p^n(x, x)) = \lambda^{-2n} v_L(p)$, where λ is the scaling constant of $(\mathcal{P}, \omega)$ and $v_L(p)$ is the pth entry of the left Perron eigenvector associated to the matrix M of partial multiplicities for AF_ω. We calculate

$$\begin{aligned}
\tau\left(1 - \sum_g a_g\right) &= \int_{\Omega_{\text{punc}}} \left(1 - \sum_g a_g\right)(T, T)\, d\mu(T) \\
&\leq \int_{\Omega_{\text{punc}}} \sum_{p \in \mathcal{P}} \sum_{i=0}^{N-1} \sum_{x \in \rho^i(s,p)} e_p^s(x, x)(T, T)\, d\mu(T) \\
&= \sum_{p \in \mathcal{P}} \sum_{i=0}^{N-1} \sum_{x \in \rho^i(s,p)} \int_{\Omega_{\text{punc}}} e_p^s(x, x)(T, T)\, d\mu(T) \\
&= \sum_{p \in \mathcal{P}} \sum_{i=0}^{N-1} \sum_{x \in \rho^i(s,p)} \tau(e_p^s(x, x)) \\
&= \sum_{p \in \mathcal{P}} \sum_{i=0}^{N-1} \sum_{x \in \rho^i(s,p)} \lambda^{-2s} v_L(p) \\
&\leq \sum_{p \in \mathcal{P}} \#\{x \in \text{Punc}(s, p) \mid D(x) < R\} \lambda^{-2s} v_L(p)
\end{aligned}$$

$$\begin{aligned} &< \varepsilon \sum_{p \in \mathcal{P}} \#\mathrm{Punc}(s,p)\lambda^{-2s} \upsilon_L(p) \\ &= \varepsilon \end{aligned}$$

where the last line is by Equation (4.6.6). ∎

Since $\mathcal{E}_2$ is a generating set for $C_r^*(\mathcal{R}_{\text{punc}})$, we can use Lemma 5.7.9 to prove the following theorem.

Theorem 5.7.10 *Suppose that G is a symmetry group for $(\mathcal{P}, \omega)$ that acts freely on $\mathcal{P}$, let Ω_{punc} be the punctured hull associated to $(\mathcal{P}, \omega)$ and let $\mathcal{R}_{\text{punc}}$ be the usual groupoid of translational equivalence on Ω_{punc}. If $C_r^*(\mathcal{R}_{\text{punc}})$ has tracial rank zero, then the action of G on $C_r^*(\mathcal{R}_{\text{punc}})$ has the tracial Rokhlin property.*

Proof: Here we invoke Theorem 5.7.4 as well as Lemmas 5.7.5 and 5.7.9. That the order on projections is determined by traces is the main theorem of [52] (Theorem 1.1). In addition to generalizing this result in [47], Phillips also proves that if $\mathscr{G}$ is an almost AF Cantor groupoid, then $C_r^*(\mathscr{G})$ has real rank zero and stable rank one. Hence we may use Lemma 5.7.5.

Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a finite set, and let $\varepsilon > 0$. For each $i \in I$, find $\lambda_j^{(i)} \in \mathbb{C}$ and $x_{j,k}^{(i)} \in \mathcal{E}_2$ with $1 \leq j \leq J_i$, $1 \leq k \leq K_i$ such that

$$\left\| f_i - \sum_{j=1}^{J_i} \lambda_j^{(i)} \left(\prod_{k=1}^{K_i} x_{k,j}^{(i)} \right) \right\| < \frac{\varepsilon}{3}.$$

Let

$$\begin{aligned} y_i &= \sum_{j=1}^{J_i} \lambda_j^{(i)} \left(\prod_{k=1}^{K_i} x_{k,j}^{(i)} \right) \\ \Lambda &= \max\{|\lambda_j^{(i)}|, 2\} \\ J &= \max\{J_i, 2\} \\ K &= \max\{K_i, 2\} \\ \delta &= \frac{\varepsilon}{3JK\Lambda}. \end{aligned}$$

Apply Lemma 5.7.9 with the positive number δ and obtain positive elements a_g for $g \in G$ such that

- $a_g a_h = 0$ for each $g, h \in G$ with $g \neq h$,
- $\|\beta_g(a_h) - a_{gh}\| < \delta$ for all $g, h \in G$,
- $\|a_g f - f a_g\| < \delta$ for all $g \in G$ and $f \in \mathcal{E}_2$, and
- $\tau\left(1 - \sum_{g \in G} a_g\right) < \delta$.

Since $\delta < \varepsilon$, Conditions 1, 2 and 4 of Lemma 5.7.5 are satisfied. We now must check Condition 3. We note that the elements and nonzero products of elements of $\mathcal{E}_2$ are partial isometries, and hence they have norm 1. For some a_g and y_i we have

$$\begin{aligned}
\|a_g y_i - y_i a_g\| &= \left\| a_g \sum_{j=1}^{J_i} \lambda_j^{(i)} \left(\prod_{k=1}^{K_i} x_{k,j}^{(i)} \right) - \sum_{j=1}^{J_i} \lambda_j^{(i)} \left(\prod_{k=1}^{K_i} x_{k,j}^{(i)} \right) a_i \right\| \\
&= \left\| \sum_{j=1}^{J_i} \lambda_j^{(i)} \left(a_g \prod_{k=1}^{K_i} x_{k,j}^{(i)} - \prod_{k=1}^{K_i} x_{k,j}^{(i)} a_g \right) \right\| \\
&\leq \sum_{j=1}^{J_i} \Lambda \left\| a_g \prod_{k=1}^{K_i} x_{k,j}^{(i)} - \prod_{k=1}^{K_i} x_{k,j}^{(i)} a_g \right\| \\
&= \Lambda \sum_{j=1}^{J_i} \left\| a_g \prod_{k=1}^{K_i} x_{k,j}^{(i)} - x_{1,j}^{(i)} a_g \prod_{k=2}^{K_i} x_{k,j}^{(i)} + x_{1,j}^{(i)} a_g \prod_{k=2}^{K_i} x_{k,j}^{(i)} - \prod_{k=1}^{K_i} x_{k,j}^{(i)} a_g \right\| \\
&\leq \Lambda \sum_{j=1}^{J_i} \left(\left\| a_g x_{1,j}^{(i)} - x_{1,j}^{(i)} a_g \right\| \left\| \prod_{k=2}^{K_i} x_{k,j}^{(i)} \right\| + \dots \right. \\
&\qquad \left. \dots + \left\| x_{1,j}^{(i)} \right\| \left\| a_g \prod_{k=2}^{K_i} x_{k,j}^{(i)} - \prod_{k=2}^{K_i} x_{k,j}^{(i)} a_g \right\| \right) \\
&= \Lambda \sum_{j=1}^{J_i} \left(\left\| a_g x_{1,j}^{(i)} - x_{1,j}^{(i)} a_g \right\| + \left\| a_g \prod_{k=2}^{K_i} x_{k,j}^{(i)} - \prod_{k=2}^{K_i} x_{k,j}^{(i)} a_g \right\| \right).
\end{aligned}$$

We can see that we can apply the same triangle inequality argument on the right portion of the sum for $2 \leq k \leq K_i$ and obtain

$$\|a_g y_i - y_i a_g\| \leq \Lambda \sum_{j=1}^{J_i} \sum_{k=1}^{K_i} \left\| a_g x_{k,j}^{(i)} - x_{k,j}^{(i)} a_g \right\|$$

$$
\begin{aligned}
&< \Lambda \sum_{j=1}^{J_i} \sum_{k=1}^{K_i} \delta \\
&= \Lambda J K \frac{\varepsilon}{3\Lambda J K} \\
&= \frac{\varepsilon}{3}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|a_g f_i - f_i a_g\| &= \|a_g f_i - a_g y_i + a_g y_i - y_i a_g + y_i a_g - f_i a_g\| \\
&\leq \|a_g\| \|f_i - y_i\| + \|a_g y_i - y_i a_g\| + \|y_i - f_i\| \|a_g\| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}
$$

∎

We notice that there was nothing special about the form of the a_g in the above proof – indeed if A is generated by a finite set $\mathcal{F}$ and one can satisfy the conditions for the tracial Rokhlin property for $\mathcal{F}$, one can satisfy them for an arbitrary finite set using a similar argument.

Corollary 5.7.11 *Suppose that G is a finite group that acts freely on $\mathcal{P}$. If $C^*_r(\mathcal{R}_{\text{punc}})$ has tracial rank zero, then $C^*_r(\mathcal{R}_{\text{punc}}) \rtimes G$ has tracial rank zero.*

Proof: This follows from Theorem 5.7.10 and [50], Theorem 2.6. ∎

We conclude this section by remarking on Corollary 5.7.11. In [47], Question 8.1, Phillips asks the question of whether every C*-algebra of an almost AF Cantor groupoid has tracial rank zero. If the answer to this question is yes, then it would appear that Corollary 5.7.11 would follow immediately, as both $\mathcal{R}_{\text{punc}}$ and $\mathcal{R}_{\text{punc}} \rtimes G$ are almost AF Cantor groupoids. However, it is a fact that if $\mathcal{R}_{\text{punc}}$ is the groupoid formed from any tiling of $\mathbb{R}^2$ consisting of polygons which meet full-edge to full-edge which has repetitivity and strong aperiodicity, then there exists a minimal transformation group groupoid $(X, \mathbb{Z}^2)$ with X homeomorphic to the Cantor set and a clopen $U \subset X$ such that $\mathcal{R}_{\text{punc}}$ is isomorphic to $(X, \mathbb{Z}^2)_U^U$, see [63]. There is no such result for the groupoid $\mathcal{R}_{\text{punc}} \rtimes G$. Hence Corollary 5.7.11 would tell us that $C_r^*(\mathcal{R}_{\text{punc}}) \rtimes G$ has tracial rank zero if one could prove that $C(X) \rtimes \mathbb{Z}^2$ has tracial rank zero for all minimal actions of $\mathbb{Z}^2$ on the Cantor set X.

Chapter 6

K-Theory

K-theory is an important invariant for C*-algebras. Topological K-theory is an invariant for topological spaces based on vector bundles over the space. This invariant turns out to generalize to "noncommutative spaces", i.e. C*-algebras. For some classes of algebras, such as AF algebras, it is a complete invariant. In this chapter we briefly define K-theory and compute it for some of our examples.

6.1 $K_0(A)$ and $K_1(A)$

This section is a brief development of K-theory culminating in the definition of $K_0(A)$ and $K_1(A)$ for a unital C*-algebra A. Each C*-algebra we consider in this thesis is either unital or strongly Morita equivalent to a unital C*-algebra, so we do not give up anything by only defining these groups in the unital case only. We follow [60] very closely.

Consider $\mathbb{M}_n(A)$, all $n \times n$ matrices with entries in A. Let $\mathcal{P}_n(A)$ be the set of all projections in $\mathbb{M}_n(A)$. There is a natural inclusion $\mathcal{P}_n(A) \hookrightarrow \mathcal{P}_{n+1}(A)$: given

$p \in \mathcal{P}_n(A)$ then

$$p \hookrightarrow \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{P}_{n+1}(A).$$

Let

$$\mathcal{P}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(A).$$

We put an equivalence relation $\sim_0$ on $\mathcal{P}_\infty(A)$ by saying that $p \in \mathcal{P}_n(A)$ is equivalent to $q \in \mathcal{P}_m(A)$ if there exists an element $v \in \mathbb{M}_{m,n}(A)$ (this is the $(m \times n)$-matrices with entries in A) such that $p = v^*v$ and $q = vv^*$. We define an addition $\oplus$ on $\mathcal{P}_\infty(A)$ as follows: given $p \in \mathcal{P}_n(A), q \in \mathcal{P}_m(A)$,

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in \mathcal{P}_{n+m}(A)$$

Let $\mathcal{D}(A) = \mathcal{P}_\infty(A)/\sim_0$. Denote by $[p]_0$ the equivalence class of a projection in $\mathcal{D}(A)$. One can show that the class of $p + q$ depends only on the classes of p and q, so that

$$[p]_0 + [q]_0 := [p \oplus q]_0$$

is a well-defined operation on $\mathcal{D}(A)$. This operation gives $\mathcal{D}(A)$ the structure of an abelian semigroup (see for example [60], Proposition 2.3.2).

Definition 6.1.1 *If A is a unital C^*-algebra, then $K_0(A)$ is the Grothendieck group associated to the abelian semigroup $\mathcal{D}(A)$.*

In [60], Proposition 3.1.7, the authors prove that

$$K_0(A) = \{[p]_0 - [q]_0 \mid p, q \in \mathcal{P}_n, n \in \mathbb{N}\}.$$

Intuitively, $K_0(A)$ is the group consisting of formal differences of equivalence classes of projections.

The group $K_0(A)$ carries a natural order structure, which we will now describe.

Definition 6.1.2 *Let G be an abelian group and G^+ be a subset of G. Then the pair (G, G^+) is an* **ordered abelian group** *if*

1. $G^+ + G^+ \subset G^+$,
2. $G^+ \cap (-G^+) = \{0\}$, *and*
3. $G^+ - G^+ = G$.

This induces an order $\leq$ on G by saying that $x \leq y$ if $y - x \in G^+$. A homomorphism of ordered abelian groups is a group homomorphism that sends positive elements to positive elements. We shall call such homomorphisms **positive** *group homomorphisms.*

The abelian group $K_0(A)$ carries a natural order structure, by declaring the elements of the form $[p]_0$ to be positive, i.e. by letting $K_0(A)^+ = \{[p]_0 \mid p \in \mathcal{P}_n, n \in \mathbb{N}\}$. For arbitrary unital C*-algebra A, $(K_0(A), K_0(A)^+)$ need not satisfy Condition 2 in Definition 6.1.2 – for instance, K_0 of the Cuntz algebras $\mathcal{O}_n$ for $n \geq 3$ does not satisfy this (in this case, we actually have $K_0(A)^+ = K_0(A)$, see [60] Exercise 4.5). Under the additional assumption that A is unital and stably finite, $(K_0(A), K_0(A)^+)$ is an ordered abelian group (see [60], Proposition 5.1.5).

An **order unit** in an ordered abelian group (G, G^+) is an element $u \in G^+$ such that for every g in G there is a positive integer n such that $-nu \leq g \leq nu$. We call the triple (G, G^+, u) an **ordered abelian group with distinguished order unit** u. Such triples form a category where a morphism between (G, G^+, u) and (H, H^+, v) is a positive group homomorphism $\varphi : G \to H$ such that $\varphi(u) = v$. If A is unital, then $[1_A]_0$ is an order unit for (G, G^+).

We now define the K_1 group of a unital C*-algebra A. Let $\mathcal{U}(A)$ denote the set of unitary elements in A. Let

$$\mathcal{U}_n(A) = \mathcal{U}(\mathbb{M}_n(A)) \qquad \mathcal{U}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{U}_n(A).$$

The inclusion of $\mathcal{U}_n(A) \hookrightarrow \mathcal{U}_m(A)$ is given by concatenating with 1 along the diagonal: given $u \in \mathcal{U}_n(A)$ then

$$u \hookrightarrow \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}.$$

As with defining K_0, we define an operation on $\mathcal{U}_\infty(A)$ by concatenation along the diagonal: if $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$, let

$$u \oplus v = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \in \mathcal{U}_{n+m}(A).$$

We define an equivalence relation $\sim_1$ on $\mathcal{U}_\infty$ based on homotopy equivalence. If $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$, we say $u \sim_1 v$ if we can find an integer $k > m, n$ such that there is a continuous path of unitaries from $u \oplus 1_{k-n}$ to $v \oplus 1_{k-m}$ in $\mathcal{U}_k(A)$. Denote by $[u]_1$ the equivalence class of u in this relation. We let

$$K_1(A) = \mathcal{U}_\infty(A)/\sim_1$$

and bestow it with an operation $[u]_1 + [v]_1 = [u \oplus v]_1$. This operation is well-defined and gives $K_1(A)$ the structure of an abelian group (see for example [60], Lemma 8.1.4).

We note that K_0 and K_1 may be defined for a nonunital C*-algebra A in terms of its unitization, see [60] for details. As defined above, it is clear that K_0 and K_1 are isomorphism invariant.

Recall from Definition 4.4.3 that two C*-algebras are stably isomorphic if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$ where $\mathcal{K}$ is the C*-algebra of compact operators on an infinite dimensional separable Hilbert space.

Theorem 6.1.3 ([60], Propositions 6.4.1 and 8.2.8) *If A is a C*-algebra, then*

$$K_i(A \otimes \mathcal{K}) \cong K_i(A), \quad i = 0, 1.$$

Corollary 6.1.4 *Suppose that A and B are separable C^*-algebras that are strongly Morita equivalent. Then*

$$K_i(A) \cong K_i(B) \quad i = 0, 1.$$

Example 6.1.5 As mentioned at the beginning of the chapter, C*-algebra K-theory is a generalization of **topological K-theory** of a space X, denoted $K^*(X)$, which we do not define here. We do note the following facts about topological K-theory:

- If $A = C(X)$ for some compact Hausdorff X, then $K_*(A) \cong K^*(X)$.
- If X is a CW complex of dimension at most two, then

  $$K^*(X) = \bigoplus_{k \text{ even}} H^{*+k}(X, \mathbb{Z})$$

  where the latter are simplicial cohomology groups with coefficients in $\mathbb{Z}$. This is a consequence of the Atiyah-Hirzebruch spectral sequence, see [1], Proposition 6.2 for an example of a calculation of such and [2] for more generality.
- If $X = \lim\limits_{\leftarrow} X_i$ is an inverse limit of spaces, then $K^*(X) = \lim\limits_{\rightarrow} K^*(X_i)$.

In [1], the authors use these facts to calculate the K-theory of $C_r^*(\mathcal{R}_{\text{punc}})$. We will use them along with Corollary 6.1.4 and Theorem 5.5.2 to calculate the K-theory of $C_r^*(\mathcal{R}_{\text{punc}}) \rtimes G$.

6.2 K-theory of AF Algebras

Both K_0 and K_1 are functors from the category of C*-algebras to the category of abelian groups, see [60] Chapter 4 and Section 8.2. Computation of $K_i(A)$ is made easier by the fact they are both continuous functors, that is if $A = \lim \left(A_n \overset{\phi_n}{\to} A_{n+1}\right)$ is an inductive limit of C*-algebras A_n, then

$$K_i(A) = \lim \left(K_i(A_n) \overset{K_i(\phi_n)}{\to} K_i(A_{n+1})\right)$$

where the limit is taken in the category of abelian groups. Both functors also respect direct sums,

$$K_i(A \oplus B) = K_i(A) \oplus K_i(B).$$

It is not hard to show that, for any $n \in \mathbb{N}$,

$$K_0(\mathbb{M}_n(\mathbb{C})) = \mathbb{Z} \qquad K_1(\mathbb{M}_n(\mathbb{C})) = 0,$$

see for example [60] Examples 3.3.2 and 8.1.8. The above combine to tell us that if $A = \overline{\cup A_n}$ is an AF-algebra with inclusion maps ι_n, then $K_1(A) = 0$ and

$$K_0(A) = \lim \left(\mathbb{Z}^{k(n)} \stackrel{K_i(\iota_n)}{\to} \mathbb{Z}^{k(n+1)} \right),$$

where $k(n)$ is the number of matrix summands in A_n. In [17] Example IV.3.1 (for example), it is proved that $K_i(\iota_n)$ is the matrix of partial multiplicities associated to the unital inclusion ι_n defined in Section 4.5. Hence calculating K_0 of an AF algebra comes down to calculating an inductive limit of abelian groups. As ordered abelian groups, the order on $K_0(A_n) \cong \mathbb{Z}^{k(n)}$ is given by saying that $v \in \left(\mathbb{Z}^{k(n)}\right)^+$ if each entry in v is nonnegative (this is called the **simplicial order** on $\mathbb{Z}^{k(n)}$). The entries of the matrices of partial multiplicities are always nonnegative, so $(K_0(A), K_0(A)^+)$ is isomorphic to the direct limit of the $(K_0(A_n), K_0(A_n)^+)$ in the category of ordered abelian groups (see [60], Chapter 6). Direct limits of free abelian finitely generated groups with the simplicial order under positive homomorphisms are quite important and are known as **dimension groups**. The theory of dimension groups has many diverse and deep applications beyond AF algebras, such as the classification of minimal $\mathbb{Z}$ actions on the Cantor set, see [27]. An abstract characterization of dimension groups was given in a celebrated theorem of Effros, Handelman, and Shen, see [21].

In [22], Elliott proved the remarkable result that the triple $(K_0(A), K_0(A)^+, [1_A]_0)$ is a complete invariant for unital AF algebras. That is, two unital AF algebras A and B are isomorphic if and only if $(K_0(A), K_0(A)^+, [1_A]_0)$ and $(K_0(B), K_0(B)^+, [1_B]_0)$ are isomorphic in the category of ordered abelian groups with distinguished order unit.

Moreover, if φ is the positive group isomorphism between $(K_0(A), K_0(A)^+, [1_A]_0)$ and $(K_0(B), K_0(B)^+, [1_B]_0)$, there is a C*-algebra isomorphism $\Phi : A \to B$ such that $K_0(\Phi) = \varphi$. For a proof one can also see [60] Theorem 7.3.4 or [17] Theorem IV.4.3.

Computation of the K_0 group for AF algebras which have constant primitive incidence matrix is well-understood; see for example [17], Example IV.3.5. We compute these groups for the octagonal and the Penrose tiling examples.

Example 6.2.1 Octagonal, $G = D_8$.

This is a continuation of Example 5.6.8. There we computed that the incidence matrix for the AF algebra $AF_\omega \rtimes D_8$ to be

$$M = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix}.$$

Hence we need to compute the limit H of

$$\mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix}} \mathbb{Z}^2 \longrightarrow \dots$$

in the category of ordered abelian groups with distinguished order unit. One calculates that the Perron-Frobenius eigenvalue and left eigenvector are

$$\rho = 3 + 2\sqrt{2} = (1 + \sqrt{2})^2, \qquad v_L = \begin{bmatrix} 2\sqrt{2} & 1 \end{bmatrix}.$$

We have the following commutative diagram

$$\begin{tikzcd}
\mathbb{Z}^2 \arrow[r, "M"] \arrow[dr, "\pi_0"'] & \mathbb{Z}^2 \arrow[r, "M"] \arrow[d, "\pi_1"] & \mathbb{Z}^2 \arrow[r, "M"] \arrow[dl, "\pi_2"] & \cdots \\
& \mathbb{R} & &
\end{tikzcd}$$

where $\pi_i(x) = \dfrac{x \cdot v_L}{\rho^i}$. Since

$$\rho^{-1} = \frac{1}{3 + 2\sqrt{2}} = 3 - 2\sqrt{2},$$

we have that the image of each π_i is $\mathbb{Z}+2\sqrt{2}\mathbb{Z}$. Since the $\pi_i(x)$ are each injective, this means that $H \cong \mathbb{Z}+2\sqrt{2}\mathbb{Z}$, and $H^+ \cong (\mathbb{Z}+2\sqrt{2}\mathbb{Z})^+$, the positive elements of $\mathbb{Z}+2\sqrt{2}\mathbb{Z}$ when viewed as a subgroup of $\mathbb{R}$. The identity element of $AF_\omega \rtimes D_8$ when viewed in $A_0 \rtimes D_8$ gets mapped to the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{Z}^2$, and

$$\pi_0\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 1+2\sqrt{2}.$$

Hence we have

$$\left(K_0(AF_\omega \rtimes D_8), K_0(AF_\omega \rtimes D_8)^+, [1]_0\right) \cong \left(\mathbb{Z}+2\sqrt{2}\mathbb{Z}, (\mathbb{Z}+2\sqrt{2}\mathbb{Z})^+, 1+2\sqrt{2}\right).$$

It is sometimes convenient to have the order unit equal to 1. In this case, we can write

$$K_0(AF_\omega \rtimes D_8) \cong \frac{\mathbb{Z}+2\sqrt{2}\mathbb{Z}}{1+2\sqrt{2}} \subset \mathbb{R}$$

with order and 1 inherited from $\mathbb{R}$.

Recall that when one does not break symmetry, one gets that $AF_\omega \rtimes D_8$ is still AF, but has incidence matrix

$$M = \begin{bmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 3 \end{bmatrix}.$$

Since M^2 has strictly positive entries, it is primitive. The matrix M also has Perron-Frobenius eigenvalue $\rho = 3+2\sqrt{2}$ with left eigenvector $v_L = \begin{bmatrix} 1 & 1 & 1 & 1 & \sqrt{2} \end{bmatrix}$. The matrix M has determinant -1, and hence is invertible over the integers. Thus as a group, $K_0(AF_\omega \rtimes D_8)$ is isomorphic to $\mathbb{Z}^5$. In fact, we once again have

$$\begin{tikzcd} \mathbb{Z}^5 \arrow[r, "M"] \arrow[dr, "\pi_0"'] & \mathbb{Z}^5 \arrow[r, "M"] \arrow[d, "\pi_1"] & \mathbb{Z}^5 \arrow[r, "M"] \arrow[dl, "\pi_2"] & \cdots \\ & \mathbb{R} & & \end{tikzcd}$$

where $\pi_i(x) = \frac{x \cdot v_L}{\rho^i}$. The $\{\pi_i\}$ induce a map $\hat{\tau} : K_0(AF_\omega \rtimes D_8) \to \mathbb{R}$ such that $[p]_0 \in K_0(AF_\omega \rtimes D_8)^+$ if and only if $\hat{\tau}([p]_0) \geq 0$. The image of $\hat{\tau}$ is clearly $\mathbb{Z} + \sqrt{2}\mathbb{Z}$, as this is the result of taking the dot product of v_L with integer vectors. Hence we can choose a generating set for $\mathbb{Z}^5$ such that

$$K_0(AF_\omega \rtimes D_8) \cong \left(\mathbb{Z} + \sqrt{2}\mathbb{Z}\right) \oplus \mathbb{Z}^3$$

where an element (a, b) is positive if and only if $a \geq 0$ when viewed as a real number. The order unit once again arises by taking the dot product of v_L by the vector of all ones. Hence our distinguished order unit in $K_0(AF_\omega \rtimes D_8)$ is $(4 + \sqrt{2}, 0)$. We may once again scale the group so that the order unit is equal to 1. In this case, we have

$$\hat{\tau}\left(K_0(AF_\omega \rtimes D_8)\right) = \frac{\mathbb{Z} + \sqrt{2}\mathbb{Z}}{4 + \sqrt{2}}.$$

This example shows that changing the substitution changes the form of $AF_\omega \rtimes G$.

Example 6.2.2 Penrose, $G = D_{10}$.

This is a continuation of Example 5.6.7. There we computed the incidence matrix to be

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Computing the stationary limit of $\mathbb{Z}^2$ under this matrix is [17], Example IV.3.5. Using the same techniques as Example 6.2.1, one gets

$$K_0(AF_\omega \rtimes D_{10}) \cong \mathbb{Z} + \gamma^{-1}\mathbb{Z} \subset \mathbb{R}$$

where γ is the golden ratio, with order and unit inherited from $\mathbb{R}$. This is in fact the same ordered group that Connes obtains in [16], Section 2.3 for an AF algebra arising from the space of Penrose tilings, see our Example 4.5.4. In his example, he considers a space homeomorphic to Ω_{punc} and declares two tilings to be equivalent if one can be carried to the other by any isometry of the plane. It may seem odd that we get the same result, since one would imagine that equivalence by any isometry could be bigger than the equivalence relation $\mathcal{R}_{AF} \rtimes D_{10}$. It is a fact for Penrose tilings that if $(T, T') \in \mathcal{R}_{\text{punc}} \setminus \mathcal{R}_{AF}$, then there exists $g \in D_{10}$ such that $(T, gT') \in \mathcal{R}_{AF}$, see for example [35], Section 4.2.1. This means that $\mathcal{R}_{AF} \rtimes D_{10}$ is in fact equivalence by any isometry on Ω_{punc}, and is a consequence of every edge in the Penrose cell complex being substitution symmetric, see Definition 5.3.4.

We make a final note on this example. By the above comments one notices that as sets,

$$\mathcal{R}_{\text{punc}} \subset \mathcal{R}_{AF} \rtimes D_{10}$$

since the left is equivalence on Ω_{punc} by any translation and the right is equivalence by any isometry. This may lead one to believe that $C^*(\mathcal{R}_{\text{punc}}) \subset AF_\omega \rtimes D_{10}$, which would imply that $C^*_r(\mathcal{R}_{\text{punc}})$ is embeddable in an AF algebra. Unfortunately, the relative topology inherited from $\mathcal{R}_{AF} \rtimes D_{10}$ does not coincide with the topology on $\mathcal{R}_{\text{punc}}$, so it is unknown whether $C^*_r(\mathcal{R}_{\text{punc}})$ can be embedded in an AF algebra.

6.3 K-theory of Crossed Products by Almost Connected Groups

The main goal for the rest of this chapter is to calculate the K-theory of the C*-algebras listed in Theorem 5.5.2. These C*-algebras are all strongly Morita equivalent, so they will have the same K-theory. This section and the next outline the techniques for calculating the K-theory of algebra (2) in Theorem 5.5.2, that is $C(\Omega) \rtimes (\mathbb{R}^2 \rtimes G)$.

In [32], Kasparov obtains a result for computing the K-theory of $C(X) \rtimes H$, where H is as follows.

Definition 6.3.1 *Let H be a locally compact group and let H_0 denote the connected component of the identity in H. Then we say that H is* **almost connected** *if H/H_0 is compact.*

If H is an almost connected group, then H has a maximal compact subgroup L and H/L is homeomorphic to a Euclidean space (see for example [67], Theorem 32.5). We denote by V the Euclidean space H/L. We will also need the following definition.

Definition 6.3.2 *Let A, B, and C be groups. By a* **central extension** *of C by A we mean a short exact sequence of groups*

$$1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1$$

such that A is contained in the centre of B. Two such extensions

$$1 \to A \to B \to C \to 1 \qquad \text{and} \qquad 1 \to A \to B' \to C \to 1$$

are said to be **isomorphic** *if there exists a group isomorphism from B to B' such that*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 1 \\ & & \| & & \downarrow & & \| & & \\ 1 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 1 \end{array}$$

commutes. The set of all isomorphism classes of central extensions of C by A is denoted $H^2(C, A)$.[1]

The following result is due to Kasparov ([32], 5.10) and is stated in the following form in [18], Theorem 3.1 and [15], §7.

Theorem 6.3.3 (**Kasparov**) *Let H be an amenable*[2] *almost connected group and let L be the maximal compact subgroup of H. Let A be an arbitrary C^*-algebra acted upon by H. Then*

$$K_i(A \rtimes H) \cong K_i((A \otimes C_0(V)) \rtimes L).$$

[1]The notation $H^2(G, A)$ is due to the fact that the isomorphism classes of extensions can be naturally identified with the cohomology group $H^2(G, A)$ where G acts trivially on A, see [70], Chapter V.

[2]Amenability is not an assumption as stated in [18], Theorem 3.1. However, to state this result without mentioning KK-theory or equivariant topological K-theory, we need to restrict to a class of groups for which the Baum-Connes conjecture holds. Amenable groups form such a class, see [30].

Suppose further that H/L is even dimensional when viewed as a real vector space, and that the action

$$\mathrm{Ad} : L \to \mathrm{Aut}(H/L)$$

$$\mathrm{Ad}_g(h + L) = ghg^{-1} + L$$

is orientation preserving. Then

$$K_i(A \rtimes H) \cong K_i((A \otimes \mathcal{K}) \rtimes L)$$

for some algebra of compact operators $\mathcal{K}$, where the action of L on $\mathcal{K}$ is determined by an element in $H^2(L, \mathbb{T})$ which vanishes if and only if H/L carries an H-equivariant spin^c *structure. If this is the case then the action of L on $\mathcal{K}$ is trivial, and we have*

$$K_i(A \rtimes H) \cong K_i(A \rtimes L).$$

With regards to tilings, we consider $C(\Omega) \rtimes (\mathbb{R}^2 \rtimes G)$ where G is a finite subgroup of $O(2)$. The connected component of the identity in $\mathbb{R}^2 \rtimes G$ is isomorphic to $\mathbb{R}^2$, and the quotient is G, a finite group. The maximal compact subgroup of $\mathbb{R}^2 \rtimes G$ is G, and $\mathbb{R}^2 \rtimes G/G \cong \mathbb{R}^2$. One notices that in Theorem 6.3.3 we do not define what it means for V to have an H-equivariant spin^c structure.[3] When G is orientation-preserving, G is isomorphic to $\mathbb{Z}_n$ for some n, and it is easy to show that $\mathbb{Z}_n$ has no nontrivial extensions by $\mathbb{T}$.

Corollary 6.3.4 *Let $(\mathcal{P}, \omega)$ be a substitution tiling system satisfying the conditions of Remark 2.5.8, and let Ω be the associated tiling space. Suppose that G is a finite symmetry group for $(\mathcal{P}, \omega)$ which consists only of rotations. Then*

$$K_i(C(\Omega) \rtimes (\mathbb{R}^2 \rtimes G)) \cong K_i(C(\Omega) \rtimes G).$$

We close this section by noting that Theorem 6.3.3 is a generalization of the following well-known theorem concerning the crossed products by $\mathbb{R}$.

[3]This is a technical condition regarding the group of complex spinors of the real vector space H/L; see [20] for a full treatment.

Theorem 6.3.5 (**Connes' Thom Isomorphism**, see for example [13] Theorem 10.2.2) *If $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$, then*

$$K_i(A \rtimes_\alpha \mathbb{R}) \cong K_{1-i}(A)$$

where the index i is read modulo 2.

Hence crossed products by $\mathbb{R}^2$ cause two dimension shifts in K-theory, and by Bott periodicity we have $K_i(A \rtimes_\alpha \mathbb{R}^2) \cong K_i(A)$.

6.4 K-theory of $C(\Omega) \rtimes G$

By Lemma 6.3.4, calculating the K-theory of $C(\Omega) \rtimes (\mathbb{R}^2 \rtimes G)$ when G is a group of rotations can be accomplished by finding the K-theory of $C(\Omega) \rtimes G$. In [19], Emerson and Echterhoff consider proper actions of general locally compact groups on locally compact spaces. They write $C_0(X) \rtimes G$ as a generalized fixed point algebra, produce an ideal which is strongly Morita equivalent to $C_0(G \backslash X)$ and use excision to write down a six-term exact sequence containing $K_i(C_0(X) \rtimes G)$. When G is finite and X is compact, their construction simplifies considerably. For instance, every action of a finite group is proper. We describe their construction in this setting.

First, we state a theorem of Baum and Connes. Let X be a locally compact Hausdorff space and let G be a finite group acting on G. For $g \in G$ let

$$X^g = \{x \in X \mid gx = x\},$$

and let

$$Z_g = \{h \in G \mid gh = hg\}.$$

Let $[g]$ denote the conjugacy class of g in G and let $[G]$ denote the set of all conjugacy classes in G. Also, if H is an abelian group, let

$$H_{\mathbb{C}} = H \otimes_{\mathbb{Z}} \mathbb{C}.$$

Then we have the following.

Theorem 6.4.1 (**Baum-Connes**, [4]) *Let G be a finite group and let X be a locally compact Hausdorff G-space. Then*

$$K_*(C(X) \rtimes G)_{\mathbb{C}} \cong \bigoplus_{[g]\in[G]} K_*(C(X^g/Z_g))_{\mathbb{C}},$$

where the isomorphism is of complex vector spaces.

For a proof, see for example [19], Section 5.1. Since tensoring with $\mathbb{C}$ eliminates torsion, this tells us the rank of the K-groups.

Definition 6.4.2 ([19], after Lemma 2.1) *Let X be a compact G-space, with G finite. Suppose B is any C^*-algebra equipped with an action $\beta : G \to \mathrm{Aut}(B)$ of G. Then we define*

$$C(X \times_{G,\beta} B) = \{f \in C(X, B) \mid f(gx) = \beta_g(f(x)) \text{ for all } g \in G\}.$$

Let $g \in G$ and let $\rho_g : L^2(G) \to L^2(G)$ be given by $\rho_g\xi(h) = \xi(hg)$. Then ρ_g defines a unitary element of $\mathcal{K}(L^2(G))$. We can then define an action of G on $\mathcal{K}(L^2(G))$

$$\mathrm{Ad}\rho : G \to \mathrm{Aut}(\mathcal{K}(L^2(G)))$$

$$\mathrm{Ad}\rho_g(K) = \mathrm{Ad}(\rho_g)(K)$$

Lemma 6.4.3 ([19], Corollary 2.11) *Let X be a compact G-space, and let $\mathcal{K} = \mathcal{K}(L^2(G))$. Then*

$$C(X) \rtimes G \cong C(X \times_{G,\mathrm{Ad}\rho} \mathcal{K}).$$

Since G is a finite group, $\mathcal{K}$ is isomorphic to $\mathbb{M}_{\#G}(\mathbb{C})$. Hence for each $g \in G$, ρ_g is a unitary $(\#G \times \#G)$-matrix, and $C(X \times_{G,\mathrm{Ad}\rho} \mathcal{K})$ can be seen as the set of continuous functions from X into the $(\#G \times \#G)$-matrices such that if $gx = x$, then $f(x)$ commutes with the unitary matrix ρ_g.

Let $p = \frac{1}{\#G} \sum_{g \in G} \delta_g$. Then p is a projection in $C(X) \rtimes G$, and $I_X := (C(X) \rtimes G)p(C(X) \rtimes G)$ is an ideal of $C(X) \rtimes G$. We describe the image of I_X under the above isomorphism. For each $x \in X$, let $G_x = \{g \in G \mid gx = x\}$ be the stabilizer subgroup of x. Let

$$L^2(G)_{1_{G_x}} = \{\xi \in L^2(G) \mid \xi(gk) = \xi(g) \forall k \in G_x\} \cong L^2(G/G_x).$$

Then when viewed as an ideal in $C(X \times_{G, \mathrm{Ad}\rho} \mathcal{K})$ it is ([19], (3.7))

$$I_X = \{f \in C(X \times_{G, \mathrm{Ad}\rho} \mathcal{K}) \mid f(x) \in \mathcal{K}(L^2(G)_{1_{G_x}})\}.$$

In [19], Lemma 3.9, the authors prove that I_X is Morita equivalent to $C(X/G)$.

Example 6.4.4 Penrose tiling, $G = \mathbb{Z}_{10}$.

First we compute the free part of the K-groups using the theorem of Baum and Connes. This means we must compute

$$K_*(C(\Omega) \rtimes G)_{\mathbb{C}} \cong \bigoplus_{[g] \in [G]} K^*(X^g/Z_g)_{\mathbb{C}}$$

For $G = \mathbb{Z}_{10} = \langle r \mid r^{10} = e \rangle$, $[G] = \mathbb{Z}_{10}$ and each of the Z_g is $\mathbb{Z}_{10}$. As discussed in Example 5.3.2, there are four tilings T_1, T_2, T_3, T_4 such that $r^2 T_i = T_i$ for all $1 \leq i \leq 4$. Also, we have that $rT_1 = T_3$ and $rT_2 = T_4$. Hence, for $g = r^{2n}$, X^g is a four-point space, and $X^g/Z_g = \{T_1, T_2\}$.

$$\begin{aligned}
K_*(C(\Omega) \rtimes \mathbb{Z}_{10})_{\mathbb{C}} &\cong K^*(\Omega/\mathbb{Z}_{10}) \oplus \bigoplus_{i=1}^{4} K^*(\{T_1, T_2\})_{\mathbb{C}} \\
&= \begin{cases} K^*(\Omega/\mathbb{Z}_{10})_{\mathbb{C}} \bigoplus_{i=1}^{4} \mathbb{C}^2 & * = 0 \\ K^*(\Omega/\mathbb{Z}_{10})_{\mathbb{C}} & * = 1 \end{cases} \\
&= \begin{cases} \mathbb{C}^3 \oplus \mathbb{C}^8 & * = 0 \\ \mathbb{C} & * = 1 \end{cases}
\end{aligned}$$

Now we calculate the K-theory using 6.4.3. We have that $\mathcal{K}(L^2(\mathbb{Z}_{10})) \cong \mathbb{M}_{10}$, and relative to the standard basis

$$\rho_r = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

If we let $z = e^{\frac{i\pi}{5}}$, then we can diagonalize the above so that relative to a suitable basis $\{v_i\}_{i=1}^{10}$ we have

$$\rho_{r^2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z^8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z^8 \end{bmatrix}.$$

Hence

$$C(\Omega) \rtimes \mathbb{Z}_{10} \cong \{f \in C(\Omega, \mathbb{M}_{10}) \mid f(gx) = \mathrm{Ad}\rho_g(f(x)), f(T_i)\rho_{r^2} = \rho_{r^2} f(T_i), i = 1, 2\}.$$

If $f(T_i)\rho_{r^2} = \rho_{r^2} f(T_i)$, then $f(T_i)$, $i = 1, 2$, must take the form

$$f(T_i) = \begin{bmatrix} B_1 & & & & \\ & B_2 & & & \\ & & B_3 & & \\ & & & B_4 & \\ & & & & B_5 \end{bmatrix}, \qquad B_i \in \mathbb{M}_2.$$

Relative to the basis which gives ρ_{r^2} the form above, we have

$$L^2(\mathbb{Z}_{10})_{1_{\langle r^2 \rangle}} = \operatorname{span}\{v_1, v_2\}.$$

Hence our I_Ω is all the functions in $C(\Omega) \rtimes \mathbb{Z}_{10}$ such that $f(T_i)$, $i = 1, 2$, is of the form

$$f(T_i) = \begin{bmatrix} B_1 & & & & \\ & 0_2 & & & \\ & & 0_2 & & \\ & & & 0_2 & \\ & & & & 0_2 \end{bmatrix}, \qquad B_1 \in \mathbb{M}_2.$$

where 0_2 denotes the 2×2 zero matrix. Let $q \in \mathbb{M}_{10}$ be the projection

$$q = \begin{bmatrix} 0_2 & \\ & I_8 \end{bmatrix}$$

where I_8 is the 8×8 identity matrix. The $*$-homomorphism

$$\begin{aligned} \varphi : C(\Omega) \rtimes \mathbb{Z}_{10} &\to \left(\bigoplus_{i=1}^{4} \mathbb{M}_2 \right) \oplus \left(\bigoplus_{i=1}^{4} \mathbb{M}_2 \right) \\ f &\mapsto (qf(T_1), qf(T_2)) \end{aligned}$$

has kernel I_Ω, and so we obtain the short exact sequence

$$0 \longrightarrow I_\Omega \longrightarrow C(\Omega) \rtimes \mathbb{Z}_{10} \xrightarrow{\varphi} \left(\bigoplus_{i=1}^{4} \mathbb{M}_2 \right) \oplus \left(\bigoplus_{i=1}^{4} \mathbb{M}_2 \right) \longrightarrow 0.$$

This short exact sequence leads to six-term exact sequence in K-theory,

$$\begin{tikzcd}
K^0(I_\Omega) \arrow[r] & K_0(C(\Omega) \rtimes \mathbb{Z}_{10}) \arrow[r] & K_0(Q_\Omega) \arrow[d] \\
K_1(Q_\Omega) \arrow[u] & K_1(C(\Omega) \rtimes \mathbb{Z}_{10}) \arrow[l] & K^1(I_\Omega) \arrow[l]
\end{tikzcd}$$

Using the fact that I_Ω is strongly Morita equivalent to $C(\Omega/\mathbb{Z}_{10})$, that $K_0(\mathbb{M}_n) = \mathbb{Z}$, and $K_1(\mathbb{M}_n) = 0$, along with the computation in Example 5.2.12, and Example 6.1.5 we obtain

$$0 \longrightarrow \mathbb{Z}^3 \longrightarrow K_0(C(\Omega) \rtimes \mathbb{Z}_{10}) \longrightarrow \mathbb{Z}^8 \xrightarrow{\partial} \mathbb{Z} \longrightarrow K_1(C(\Omega) \rtimes \mathbb{Z}_{10}) \longrightarrow 0$$

We know that $K_1(C(\Omega) \rtimes \mathbb{Z}_{10}) \otimes \mathbb{C} \cong \mathbb{C}$, so $K_1(C(\Omega) \rtimes \mathbb{Z}_{10})$ must be isomorphic to a subgroup of the rationals direct sum with some torsion group. We know that $\mathbb{Z}$ maps surjectively onto $K_1(C(\Omega) \rtimes \mathbb{Z}_{10})$, and hence it must be singly generated. Thus it must be isomorphic to $\mathbb{Z}$, making the second to last map an isomorphism and $\partial = 0$. This means the left hand side is a short exact sequence, and since $\mathbb{Z}^8$ is free we have

$$K_0(C(\Omega) \rtimes \mathbb{Z}_{10}) = \mathbb{Z}^3 \oplus \mathbb{Z}^8$$

$$K_1(C(\Omega) \rtimes \mathbb{Z}_{10}) = \mathbb{Z}$$

By Corollary 6.3.4, Theorem 5.5.2, and Corollary 6.1.4 we conclude that

$$K_0(C_r^*(\mathcal{R}_{\text{punc}}) \rtimes \mathbb{Z}_{10}) = \mathbb{Z}^3 \oplus \mathbb{Z}^8$$

$$K_1(C_r^*(\mathcal{R}_{\text{punc}}) \rtimes \mathbb{Z}_{10}) = \mathbb{Z}.$$

Example 6.4.5 Octagonal, $G = \mathbb{Z}_8$.

Again we consider a full rotation group $\mathbb{Z}_8 = \langle r \rangle$, this time on the octagonal tiling. The computations are similar to Example 6.4.4 so here we are more brief. As discussed in Example 5.3.3 there are two orbits, that of T_1 with stabilizer $\mathbb{Z}_8$ and T_2

with stabilizer $\mathbb{Z}_2$. By Example 5.2.11 and Example 6.1.5 we know that the K-theory of the orbit space is

$$K^0(\Omega/\mathbb{Z}_8) \cong \mathbb{Z}^3,$$

$$K^1(\Omega/\mathbb{Z}_8) \cong \mathbb{Z}.$$

We first calculate the free part by using the theorem of Baum and Connes

$$K^*_{\mathbb{Z}_8}(\Omega)_{\mathbb{C}} \cong \bigoplus_{[g]\in[\mathbb{Z}_8]} K^*(\Omega^g/Z_g)_{\mathbb{C}}$$

We have

$$X^e/Z_e = \Omega/\mathbb{Z}_8 \qquad \Omega^{r^4}/Z_{r^4} = \{T_1, T_5\}$$

$$\Omega^{r^i}/Z_{r^i} = \{T_5\} \quad i \neq 0, 4$$

Hence

$$\begin{aligned} K_*(C(\Omega) \rtimes \mathbb{Z}_8)_{\mathbb{C}} &\cong \bigoplus_{[g]\in[\mathbb{Z}_8]} K^*(X^g/Z_g)_{\mathbb{C}} \\ &= K^*(\Omega/\mathbb{Z}_8)_{\mathbb{C}} \oplus \left(\bigoplus_{i=1}^{6} K^j(\{T_1\})_{\mathbb{C}} \right) \oplus K^*(\{T_1, T_2\})_{\mathbb{C}} \\ &= \begin{cases} \mathbb{C}^3 \oplus \mathbb{C}^8 & * = 0 \\ \mathbb{C} & * = 1 \end{cases} \end{aligned}$$

Here we get

$$\begin{aligned} C(\Omega) \rtimes \mathbb{Z}_8 \cong \{ & f \in C(\Omega, \mathbb{M}_8) \mid f(gx) = \mathrm{Ad}\rho_g f(x), f(T_1)\rho_r = \rho_r f(T_1), \\ & f(T_2)\rho_{r^4} = \rho_{r^4} f(T_2)\} \end{aligned}$$

where relative to a suitable basis and $z = e^{\frac{\pi i}{4}}$, we have

$$\rho_r = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z^5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z^7 \end{bmatrix},$$

$$\rho_r^4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

This implies that $f(T_1)$ is diagonal and $f(T_2) \in \mathbb{M}_4 \oplus \mathbb{M}_4 \subset \mathbb{M}_8$. Further, I_Ω is the functions in $C(\Omega) \rtimes \mathbb{Z}_8$ where $f(T_1)$ is only possibly nonzero in the top-left entry and $f(T_2)$ is only possibly nonzero in the top-left 4×4 block. Hence, as before,

$$Q_\Omega = C(\Omega) \rtimes \mathbb{Z}_8 / I_\Omega \cong \mathbb{C}^7 \oplus \mathbb{M}_4.$$

Once again we obtain an exact sequence

$$0 \longrightarrow \mathbb{Z}^3 \longrightarrow K_0(C(\Omega) \rtimes \mathbb{Z}_8) \longrightarrow \mathbb{Z}^8 \xrightarrow{\partial} \mathbb{Z} \longrightarrow K_1(C(\Omega) \rtimes \mathbb{Z}_8) \longrightarrow 0$$

where an argument similar to Example 6.4.4 gives us that $K_1(C(\Omega) \rtimes \mathbb{Z}_8) \cong \mathbb{Z}$ and hence

$$K_0(C(\Omega) \rtimes \mathbb{Z}_8) \cong \mathbb{Z}^8 \oplus \mathbb{Z}^3.$$

Once again, Corollary 6.3.4, Theorem 5.5.2, and Corollary 6.1.4 allow us to conclude that

$$K_0(C_r^*(\mathcal{R}_{\text{punc}}) \rtimes \mathbb{Z}_8) = \mathbb{Z}^3 \oplus \mathbb{Z}^8$$

$$K_1(C_r^*(\mathcal{R}_{\text{punc}}) \rtimes \mathbb{Z}_8) = \mathbb{Z}.$$

Notice in these two examples that computation of the K-theory was possible due to the fact that $K^1(\Omega/G)$ was a free abelian group. This simplified the exact sequence which arose. We know of no examples where $K_1(\Omega/G)$ is not free abelian, though we do not claim to have a proof of this statement in generality.

Remark 6.4.6 The results of Examples 6.4.4 and 6.4.5 may be alarming, since we obtain the same groups in the case of both the Penrose tiling and the octagonal tiling. These algebras are almost certainly not isomorphic, as their ordered K_0 groups are almost certainly different (though we do not claim to have calculated these). If we ignore the fact that they have different symmetry groups, these two tilings are philosophically very similar. While the inflation constant for the Penrose is the golden ratio γ, the inflation constant for the octagonal is $1+\sqrt{2}$, the so-called "silver ratio". The continued fraction expansions of these numbers are

$$\gamma = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}},$$

$$1 + \sqrt{2} = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}.$$

Chapter 7

Conclusion

We have studied the action of finite symmetry groups G on substitution tiling systems $(\mathcal{P}, \omega)$ satisfying the conditions of Remark 2.5.8 and have obtained many new results, which we summarize over the next three paragraphs. In the case where G acts freely on $\mathcal{P}$, we have shown that $C_r^*(\mathcal{R}_{\text{punc}}) \rtimes G$ is isomorphic to the C*-algebra of an almost AF Cantor groupoid, and hence it has real rank zero, stable rank one, and the order on its projections is determined by its unique trace. We have proved that the action of G on AF_ω has the Rokhlin property and that the action of G on $C_r^*(\mathcal{R}_{\text{punc}})$ has the tracial Rokhlin property if one assumes that $C_r^*(\mathcal{R}_{\text{punc}})$ has tracial rank zero. As the remark after Corollary 5.7.11 indicates, this is tied to the question of whether the crossed product associated to a minimal action of $\mathbb{Z}^2$ on the Cantor set has tracial rank zero.

In addition, we have shown that the K-theory is computable for the crossed products in many of the cases of interest. The K-theory for the AF algebra $AF_\omega \rtimes G$ can be computed whether or not G acts freely on $\mathcal{P}$ and whether or not the action of G on $\mathbb{R}^2$ preserves orientation. Example 5.6.7 illustrates that computing the matrix of partial multiplicities of $AF_\omega \rtimes G$ is quite straightforward if G acts freely on $\mathcal{P}$, whereas the second part of Example 5.6.8 shows that the calculation gets longer if G does not act freely on $\mathcal{P}$. Furthermore, Example 6.2.1 shows that breaking symmetry on the prototiles to make the action of a given G on $\mathcal{P}$ free results in possibly nonisomorphic $AF_\omega \rtimes G$.

We have shown that when the action of G on $\mathbb{R}^2$ preserves orientation, we can compute K-theory of $C_r^*(\mathcal{R}_{\text{punc}}) \rtimes G$. As in the case of the AF crossed products, we can compute these groups whether G acts freely on $\mathcal{P}$ or not. In both of our examples $K^1(\Omega/G)$ was a free abelian group; this allowed us to easily calculate the K-theory of $C_r^*(\mathcal{R}_{\text{punc}}) \rtimes G$ from the cohomology of Ω/G and the points in Ω fixed under some element of G. We do not know of any case where $K^1(\Omega/G)$ is not free, so at least in known examples the techniques of Examples 6.4.4 and 6.4.5 should allow computation of the K-theory of $C_r^*(\mathcal{R}_{\text{punc}}) \rtimes G$ when G is a rotation group. This leads to the question of how to proceed when G does not preserve orientation; this is an interesting question for further study.

This thesis raises many other questions for future work. One obvious question to consider is how our results generalize to tilings in which tiles sit in an infinite number of orientations; an example of such a tiling is the pinwheel tiling. The C*-algebra associated to such a tiling was considered by Whittaker [72], and he showed that in this case $C_r^*(\mathcal{R}_{\text{punc}})$ has a large subalgebra which is an $A\mathbb{T}$ algebra – i.e., an inductive limit of matrix algebras over $C(\mathbb{T})$. Obvious candidates for symmetry groups here are either the torus $\mathbb{T}$ or $\mathbb{T} \rtimes \mathbb{Z}_2$. In this case the groups are not finite and Ω is not an inverse limit of 2-dimensional CW complexes – this means that our techniques do not immediately carry over.

Another direction to consider is the case of cut-and-project tilings considered by the authors in [24]. Putnam [51] provides a framework to study the C*-algebras associated to such a tiling on which a symmetry group would naturally act. Some of the techniques in Chapter 6 apply to crossed products by nondiscrete groups and in [51] the algebras considered are crossed products by a discrete lattice, so different techniques will be needed to compute the K-theory of these crossed products.

Since the physical motivation for studying tilings is quasicrystals, and these objects are modelled by three dimensional tilings, another question which one might naturally consider is the extent to which our results apply to higher dimensions. One has to take slightly more care with the CW structure of the orbit space in this case, but this should certainly be within reach. Furthermore, finite symmetry groups in this case need no longer be either cyclic or a semidirect product of two cyclic groups, and even in the orientation-preserving case the question of satisfying the spin^c condition in Theorem 6.3.3 (for example) is less clear.

A fourth problem to pursue in relation to this work is attempting to prove that $C_r^*(\mathcal{R}_{\text{punc}})$ has tracial rank zero by using the definition. We believe that the techniques of Lemma 5.7.9 could possibly be adapted to produce a commuting set of projections for the finite set $\mathcal{E}_2$. The main obstruction seems to be ensuring that the elements constructed are projections – indeed, Lemma 5.7.9 produces elements that satisfy all the conditions of the tracial Rokhlin property *except* being projections.

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