Aperiodic Order in Dynamical Systems and Operator Algebras

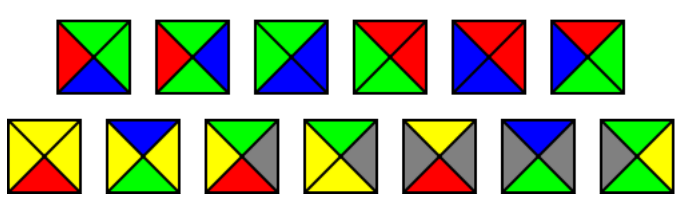
Charles Starling

February 22, 2012

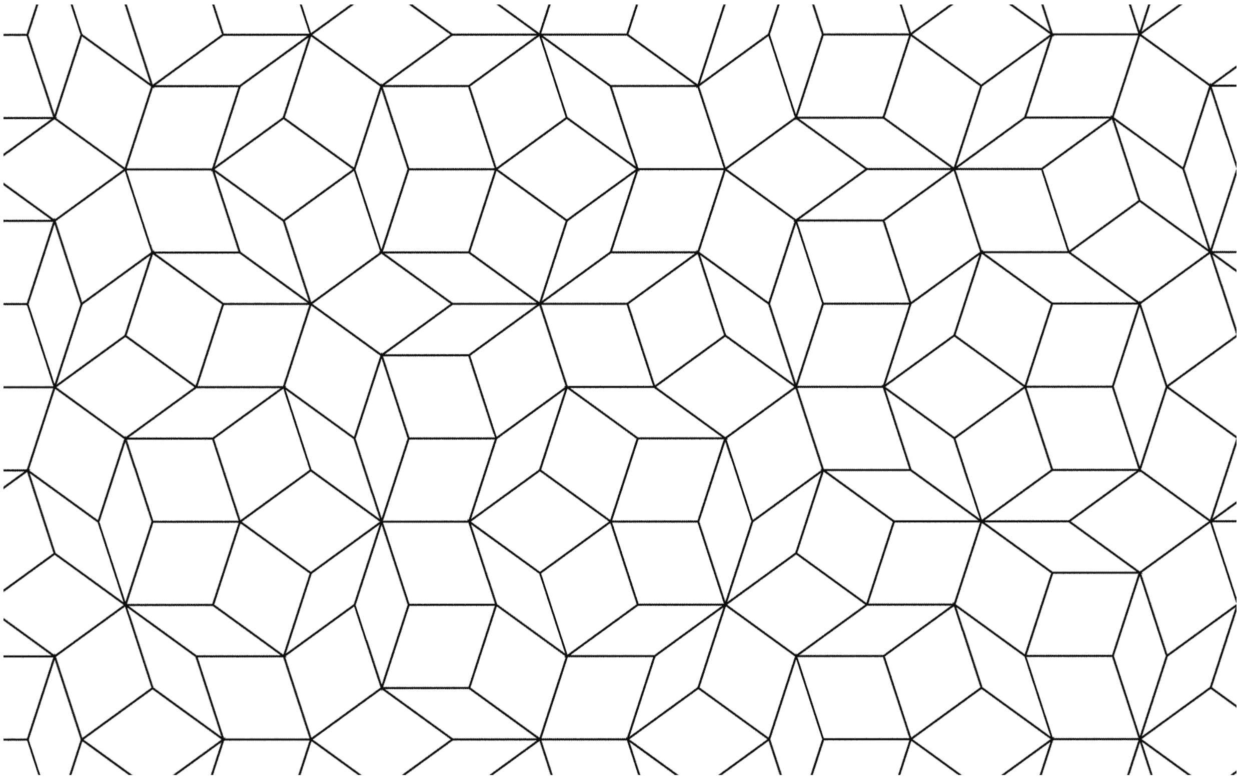
1 Introduction and History

This minicourse is meant to be an introduction to the theory of aperiodic tilings with an eye towards dynamical systems and operator algebras. We begin giving some history on the subject, and then give definitions standard in the study of tilings.

Mathematical interest in tilings began with Wang [14] in 1961. Wang considered a problem in logic known as the "domino problem": given a finite set $\mathcal{F}$ of "dominos" (unit squares with rules governing which edges could be next to each other), is it possible to tile the plane with translates of $\mathcal{F}$? He conjectured that if such a set admits a tiling of the plane, then it must admit a periodic tiling (that is, a tiling which repeats in some direction). This conjecture was proven false by Wang's student Berger [2] in 1966 when he found a set of 20426 dominos which could only tile the plane aperiodically; such a set he called an **aperiodic set**. Study then began on finding smaller aperiodic sets; the figure below gives an aperiodic set with 13 tiles with matching rules given by allowing two edges to meet if they are the same colours.



In 1974, Roger Penrose [8] introduced what are now the very famous Penrose tilings. These are tilings of the plane with two rhombs rotated by multiples of $\pi/5$.

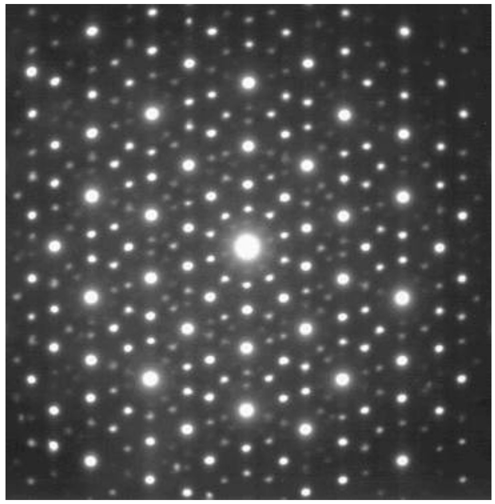


As with Berger's dominos, there are rules about which edges may meet in a tiling. Given these rules, these rhombs can only tile the plane aperiodically. In addition, every Penrose tiling has the following properties:

- Given a finite radius $r > 0$, there are only a finite number of different patches with radius smaller than r among all Penrose tilings (modulo translation).
- Given some patch P in a Penrose tiling, there is a radius $R > 0$ such that every ball of radius R in any Penrose tiling contains a copy of P.

So while tilings do not repeat exactly, the same patches appear infinitely often in all Penrose tilings, do not appear too far apart. These properties are what are now loosely referred to as "Aperiodic Order".

Perhaps the most important event in the study of aperiodic order is an experiment by Dan Shechtman et al [13] in 1984. Shechtman was studying the properties of certain alloys when he found a substance with the following diffraction pattern:



The fact that there are peaks of concentration meant that the material must have some long-range order at the atomic level. The above pattern is rotation symmetric by the angle $\pi/5$, and this would imply that at the arrangement of the atoms has rotational symmetry by that angle. However, this type of rotational symmetry is impossible for crystals! This discovery was so unexpected that when Shechtman reported his findings he was ridiculed and fired from his lab. He was eventually vindicated, as hundreds of examples of such materials (called **quasicrystals**) now exist, and in 2011 Shechtman was awarded the Nobel Prize in Chemistry for his work.

At the atomic level, quasicrystals have similar properties to the Penrose tilings discussed above – they are aperiodic, but display some long-range order. These tilings and their 3D analogues were therefore seen as natural models for these materials; this led to much mathematical research into aperiodic tilings.

Despite this seemingly applied perspective, there will be essentially no physics in these talks. Aperiodic tilings give rise to interesting topological spaces, dynamical systems, and C*-algebras. Some of these objects of course do have some physical interpretation, but are presented and developed "in their own right".

2 Tilings

For us, tilings will exist in the Euclidean space $\mathbb{R}^d$. We let $B_r(x)$ denote the open ball of radius r centred at x in $\mathbb{R}^d$.

Definition 1. In $\mathbb{R}^d$, we define the following:

- A **tile** is a subset of $\mathbb{R}^d$ homeomorphic to the closed unit ball $\overline{B_1(0)}$. Tiles may carry labels to distinguish identical sets.
- A **partial tiling** is a set of tiles P such that $t_1, t_2 \in P$ and $t_1 \neq t_2$ implies that $\mathrm{Int}(t_1 \cap t_2) = \emptyset$.
- The **support** of a partial tiling P, denoted $\mathrm{supp}(P)$, is the union of its tiles, that is $\mathrm{supp}(P) = \bigcup_{t \in P} t$.
- A **patch** is a finite tiling.
- Finally, a **tiling** is a partial tiling T with $\mathrm{supp}(T) = \mathbb{R}^d$.

For us, d will always be either 1 or 2. For T a partial tiling, we use the following notation: for $U \subset \mathbb{R}^d$ we let

$$T(U) = \{t \in T \mid U \cap t \neq \emptyset\}.$$

For $x \in \mathbb{R}^d$, the set $T(\{x\})$ is abbreviated to $T(x)$. Two partial tilings T and T' are said to **agree** on a set $U \subset \mathbb{R}^d$ if $T(U) = T'(U)$. If t is a tile and $x \in \mathbb{R}^d$, then $t + x = \{u + x \mid u \in t\}$ is also a tile. If T is a (partial) tiling, then $T + x = \{t + x \mid t \in T\}$ is also a (partial) tiling. We say that a tiling T is **aperiodic** if $x \in \mathbb{R}^d$ and $T + x = T$ implies that $x = 0$.

Example 1. Let $T = \{[n, n+1] \mid n \in \mathbb{Z}\}$. Then T is a tiling of $\mathbb{R}$ consisting of the closed intervals between consecutive integers.

Example 2. Let for $m, n \in \mathbb{Z}$, let $U_{m,n} = \{(x, y) \in \mathbb{R}^2 \mid x \in [m, m+1], y \in [n, n+1]\}$, and let

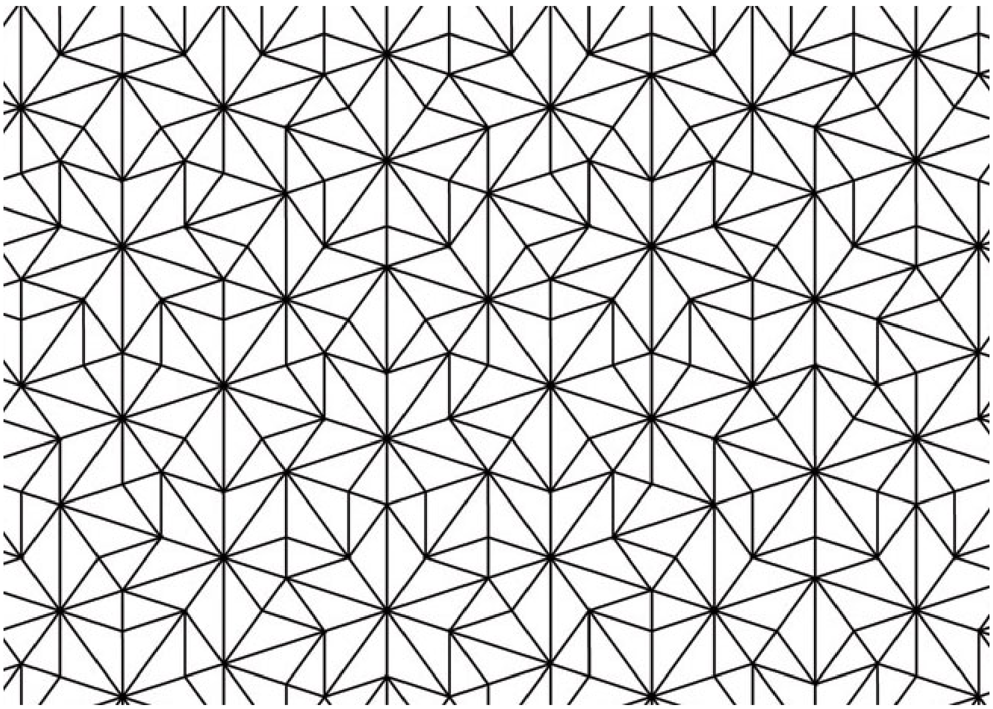
$$T = \{U_{m,n} \mid m, n \in \mathbb{Z}\}.$$

Then T is the usual grid in $\mathbb{R}^2$.

These two examples are periodic. We now give an example of the simplest type of aperiodic tiling.

Example 3. Let T be as in Example 2, except suppose we colour the square $U_{0,0}$ black. Then $T + x \neq T$ unless $x = 0$, so T is aperiodic.

Example 4. The Penrose tiling.



In this picture we have divided the rhombs in the usual Penrose tiling along their diagonals. We will refer to this tiling as the Penrose tiling as well, though it is also sometimes referred to as a tiling by **Robinson triangles**.

Frequently, a tiling will consist of translates of a finite number of "tile types". A finite set of tiles $\mathcal{P}$ is called a set of **prototiles** for a tiling T if for every $t \in T$ there exists $x \in \mathbb{R}^d$ and $p \in \mathcal{P}$ such that $t = p + x$. Notice that we do not allow for rotation of prototiles, so for the Penrose tiling one would have to choose a prototile set consisting of the two triangle shapes plus all their rotates by multiples of $\pi/5$. Prototiles may carry labels to distinguish congruent shapes – this was the case in Example 3 where there are two prototiles which are congruent but which are labeled differently (one "black" and one "white").

Tiling Spaces

Given a set of tilings X, one can put the following metric on X. For $T, T' \in X$, let

$$\begin{aligned} d(T,T') = \inf\{1, \varepsilon \mid \exists\, x, x' \in \mathbb{R}^d \ni |x|, |x'| < \varepsilon, \\ (T-x)(B_{1/\varepsilon}(0)) = (T'-x')(B_{1/\varepsilon}(0))\}. \end{aligned}$$

This is called the **tiling metric**. In this metric, two tilings are close if they agree on a large ball around the origin, up to a small translation. In this metric, there are essentially two ways that T and T' can be close:

- $T = T' + x$ for some small x',
- T and T' agree on a large ball around the origin.

A common way to obtain a collection of tilings is by taking all possible translates of a single tiling T.

Definition 2. Let T be a tiling. Then we define the **continuous hull** of T (or the **tiling space** of T) to be the completion of $T + \mathbb{R}^d = \{T + x \mid x \in \mathbb{R}^d\}$ in the tiling metric. This space is denoted Ω_T.

For some examples, the topology of the hull can be quite weird. It is a fact that given a Cauchy sequence in $T + \mathbb{R}^d$ one can find a tiling T' which it converges to, so the elements of Ω_T are all tilings. It is clear that if $T' \in \Omega_T$, then $T' + x \in \Omega_T$ for all $x \in \mathbb{R}^d$, since any Cauchy sequence converging to T' can be translated by x to a Cauchy sequence converging to $T' + x$.

Definition 3. Let T be a tiling. Then we say that T has **finite local complexity** (or **FLC**) if for every $r > 0$ there are only a finite number of different patches in T whose supports have radius less than r.

Proposition 1. ([11], Lemma 2) If T has FLC and admits a finite set of prototiles, then Ω_T is compact.

Example 5. Consider the tiling of $\mathbb{R}$ by unit intervals from Example 1. A translate of this tiling only depends on the placement of the origin within an interval. Furthermore, translating the origin to any tile boundary yields the same tiling. In this way, one sees that $\Omega_T \cong S^1$, the unit circle.

We see from Example 3 that it can be relatively straightforward to produce aperiodic tilings – we can simply label or cut up one tile in a periodic tiling to break periodicity. Producing aperiodic order can be a little more involved. One method of producing aperiodic order is what is called the **substitution** method which we now describe.

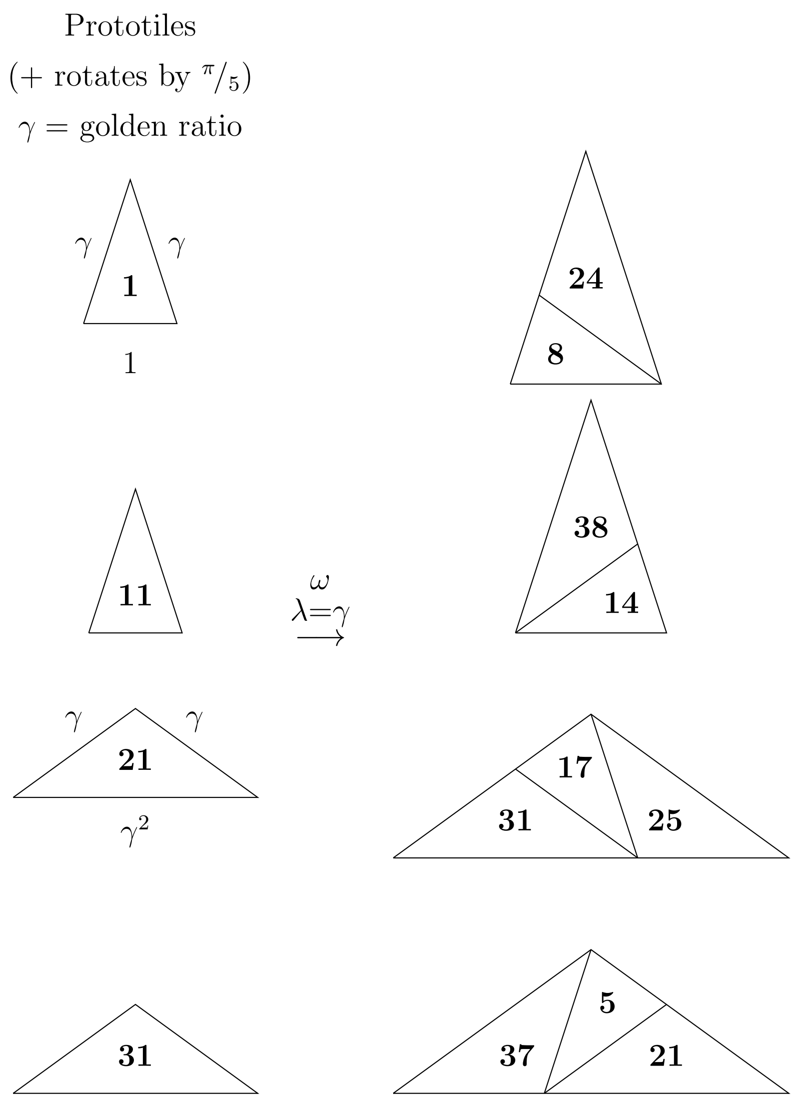
We start with a finite set of tiles $\mathcal{P} = \{p_1, p_2, \ldots, p_{N_{\text{pro}}}\}$. The similarity of this notation and that for prototiles is no accident; we will produce tilings for which $\mathcal{P}$ is a prototile set. For now however, there are no tilings in sight.

We let $\mathcal{P}^*$ be the set of all partial tilings consisting of translates of elements of $\mathcal{P}$. A **substitution rule** is a function $\omega : \mathcal{P} \to \mathcal{P}^*$ such that there exists $\lambda > 1$ such that $\operatorname{supp}(\omega(p)) = \lambda p$ for all $p \in \mathcal{P}$. In words, a substitution rule is a prescription for splitting up each prototiles into smaller copies of the prototiles, and then scaling up by a factor of λ so that the elements of $\omega(p)$ are of the same sizes as the originals. If $t = p + x$ for $p \in \mathcal{P}$ and $x \in \mathbb{R}^d$, then we can extend the definition of ω by setting $\omega(p + x) = \omega(p) + \lambda x$. In this way, we can see ω as a function $\omega : \mathcal{P}^* \to \mathcal{P}^*$. We call the pair $(\mathcal{P}, \omega)$ a **substitution tiling system**.

Example 6. The following picture illustrates a substitution on a set of prototiles

$$\mathcal{P}_{\text{Pen}} = \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{40}\},$$

all of which are Robinson triangles. Only four prototiles are shown; the others are obtained by rotation. Let r denote the counterclockwise rotation of $\mathbb{R}^2$ by ${}^{\pi}/_{5}$ and let $\mathbf{2} = r\mathbf{1}$, $\mathbf{12} = r\mathbf{11}$, and so on.



We let $\Omega \subset \mathcal{P}^*$ be the set of all tilings T such that every patch in T is contained in a translate of $\omega^n(p)$ for some $n \in \mathbb{N}$ and $p \in \mathcal{P}$.

Assumption 1. The substitution tiling system $(\mathcal{P}, \omega)$ is **primitive**, that is, there exists $N \in \mathbb{N}$ such that for all $p, q \in \mathcal{P}$, $\omega^n(p)$ contains a translate of q.

Proposition 2. ([1], Proposition 2.1) The set Ω is nonempty.

Proof. Let $p \in \mathcal{P}$, and find $k \in \mathbb{N}$ such that $\omega^k(p)$ contains a tile in the interior of its support. Since the substitution is primitive, there exist $x \in \mathbb{R}^d$ and $m \in \mathbb{N}$ such that $p + x \in \omega^{k+m}(p)$ and $p + x$ is contained in the interior of the $\text{supp}(\omega^{k+m}(p)) = \lambda^{k+m}p$. The function

$$f : \lambda^{k+m}p \to p + x$$

$$z \mapsto \lambda^{-(k+m)}(z + x)$$

is continuous and onto, and $p + x \subset \lambda^{k+m}p$. Hence by the Brouwer fixed point theorem f has a fixed point in its interior; call it z_0. That is, z_0 satisfies

$$\begin{aligned} \lambda^{k+m}z_0 &= z_0 + x \\ \implies \qquad x &= \lambda^{k+m}z_0 - z_0 \end{aligned}$$

We have

$$\begin{aligned} p + x &\in \omega^{k+m}(p) \\ p + \lambda^{k+m}z_0 - z_0 &\in \omega^{k+m}(p) \\ p - z_0 &\in \omega^{k+m}(p) - \lambda^{k+m}z_0 \\ p - z_0 &\in \omega^{k+m}(p - z_0). \end{aligned}$$

Hence,

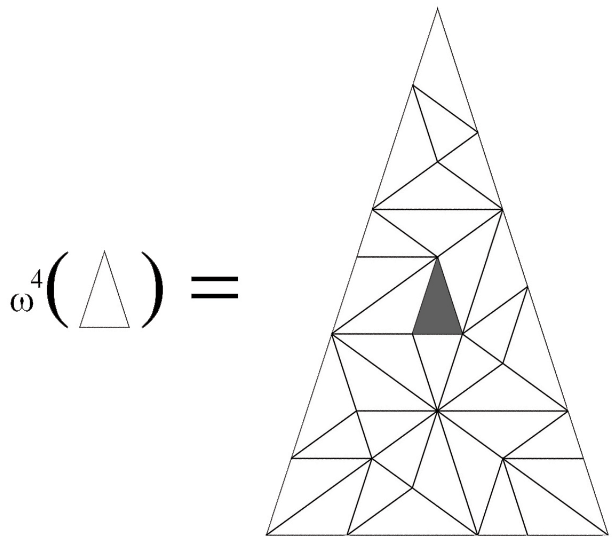
$$\{p - z_0\} \subset \omega^{k+m}(p - z_0) \subset \cdots \subset \omega^{i(k+m)}(p - z_0) \subset \omega^{(i+1)(k+m)}(p - z_0) \subset \ldots$$

is an increasing nested sequence of patches. Further, since $p - z_0$ is in the interior of $\omega^{k+m}(p - z_0)$, the supports of these patches are an increasing nested sequence of sets in $\mathbb{R}^d$ whose union is $\mathbb{R}^d$. Thus

$$T = \bigcup_{i=1}^{\infty} \omega^{i(k+m)}(p - z_0)$$

is a tiling. $\square$

This proof is best illustrated by the picture below



Here $\omega^{4k}(p)$ is an increasing nested sequence of patches whose supports eventually cover $\mathbb{R}^2$.

Not only is Ω nonempty, but it is translation invariant. If $T \in \Omega$, then the patches of $T + x$ are the same as those of T only translated, hence $T + x \in \Omega$. We may restrict ω to a function on Ω, and it is easy to see that $\omega(\Omega) \subset \Omega$. In fact, the other inclusion is also true (see [1] Proposition 2.2), so the map $\omega : \Omega \to \Omega$ is surjective.

Assumption 2. The map $\omega : \Omega \to \Omega$ is injective.

This assumption is also commonly referred to as "recognizability". Given the above assumptions, ω is a bijection and hence has an inverse ω^{-1}. These have the following interactions with translation:

$$\omega(T + x) = \omega(T) + \lambda^n x$$
$$\omega^{-1}(T + x) = \omega(T) + \lambda^{-n} x.$$

Proposition 3. ([1] Proposition 2.3) The set Ω contains no periodic tilings.

Proof. Suppose that $T \in \Omega$ such that $T + x = T$ for some nonzero $x \in \mathbb{R}^d$. Find $n \in \mathbb{N}$ such that a ball of radius $\lambda^{-n}\|x\|$ is contained in every prototile. Then we have that $\omega^{-n}(T) + \lambda^{-n}x = \omega^{-n}(T)$. This is a contradiction, for if t is any tile in $\omega^{-n}(T)$, then t meets $t + \lambda^{-n}x$ in its interior, which is impossible for two tiles in the same tiling. $\square$

Assumption 3. Every element of Ω has finite local complexity.

Proposition 4. If $T \in \Omega$, then $\Omega = \Omega_T$.

Proof. The inclusion $\Omega_T \subset \Omega$ is obvious, for if a tiling T' is approximated around the origin by translates of T, every patch in T' must be a patch in T.

To get the other inclusion, let $\varepsilon > 0$ and let $T' \in \Omega$. Find $N \in \mathbb{N}$ as in the definition of primitivity, and find $k \in \mathbb{N}, p \in \mathcal{P}$, and $x \in \mathbb{R}^d$ such that $T'(B_{1/\varepsilon}(0)) \subset \omega^k(p) + x$. Since $\omega^{-N-k}(T)$ is a tiling, it contains a tile. Hence $\omega^N(\omega^{-N-k}(T)) = \omega^{-k}(T)$ contains a translate of p, and so $\omega^k(\omega^{-k}(T)) = T$ contains a translate of $\omega^k(p)$. Translating T so that this patch sits at the origin aligned with T' gives us a translate of T which is ε-close to T', and so we are done. $\square$

Since every element of Ω has FLC, this means that Ω is a compact metric space. The bijection ω satisfies (see [1] Proposition 3.1) that

$$d(\omega(T), \omega(T')) \leq \lambda d(T, T')$$
$$d(\omega^{-1}(T), \omega^{-1}(T')) \leq \lambda d(T, T')$$

and so $\omega : \Omega \to \Omega$ is a homeomorphism.

From now on, we will assume that we are in the setting of a substitution system and that Assumptions 1–3 hold. The Penrose substitution of Example 6 satisfies these assumptions.

Definition 4. We say that a tiling T has **repetitivity** (or is **repetitive**) if for every patch $P \subset T$ there exists $R > 0$ such that $B_R(x)$ contains a translate of P for all $x \in \mathbb{R}^d$.

Proposition 5. If $T \in \Omega$, then T has repetitivity.

Proof. This follows from primitivity. Let $T \in \Omega$ and let $P \subset T$ be a patch. Then there exists $k \in \mathbb{N}$ and $p \in \mathcal{P}$ such that a translate of P appears in $\omega^k(p)$. By primitivity, there exists $N \in \mathbb{N}$ such that $\omega^N(t)$ contains a translate of p for all tiles t which are translates of prototiles. Take $r > 0$ such that for every $T' \in \Omega$, there is a tile $t \in T'$ contained in the interior of $B_r(x)$ for all $x \in \mathbb{R}^d$ – this is possible because there are a finite number of prototiles. Thus every ball of radius r contains a tile in $\omega^{-N-k}(T)$, and so every ball of radius $\lambda^{N+k}r$ in T contains a translate of P. $\square$

In this case, we can prove that Ω has some very interesting local structure. Let $B^{\Omega}_R(T)$ denote the open ball of radius R around T in the tiling metric. Let $r > 0$, $T \in \Omega$, and define

$$C(T, r) = \{T' \in \Omega \mid T(B_r(0)) = T'(B_r(0))\} \subset B^{\Omega}_{1/r}(T).$$

Proposition 6. In the relative topology, $C(T, r)$ is homeomorphic to the Cantor set.

Proof. Recall that a metric space X is homeomorphic to the Cantor set if and only if X is compact, totally disconnected, and has no isolated points. That $C(T, r)$ is compact is straightforward – if there is a convergent sequence of tilings that agree with T on a ball of radius r, its limit must agree with T on the same ball.

To see that $C(T, r)$ is totally disconnected, take $T_1, T_2 \in C(T, r)$ such that $T_1 \neq T_2$. Since these are not equal, there exists $R > 0$ such that $T_1(B_R(0)) \neq T_2(B_R(0))$, ie $T_1 \notin C(T_2, R)$. It is not hard to see that (after perhaps taking R larger) that we have

$$C(T_2, R) = C(T, r) \cap B^{\Omega}_{1/R}(T_2).$$

Thus $C(T_2, R)$ is closed and open in $C(T, r)$, and so T_1 is not in the connected component of T_2. Since T_1 and T_2 were arbitrary, this shows that the only connected components in $C(T, r)$ are singletons, and so $C(T, r)$ is totally disconnected.

To see that $C(T, r)$ has no isolated points, we use repetitivity. Let $T_1 \in \Omega$ and $\varepsilon > 0$. Let $P = T_1(B_{1/\varepsilon}(0))$. Then we can find $R > 0$ and an $x \in \mathbb{R}^d$ with $\|x\| > 2R$ such that $(T_1 + x)(B_{1/\varepsilon}(0)) = P$. Since T_1 is aperiodic, $T_1 + x \neq T_1$ and so T_1 is not isolated. $\square$

For $T \in \Omega$ and $\delta > 0$ small enough relative to the size of the prototiles, we have that

$$D(T, \delta) = \{T + x \mid \|x\| < \delta\} \cong B_\delta(0)$$

and

$$D(T, \delta) \subset B^{\Omega}_\delta(T).$$

One can easily show that the map

$$\begin{aligned} \sigma : B_\delta(0) \times C(T, r) &\to \Omega \\ (x, T') &\mapsto T' + x \end{aligned}$$

is continuous and a homeomorphism onto its image. Its image is an open set containing T. Hence each $T \in \Omega$ has a neighbourhood homeomorphic to the Cartesian product of an open disc and a Cantor set. This kind of local structure may seem a little odd, but spaces of this kind show up quite naturally in the study of Dynamical Systems.

3 Dynamical Systems

Let X be a locally compact Hausdorff space and let $\varphi : X \to X$ be a homeomorphism. Then we call the pair (X, φ) a **dynamical system**. A good reference on dynamical systems is the book [4] by M. Brin and G. Stuck. Definitions, results, and examples in this section are taken from that book unless otherwise stated.

Definition 5. Let (X, φ) be a dynamical system and let $x \in X$. Then the set

$$\mathcal{O}(x) = \{\varphi^n(x) \mid n \in \mathbb{Z}\}$$

is called the **orbit** of x.

Definition 6. If $\varphi^n(x) = x$ then we say that x is **periodic** with **period** n.

We now define the natural notion of equivalence for dynamical systems.

Definition 7. We say that two dynamical systems (X, φ) and (Y, ψ) are **topologically conjugate** (or simply **conjugate**) if there exists a homeomorphism $h : X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "h"] & X \arrow[d, "h"] \\ Y \arrow[r, "\psi"] & Y \end{tikzcd}$$

in other words, if $\varphi \circ h = h \circ \psi$.

Definition 8. Let (X, φ) be a dynamical system.

- We say that (X, φ) is **topologically mixing** (or simply **mixing**) if for every pair of open sets $U, V \subset X$ there exists $N \in \mathbb{N}$ such that $\varphi^n(U) \cap V \neq \emptyset$ for all $n > N$.

- Finally, we say that (X, φ) is **minimal** if $\overline{\mathcal{O}(x)} = X$ for all $x \in X$.

These two properties are notions of recurrence for dynamical systems. They are both preserved by topological conjugacy.

Example 7. Circle rotations

Let $S^1 = [0, 1]$ mod 1 be the circle. For $\alpha \in (0, 1)$, the map $R_\alpha : S^1 \to S^1$ defined by $R_\alpha(x) = x + \alpha$ mod 1 can be seen as rotation of x clockwise through an angle of $2\pi\alpha$, and is an isometry and hence a homeomorphism. When α is rational, say $\alpha = {}^{p}/_{q}$ in lowest terms, then every point is periodic with period q, as R_α^q is addition of the integer p. When α is irrational, every orbit is infinite. To see this, if x were some point with finite orbit, we would be able to find $n \in \mathbb{N}$ such that $R_\alpha^n(x) = x + n\alpha = x \bmod 1$, which would imply that $n\alpha \in \mathbb{Z}$.

We claim that (S^1, R_α) is minimal when α is irrational. It is not hard to see that $\mathcal{O}(x) = \mathcal{O}(0) + x \bmod 1$, and so it suffices to show that the orbit of 0 is dense. Let $\varepsilon > 0$. Compactness of S^1 implies that we may cover S^1 with a finite number of open ε-balls. Since $\mathcal{O}(x)$ is infinite, there is at least one ε-ball in our cover which contains two points from $\mathcal{O}(0)$, let these two points be $R_\alpha^n(0)$ and $R_\alpha^m(0)$. The distance between $R_\alpha^n(0)$ and $R_\alpha^m(0)$ is less than ε and since R_α is an isometry, the distance between $R_\alpha^{n-m}(0)$ and 0 must be less than ε. Hence for every ε-ball U there exists $k \in \mathbb{Z}$ such that $R_\alpha^{k(n-m)}(0) \in U$, and so $\mathcal{O}(0)$ is dense.

The system (S^1, R_α) is not mixing, in fact no isometry of a space with more than one point is mixing (see [4], Exercise 2.3.1).

Example 8. The 2^∞ odometer

Let $X = \{0,1\}^{\mathbb{N}} = \{(x_n)_{n\in\mathbb{N}} \mid x_i \in \{0,1\}\}$, the set of all sequences of 0's and 1's. For $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ the formula

$$d(x,y) = 2^{-n} \quad \text{where} \quad n = \max\{k \in \mathbb{N} \mid x_i = y_i \text{ for all } i \neq k\}$$

defines a metric on X which generates the product topology. This topology is also generated by **cylinder sets**: let $y_1, y_2, \ldots, y_k$ be a sequence in $\{0,1\}$ and define

$$C(y_1, y_2, \ldots, y_k) = \{(x_n)_{n\in\mathbb{N}} \mid x_i = y_i \text{ for all } 1 \leq i \leq k\}.$$

Under this topology X is compact, totally disconnected, and has no isolated points, and hence is homeomorphic to the Cantor set. We define a map φ on X by the following rule: if $x = (x_n)_{n\in\mathbb{N}}$ is the constant sequence $x_i = 1$ for all i, then we define $\varphi(x)$ to be the constant sequence $\varphi(x)_i = 0$ for all i. Otherwise, let $k = \min\{i \mid x_i = 0\}$, and define

$$\varphi(x)_i = \begin{cases} x_i & i > k \\ 1 & i = k \\ 0 & i < k \end{cases}$$

This can be viewed as "addition of 1 to the first coordinate with rollover". For instance,

$$\begin{aligned}
\varphi(0,0,0,\dots) &= (1,0,0,\dots) \\
\varphi(1,0,0,\dots) &= (0,1,0,0,\dots) \\
\varphi(0,1,0,0,\dots) &= (1,1,0,0,\dots) \\
\varphi(1,1,0,0,\dots) &= (0,0,1,0,0,\dots).
\end{aligned}$$

We see from the above that if

$$z \in C(\underbrace{0,0,\dots,0}_{k})$$

and $n < 2^{k+1}$ then if one reverses the first k entries of the sequence $\varphi^n(z)$ then one obtains the binary representation of the integer n. Hence if $C(y_1,\dots,y_k)$ is a cylinder set and n is the integer whose binary representation is $y_k y_{k-1}\dots y_1$, we have that $\varphi^n(z) \in C(y_1,\dots,y_k)$. Furthermore, if $x = (x_i)_{i\in\mathbb{N}} \in X$ and n is the integer whose binary representation is $x_k x_{k-1}\dots x_1$, then

$$\varphi^{2^k-n}(x) \in C(\underbrace{0,0,\dots,0}_{k}).$$

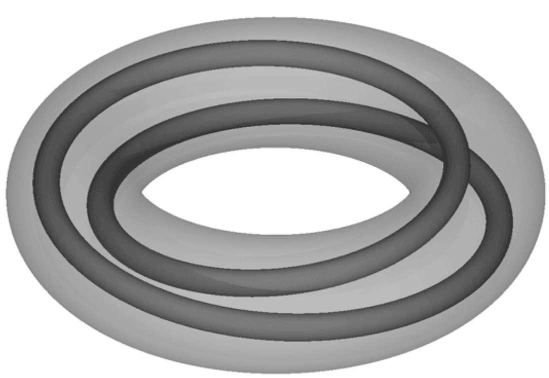
Together, these two observations show that the dynamical system (X,φ) is minimal.

Example 9. The solenoid

Let $\mathcal{T} = S^1 \times D^2$ be the solid torus, with $D^2 = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \le 1\}$ and $S^1 = [0,1]$ mod 1. Fix $\lambda \in (0,{}^1/_2)$ define a map $F : \mathcal{T} \to \mathcal{T}$ by

$$F(\phi,x,y) = (2\phi, \lambda x + \tfrac{1}{2}\cos 2\pi\phi, \lambda y + \tfrac{1}{2}\sin 2\pi\phi).$$

It is perhaps easier to understand what F does visually: F stretches $\mathcal{T}$ by a factor of 2 in the circle direction, contracts $\mathcal{T}$ by a factor of λ in the disc direction, and then wraps the result around twice inside $\mathcal{T}$.



The map F is injective and continuous. If we let

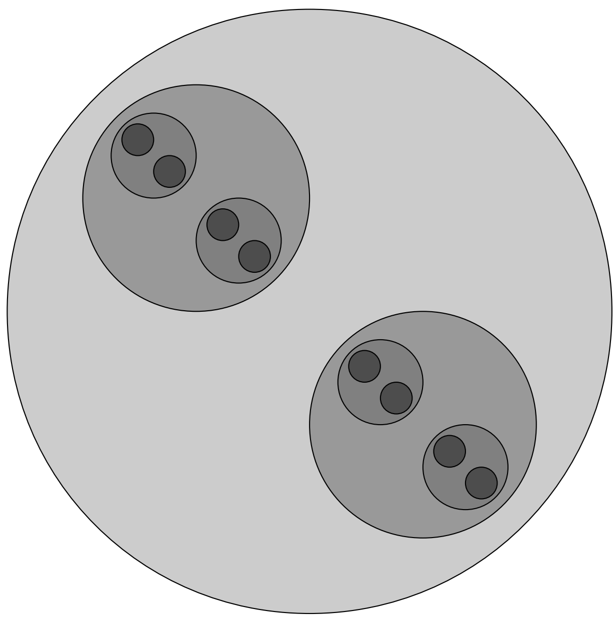
$$X = \bigcap_{n \in \mathbb{N}} F^n(\mathcal{T})$$

then the restriction of F to X is a homeomorphism and the dynamical system (X, F) is known as the **solenoid**.

The space X is a subspace of the solid torus. If we fix an angle ϕ_0 then the cross-section

$$X_{\phi_0} = \{(\phi, x, y) \in X \mid \phi = \phi_0\}$$

is contained in the infinite intersection of discs as pictured below.



It is not hard to see that X_{ϕ_0} is totally disconnected, compact and has no isolated points, and so is homeomorphic to the Cantor set. Furthermore, every point in X has a neighbourhood of the form $X_{\phi_0} \times (0, 1)$. Hence X has the same kind of local structure as our tiling space Ω.

There is another dynamical system which is topologically conjugate to the solenoid which is easier to describe. Let

$$Y = \{(y_i)_{i \geq 0} \mid y_i \in S^1, y_i = 2y_{i+1} \bmod 1\},$$

and let Y inherit the product topology from $(S^1)^{\mathbb{N}_0}$. Let $\sigma : Y \to Y$ be the map

$$\sigma(y_0, y_1, \dots) = (2y_0, y_0, y_1, \dots).$$

Then σ is a homeomorphism. The space Y is called the **inverse limit** or **projective limit** of S^1 under the map "multiplication by 2".

For $x \in X$, one can show that the first coordinates of the preimages $F^{-n}(x) = (\phi_n, x_n, y_n)$ form a sequence $h(x) = (\phi_0, \phi_1, \dots)$ which is an element of the space Y. This defines a map $h : X \to Y$. One can show that h is a homeomorphism and that the diagram

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "h"] & X \arrow[d, "h"] \\ Y \arrow[r, "\sigma"] & Y \end{tikzcd}$$

commutes, which means that the solenoid is conjugate to (Y, σ). We refer the reader to [4], Section 1.9 for the details.

We claim that (Y, σ) is mixing. Since Y inherits the product topology, it is enough to let

$$\begin{aligned} U &= (U_1 \times U_2, \times \cdots \times U_r \times S^1 \times S^1 \times \cdots) \cap Y \\ V &= (V_1 \times V_2, \times \cdots \times V_s \times S^1 \times S^1 \times \cdots) \cap Y \end{aligned}$$

with the U_i and V_i open in S^1 and find $N \in \mathbb{N}$ such that $\sigma^n(U) \cap V$ is nonempty for all $n > N$. It is easy to see that

$$\sigma(U) = (2U_1, U_1 \times U_2, \times \cdots \times U_m \times S^1 \times S^1 \times \cdots) \cap Y.$$

Since U_1 is open in S^1, we may find an open interval in S^1 contained in U. Thus there exists $k \in \mathbb{N}$ such that $2^k U_1 = S^1$. Let $N = k + s$. Then for $n > N$ the first s entries of

$$\sigma^n(U) = (2^n U_1, 2^{n-1} U_1 \times \cdots \times U_1 \times U_2 \times \cdots \times U_m \times S^1 \times S^1 \times \cdots) \cap Y.$$

are equal to S^1, and so $\sigma^n(U) \cap V$ is nonempty.

Example 9 shows that spaces which are locally the product of a Cantor set and an open ball can occur naturally as the inverse limits of simpler spaces. It was an important discovery by Anderson and Putnam in [1] that the space Ω of tilings can also be expressed in this form. In the presence of the following condition, the space in the inverse limit is the easiest to describe.

Definition 9. A substitution tiling system $(\mathcal{P}, \omega)$ is said to **force its border** if there exists an $n \in \mathbb{N}$ such that for all $p \in \mathcal{P}$ if we have that whenever $\omega^n(p) + x \subset T$ and $\omega^n(p) + x' \subset T'$ then we can conclude that

$$T\,(\mathrm{supp}(\omega^n(p) + x)) - x = T'\,(\mathrm{supp}(\omega^n(p) + x')) - x'.$$

In words, a substitution forces its border if there exists an n such that the tiles touching the patch $\omega^n(p)$ are the same no matter where in any given tiling one sees a translate of it. The Penrose substitution of Example 6 forces its border.

Consider

$$Y = \{(x,p) \in \mathbb{R}^d \times \mathcal{P} \mid x \in p\}$$

i.e., the disjoint union of the prototiles. We define an equivalence relation on this set as follows: we declare (x,p) and (y,q) to be equivalent if there is a tiling T in Ω such that, for some $z_p, z_q \in \mathbb{R}^d$ we have $p + z_p, q + z_q \in T$ and $z_p + x = z_q + y$. In words, we treat the prototiles as disjoint sets and then glue them together wherever they could possibly meet up in any tiling. If $\mathcal{R}$ is the equivalence relation generated by the above, we let

$$\Gamma = Y/\mathcal{R}.$$

When our prototiles are polytopes meeting full-face to full-face, then in [1] it is shown that Γ is a d-dimensional CW complex whose d-cells are the prototiles. The substitution induces a map γ on Γ in the obvious way – if x is in some prototile p, then $\omega(p)$ is a patch consisting of translates of prototiles, so λx lies inside at least one translate of a prototile $p_i + y$. So then $\lambda x - y$ is in p_i, and we define $\gamma((x,p)) = (\lambda x - y, p_i)$. Even though λx could lie in more than one tile, and hence the image could be in more than one prototile, this map is well-defined precisely because such points are identified. It is proved in [1] that γ is continuous and surjective.

We let

$$\Omega_0 = \{(x_1, x_2, \ldots) \mid x_i \in \Gamma, x_i = \gamma(x_{i+1})\}.$$

This is an inverse limit space similar to that constructed for the solenoid in Example 9. The map

$$\gamma_0(x_1, x_2, \ldots) = (\gamma(x_1), x_1, x_2, \ldots)$$

is a homeomorphism of (Ω_0, γ_0).

Theorem 1. ([1], Theorem 4.3) If $(\mathcal{P}, \omega)$ forces its border, then the dynamical systems (Ω, ω) and (Ω_0, ω_0) are topologically conjugate.

Remark 1. The need to force the border is not as restrictive as it looks. The following argument from ([1], §4) explains why. From a substitution tiling system $(\mathcal{P}, \omega)$ we form a new one $(\mathcal{P}', \omega')$ as follows: for each prototile $p \in \mathcal{P}$, look at the set of all patches

$\Omega(p) = \{T(p) \mid T \in \Omega\}$. By finite local complexity, this set is finite. We let

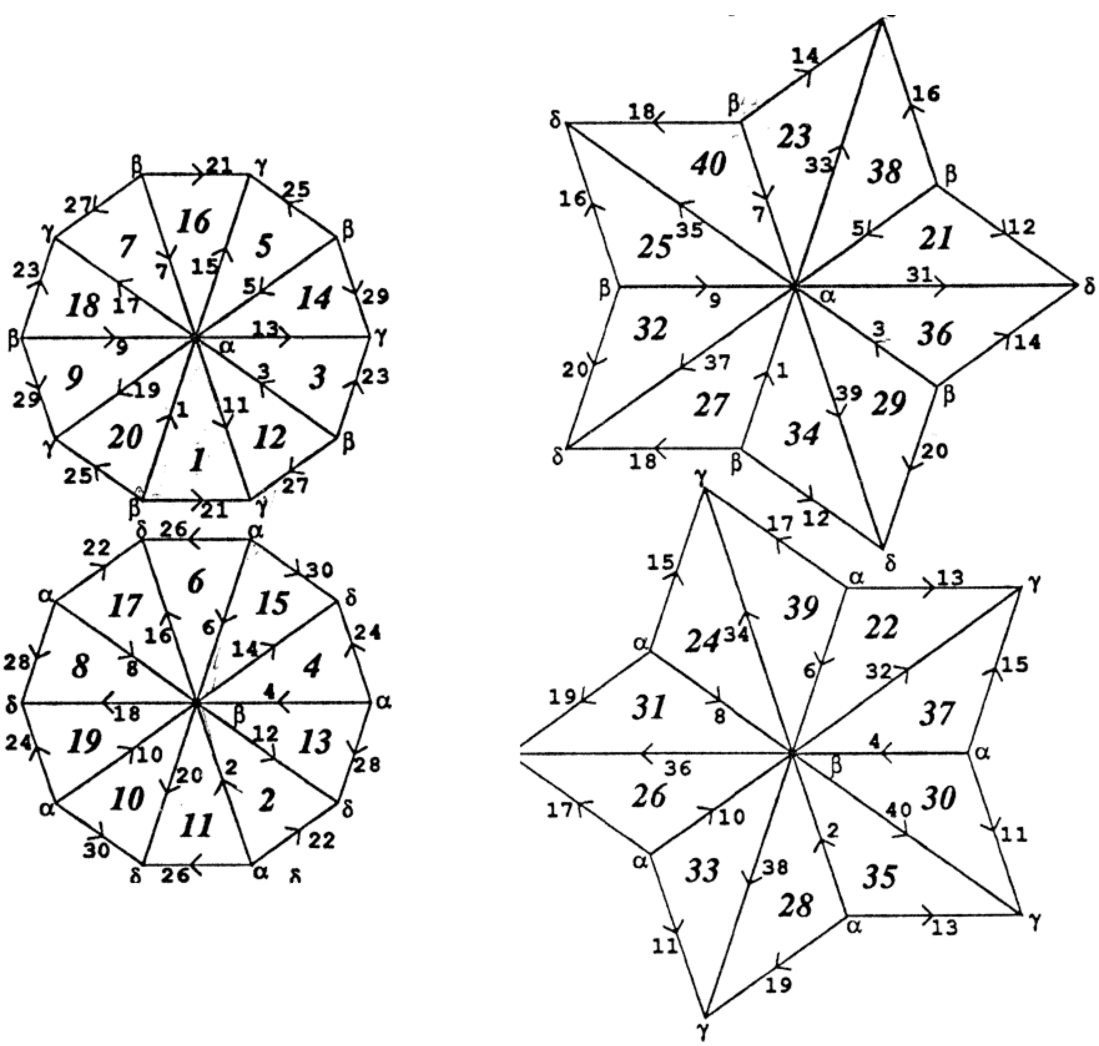
$$\mathcal{P}' = \{(p, P) \mid p \in \mathcal{P}, P \in \Omega(p)\}.$$

In words, we create a labeled copy of p for each patch consisting of tiles that intersect p that could possibly surround it in any tiling in Ω. The substitution extends to this in the natural way. If we let Γ_1 and γ_1 be the CW complex and map formed as above but from $(\mathcal{P}', \omega')$, and form

$$\Omega_1 = \{(x_1, x_2, \dots) \mid x_i \in \Gamma_1, x_i = \gamma(x_{i+1})\}.$$

then, with ω_1 the shift map on the above, (Ω_1, ω_1) is always topologically conjugate to (Ω, ω). This procedure is called *collaring*.

Example 10. The CW complex of the Penrose tiling is given below.



4 C*-algebras from Tilings

In this final section we describe some C*-algebras which may be associated to aperiodic order. These C*-algebras have some interest from the perspective of physics, as their self-adjoint elements may be seen as observables for a particle moving through a quasicrystal modeled by an aperiodic tiling. These C*-algebras are also quite interesting in their own right, as they are simple, have a unique trace, and have computable K-theory.

The first C*-algebra one may think to associate to our tiling space Ω is the crossed product associated to translation. Specifically, the map

$$\begin{aligned} \Omega \times \mathbb{R}^d &\to \Omega \\ (T, x) &\mapsto T + x \end{aligned}$$

is jointly continuous, and so induces an action α of $\mathbb{R}^d$ on the continuous functions on Ω. From this action we may form the crossed product C*-algebra

$$C(\Omega) \rtimes_\alpha \mathbb{R}^d.$$

Since every $\mathbb{R}^d$ orbit in Ω is dense, $C(\Omega) \rtimes_\alpha \mathbb{R}^d$ is simple (that is, it has no closed two-sided ideals). The K-theory of this C*-algebra can be computed through the use of the following theorem.

Theorem 2. (**Connes' Thom Isomorphism**, see for example [3] Theorem 10.2.2) If $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$, then

$$K_i(A \rtimes_\alpha \mathbb{R}) \cong K_{1-i}(A)$$

where the index i is read modulo 2.

Hence $K_i(C(\Omega) \rtimes_\alpha \mathbb{R}^d) \cong K_{i-d}(C(\Omega))$. For a reference on computing the latter, see [1].

We now present what is usually referred to as the C*-algebra of a tiling. It will be Morita equivalent to $C(\Omega) \rtimes_\alpha \mathbb{R}^d$ but will be much easier to work with. The following construction was introduced in [6], although [7] is another excellent reference.

For each $p \in \mathcal{P}$, choose a point x_p in the interior of p. This point shall be called the **puncture** of p. If t is a tile which is a translate of a prototile p, then $t + x = p$ and we let $x_t = x_p + x$ and call this the puncture of t. After possibly labelling prototiles, the puncture of a tile is unique. We let $\Omega_{\text{punc}} \subset \Omega$ be the set of all tilings in Ω such that the origin is the puncture of a tile in T. Locally, this eliminates the "open disc direction" of the local product structure, and by an argument similar to Proposition 6 Ω_{punc} is homeomorphic to the Cantor set in the relative topology. This topology on Ω_{punc} admits a basis of clopen sets of the following form. Let P be a patch in some tiling in Ω and let $t \in P$. Then we let

$$U(P,t) = \{T \in \Omega_{\text{punc}} \mid P - x_t \subset T\}.$$

We note that for $x \in \mathbb{R}^d$, $U(P,t) = U(P+x, t+x)$.

We now let

$$\mathcal{R}_{\text{punc}} = \{(T, T+x) \mid x \in \mathbb{R}^d; T, T+x \in \Omega_{\text{punc}}\}.$$

Then $\mathcal{R}_{\text{punc}}$ is an equivalence relation on Ω_{punc}. There is an embedding of $\mathcal{R}_{\text{punc}}$ into $\Omega \rtimes \mathbb{R}^d$, and so we give $\mathcal{R}_{\text{punc}}$ the topology inherited from $\Omega \rtimes \mathbb{R}^d$. Given this topology, $\mathbb{R}^d$ has the structure of what is known as an **étale equivalence relation**, that is:

- $\mathcal{R}_{\text{punc}}$ is locally compact and Hausdorff,
- the subspace $\Delta = \{(T,T) \mid T \in \Omega_{\text{punc}}\}$ is open in $\mathcal{R}_{\text{punc}}$, and
- the maps $r : \mathcal{R}_{\text{punc}} \to \Omega_{\text{punc}}$ and $s : \mathcal{R}_{\text{punc}} \to \Omega_{\text{punc}}$ given by $r(T,T') = T$ and $s(T,T') = T'$ are local homeomorphisms.

The second item follows from the fact that the punctures of a given tiling T form a discrete subset of $\mathbb{R}^d$. For a patch P and tiles $t_1, t_2 \in P$, we define

$$V(P, t_1, t_2) = \{(T, T+x) \in \mathcal{R}_{\text{punc}} \mid T \in U(P, t_1), x = x(t_1) - x(t_2)\}.$$

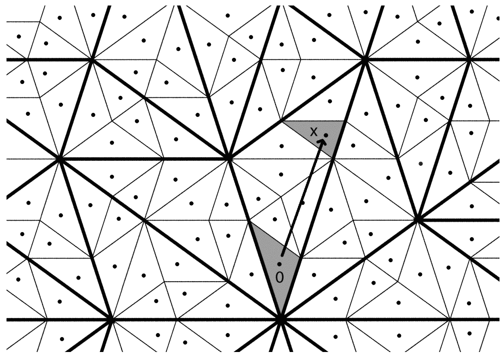
Sets of this form generate the topology on $\mathcal{R}_{\text{punc}}$. We once again note that for $x \in \mathbb{R}^d$, we have that $V(P, t_1, t_2) = V(P+x, t_1+x, t_2+x)$. In addition,

$$r(V(P,t_1,t_2)) = U(P,t_1), \qquad s(V(P,t_1,t_2)) = U(P,t_2)$$

and r and s are homeomorphisms when restricted to this domain.

We build another étale equivalence relation on Ω_{punc} from the substitution. For a tile t and $n \in \mathbb{N}$, we call $\omega^n(t)$ an **nth-order supertile**. Since ω is invertible, each tiling $T \in \Omega$ has a unique decomposition into nth-order supertiles. Furthermore, these decompositions are "nested" in the sense that if $\omega^n(t) \subset T$ then $\omega^n(t)$ is contained in a unique $(n+1)$th-order supertile in T.

For each $n \in \mathbb{N}$, let $\mathcal{R}_n$ be the set of pairs $(T, T - x)$ from $\mathcal{R}_{\text{punc}}$ such that 0 and x are punctures inside the same nth-order supertile in T. The figure below shows an example where $(T, T - x) \in \mathcal{R}_2$.



For each n we must have that $\mathcal{R}_n \subset \mathcal{R}_{n+1}$, since if 0 and x are punctures in the same nth order supertile, they must be in the same $(n+1)$th-order supertile. For each n, $\mathcal{R}_n$ is a compact open subequivalence relation of $\mathcal{R}_{\text{punc}}$. We let

$$\mathcal{R}_{AF} = \bigcup_{n \in \mathbb{N}} \mathcal{R}_n.$$

It is from these equivalence relations that we build our C*-algebras. The following is a construction due to Renault; for the more general situation see [12].

Consider $C_c(\mathcal{R}_{\text{punc}})$, the continuous compactly-supported complex-valued functions on $\mathcal{R}_{\text{punc}}$. We define a product and involution on $C_c(\mathcal{R}_{\text{punc}})$ by the following formulas:

$$f^*(T, T') = \overline{f(T', T)}$$

$$fg(T, T') = \sum_{T'' \in [T]} f(T, T'')g(T'', T),$$

where $[T]$ denotes the equivalence class of T in $\mathcal{R}_{\text{punc}}$. These operations give $C_c(\mathcal{R}_{\text{punc}})$ the structure of a $*$-algebra. We notice that the sum in the formula for the product must be finite, since each of f and g is compactly supported. This $*$-algebra may not be complete, so we complete it in a suitable norm (for details on this norm, see [12] or [9], Definition 1.6). We denote the completion of $C_c(\mathcal{R}_{\text{punc}})$ by $C^*(\mathcal{R}_{\text{punc}})$ and call this the C*-algebra of $\mathcal{R}_{\text{punc}}$.

The algebra $C^*(\mathcal{R}_{\text{punc}})$ admits a convenient generating set based on patches and tiles. Fpr a patch P and tiles $t_1, t_2 \in P$ the set $V(P, t_1, t_2)$ is a compact open set in $\mathcal{R}_{\text{punc}}$, and so its characteristic function is an element of $C_c(\mathcal{R}_{\text{punc}}) \subset C^*(\mathcal{R}_{\text{punc}})$. We let $e(P, t_1, t_2)$ denote the characteristic function of $V(P, t_1, t_2)$, and we now describe some algebraic relations on these elements. Let P, P' be patches and let $t_1, t_2, t \in P$ and $t_1', t_2' \in P'$. Assume without loss of generality that $x_{t_2} = 0$ and that $x_{t_1'} = 0$. Then we have the following.

- The product $e(P, t_1, t_2)e(P', t_1', t_2')$ is nonzero precisely when $U(P, t_1) \cap U(P, t_2) \neq \emptyset$ and the patches P and P' agree on the overlap of their supports, i.e., $P \cup P'$ is a patch. In this case the product is $e(P \cup P', t_1, t_2')$.
- $e(P, t_1, t_2)^* = e(P, t_2, t_1)$.
- $e(P, t, t)e(P, t, t) = e(P, t, t)$.

Hence each $e(P, t, t)$ is a projection and $e(P, t_1, t_2)$ is a partial isometry from $e(P, t_2, t_2)$ to $e(P, t_1, t_1)$ in $C_c(\mathcal{R}_{\text{punc}})$.

Let

$$\mathcal{E} = \{e(P, t_1, t_2) \mid P \text{ is a patch with } t_1, t_2 \in P\}.$$

Then $\operatorname{span}_{\mathbb{C}} \mathcal{E}$ is a $*$-subalgebra of $C_c(\mathcal{R}_{\text{punc}})$. It has identity $\sum_{p \in \mathcal{P}} e(\{p\}, p, p)$. In [15], Lemma 4.13, Whittaker uses a Stone-Weierstrass argument to show that $\operatorname{span}_{\mathbb{C}} \mathcal{E}$ is dense in $C_c(\mathcal{R}_{\text{punc}})$.

The C*-algebra of the subgroupoid $\mathcal{R}_{AF} \subset \mathcal{R}_{\text{punc}}$ is $C^*(\mathcal{R}_{AF})$, and we have $C^*(\mathcal{R}_{AF}) \subset C^*(\mathcal{R}_{\text{punc}})$. This C*-algebra is an **AF algebra**, that is, it is an inductive limit of finite dimensional C*-algebras. The identity is an element of $C^*(\mathcal{R}_{AF})$. As a unital AF algebra, it is completely described by its **Bratteli diagram** (for a good reference on AF algebras and Bratteli diagrams, see [5] or see Appendix A). If $\mathcal{P} = \{p_1, p_2, \ldots, p_{N_{\text{pro}}}\}$, then the Bratteli diagram associated to $C^*(\mathcal{R}_{AF})$ has $p_{N_{\text{pro}}}$ vertices at each level, and there is an edge from vertex i at level n to vertex j at level $(n+1)$ for each translate of p_i in $\omega(p_j)$. The incidence matrix for each level of the Bratteli diagram is constant, and is primitive precisely because ω is primitive. This implies that $C^*(\mathcal{R}_{AF})$ is simple and has unique trace. It is a result of Putnam [10] (generalized by Phillips in [9]) that $C^*(\mathcal{R}_{AF})$ is “large enough” inside $C^*(\mathcal{R}_{\text{punc}})$ for this to imply that $C^*(\mathcal{R}_{\text{punc}})$ has unique trace as well. In [9], Phillips also proves that $C^*(\mathcal{R}_{\text{punc}})$ has real rank zero and tracial rank one.

5 Appendix: AF Algebras

In this appendix we briefly present a well-studied class of C*-algebras, the AF algebras. Recall that a finite dimensional C*-algebra A is isomorphic to a direct sum of full matrix algebras, i.e.

$$A = \bigoplus_{i=1}^{k} \mathbb{M}_{n_i}(\mathbb{C}).$$

In particular, a finite dimensional C*-algebra is unital. If

$$B = \bigoplus_{i=1}^{l} \mathbb{M}_{m_i}(\mathbb{C})$$

is another finite dimensional algebra, and $\varphi : A \to B$ is a unital $*$-homomorphism, then φ is determined up to unitary equivalence in B by an $l \times k$ matrix M of nonnegative integers such that

$$M \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_k \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_l \end{bmatrix}.$$

The matrix M is called the matrix of **partial multiplicities**. If $M = [M_{ij}]$, then the integer M_{ij} is the multiplicity of the embedding of the summand $\mathbb{M}_{n_j}(\mathbb{C})$ of A into the summand $\mathbb{M}_{m_i}(\mathbb{C})$ of B. For details see [5] Lemma III.2.1.

One way of obtaining the matrix of partial multiplicities is through traces. If τ is a trace on $\mathbb{M}_n(\mathbb{C})$, then it is a positive scalar multiple of the usual matrix trace Tr (this is the sum of the diagonal entries). If A is a finite dimensional algebra written as before,

$$A = \bigoplus_{i=1}^{k} \mathbb{M}_{n_i}(\mathbb{C}).$$

then for each j,

$$\tau_j^A \left((a_i)_{i=1}^k \right) = \mathrm{Tr}(a_j)$$

is a trace on A. Furthermore, every trace on A can be written as a positive linear combination of the τ_j^A since restricting to a summand yields a trace on that summand. Let

$$B = \bigoplus_{i=1}^{l} \mathbb{M}_{m_i}(\mathbb{C})$$

and suppose that $\varphi : A \to B$ is a unital injective homomorphism of C*-algebras. Then for each i between 1 and l, $\tau_i^B \circ \varphi$ is a trace on A. Furthermore, if we denote by q_i the identity on the ith summand in A, $\tau_i^B \circ \varphi(q_s)$ should be the trace of q_s multiplied by the multiplicity of the embedding of the summand $\mathbb{M}_{n_s}(\mathbb{C})$ of A into the summand $\mathbb{M}_{m_i}(\mathbb{C})$ of B. On the other hand, we know that

$$\tau_i^B \circ \varphi = \sum_{j=1}^{k} M_{ij} \tau_j^A \tag{1}$$

for some positive scalars M_{ij}. Hence,

$$\tau_i^B \circ \varphi(q_s) = \sum_{j=1}^{k} M_{ij} \tau_j^A(q_s) = M_{is} \tau_s^A(q_s) = M_{is} n_s,$$

and so $M = [M_{ij}]$ is the matrix of partial multiplicities of the inclusion. A formula for its entries is given by manipulating the above,

$$M_{ij} = \frac{\tau_i^B \circ \varphi(q_j)}{\tau_j^A(q_j)} \tag{2}$$

A C*-algebra A is called **approximately finite dimensional** or **AF** if it is the closure of an increasing union of finite dimensional subalgebras A_n. When A is unital, it is required that the A_0 consist only of the scalar multiples of the identity of A. Thus in the unital case, each A_n contains the identity. Given an AF algebra $A = \overline{\cup A_n}$, the inclusion of A_n in A_{n+1} is determined up to unitary equivalence in A_{n+1} by the matrix of partial multiplicities. We may describe this series of inclusions by what is known as a **Bratteli diagram**.

Definition 10. A **Bratteli diagram** is an infinite directed graph (E, V), where E is the set of edges and V is the set of vertices, with the following properties:

1. The vertex set is a disjoint union finite subsets $V_n \subset V$ for $n \geq 0$,

2. the set V_0 consists of one vertex v_0, called the root,

3. if $e \in E$ then there exists $n \geq 0$ such that $i(e) \in V_n$ and $t(e) \in V_{n+1}$,

4. for $v \in V \setminus V_0$, there exist $e_1, e_2 \in E$ such that $t(e_1) = i(e_2) = v$.

In the above, $i(e)$ and $t(e)$ denote the initial vertex and terminal vertex of the edge e respectively. We say a Bratteli diagram is **simple** if for every $v \in V_n$ and $u \in V_{n+1}$ there exists $e \in E$ such that $i(e) = v$ and $t(e) = u$.

A Bratteli diagram is built from an AF algebra $A = \overline{\cup A_n}$ as follows: the set V_n consists of one vertex for every full matrix summand in A_n. If $M(n)$ is the matrix of partial multiplicities for the inclusion $A_n \subset A_{n+1}$, then we draw $M(n)_{ij}$ edges from the jth vertex in V_n to the ith vertex in V_{n+1}. The requirement that A_0 consist of the scalar multiples of the identity implies that $A_0 \cong \mathbb{C}$, and so V_0 has one vertex as required.

Example 11. Let $A_n = \mathbb{M}_{2^n}(\mathbb{C})$, and let each inclusion $A_n \subset A_{n+1}$ be determined by

$$a \mapsto \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

Then the matrix of partial multiplicities of each inclusion is the 1×1 matrix $[2]$. The Bratteli diagram of this sequence of finite dimensional algebras is

$$\cdot \rightrightarrows \cdot \rightrightarrows \cdot \rightrightarrows \cdot \rightrightarrows \quad \cdots$$

This algebra is what is known as the **CAR algebra**, see [5], Example III.2.4.

We note the following.

Proposition 7. ([5], Proposition III.2.7) Let $A = \overline{\cup A_n}$ and $B = \overline{\cup B_n}$ be two AF algebras. Then if A and B have the same Bratteli diagram, they are isomorphic.

The act of **telescoping** also results in isomorphic AF algebras. If (V, E) is a Bratteli diagram, we may form another by deleting one of the vertex sets. Pick any $n \in \mathbb{N}$ and let

$$V' = \bigcup_{\substack{i \geq 0 \\ i \neq n}} V_i.$$

Our new edge set E' will consist of all the edges from E which did not have source or range in V_n. We create a new edge in E' for every pair of edges e_1, e_2 with $t(e_1) = i(e_2) \in V_n$. The result (E', V') will be a Bratteli diagram. The incidence matrix between V_{n-1} and V_{n+1} will simply be the product of $M(n)$ and $M(n+1)$. The diagrams (E, V) and (E', V') will have isomorphic AF algebras.

One may also form an étale equivalence relation from a Bratteli diagram. Let (V, E) be a Bratteli diagram. Define

$$X = \{(x_i)_{i \in \mathbb{N}} \mid x_i \in E, s(x_1) = v_0, i(x_{i+1}) = t(x_i)\},$$

the set of all infinite paths in (V, E) which start at the root. If $x \in X$, we define

$$U(x, k) = \{(y_i)_{i \in \mathbb{N}} \mid y_i = x_i, 1 \leq i \leq k\}.$$

This is the set of all infinite paths which look like x up to the kth term. We endow X with the topology generated by sets of this form as x and k vary. If (V, E) is simple, then X with this topology is homeomorphic to the Cantor set. We let

$$\mathcal{R}_k = \{(x, y) \in X \times X \mid x_i = y_i \text{ for all } i \geq k\},$$

$$\mathcal{R} = \bigcup_{n \in \mathbb{N}} \mathcal{R}_n.$$

We see that $\mathcal{R}$ is an equivalence relation on X, and two sequences are equivalent if they are eventually equal. This relation is known as **tail equivalence**. We note that $\mathcal{R}_k \subset \mathcal{R}_{k+1}$ for all $k \in \mathbb{N}$ and that each $\mathcal{R}_k$ contains the diagonal.

Let $x, y \in X$ be such that $t(x_k) = t(y_k)$, i.e., they pass through the same vertex at stage k. Define

$$V(x, y, k) = \{(z, w) \in X \times X \mid z \in U(x, k), w \in U(y, k), z_i = w_i, i > k\}.$$

Then $V(x, y, k) \subset \mathcal{R}_k$. We give $\mathcal{R}$ the topology generated by the $V(x, y, k)$ as x, y, and k vary, keeping in mind it is only defined if $t(x_k) = t(y_k)$. In this topology, $\mathcal{R}_k$ is compact and open in $\mathcal{R}$ for all k. We have that $r(V(x, y, k)) = U(x, k)$, and restricted to this domain r is easily checked to be a homeomorphism. It is a fact that $C^*(\mathcal{R})$ is isomorphic to the AF algebra associated to (V, E). For details on the above construction, see [12], Section III.1.

References

[1] J. E. Anderson and I.F. Putnam, *Topological Invariants for Substitution Tilings and their Associated C^*-algebras*, Ergodic Theory and Dynamical Systems **18**, (1998) 509-537

[2] R. Berger, *The undecidability of the domino problem*, Mem. Amer. Math. Soc. 66 (1966), 1-72.

[3] B. Blackadar, *K-theory for Operator Algebras*, Second edition. Mathematical Sciences Research Institute Publications, 5. Cambridge University Press, Cambridge, 1998. xx+300 pp.

[4] M. Brin, G. Stuck, *Introduction to Dynamical Systems*, Cambridge University Press, Cambridge, 2002.

[5] K.R. Davidson, *C*-algebras by Example*, Fields Inst. Monographs, vol 6,American Math. Soc., Providence, RI, 1996.

[6] J. Kellendonk, *Noncommutative Geometry of Tilings and Gap Labeling*, Rev. Math. Phys. **7** (1995), 1133-1180.

[7] J. Kellendonk and I.F. Putnam, *Tilings, C*-algebras, and K-Theory*, CRM Monograph Series, vol 13 (2000), 177-206.

[8] R. Penrose. *The role of aesthetics in pure and applied mathematical research.* Bull. Inst. Math. Appl., 10(7/8):266-71, 1974.

[9] N. C. Phillips, *Crossed Products of the Cantor Set by Free Minimal Actions of* $\mathbb{Z}^d$, Comm. Math. Phys. 256 (2005), no. 1, 1-42.

[10] I.F. Putnam, *The ordered K-theory of C*-algebras associated with substitution tilings*, Comm. Math. Phys., 214 (2000), 593-605.

[11] C. Radin and M. Wolff, *Space Tilings and Local Isomorphism*, Geom. Dedicata, 42(1992), 355-360.

[12] J. Renault, *A Groupoid Approach to C*-algebras*, Springer Lecture Notes in Math, no. 793, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[13] D. Shechtman, I. Blech, D. Gratias, and J.V. Cahn. *Metallic phase with long range orientational order and no translational symmetry.* Phys. Rev. Lett., 53:1951-1953, 1984.

[14] H. Wang *Proving theorems by pattern recognition-II*, Bell System Tech. Journal 40(1):1-41. 1961

[15] M. F. Whittaker, *Groupoid C*-algebras of the pinwheel tiling*, Masters thesis, University of Victoria 2005.