Finite Group Actions on Substitution Tilings

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Overview

We study tilings of $\mathbb{R}^2$ which are aperiodic, but display long-range order.

We produce such tilings using substitution rules.

Tiling $T \longrightarrow$ topological space Ω_T

Elements of Ω_T are tilings, and $\mathbb{R}^2$ acts by translating them.

We replace $(\Omega_T, \mathbb{R}^2)$ with an étale groupoid $\mathcal{R}_{punc}$ which is transverse to the action.

Étale groupoid $\mathcal{R}_{punc} \longrightarrow$ C*-algebra $C^*(\mathcal{R}_{punc})$.

Finite symmetry group actions on tilings $\longrightarrow$ crossed products.

Tilings

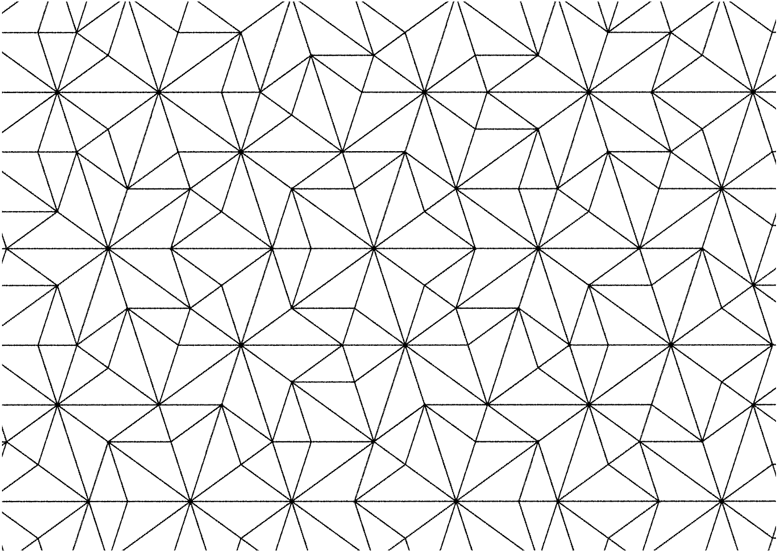
Definition

A **tiling** T of $\mathbb{R}^2$ is a countable set $T = \{t_1, t_2, \dots\}$ of subsets of $\mathbb{R}^2$, called **tiles** such that

- Each tile is homeomorphic to the closed ball (they are usually polygons),
- $t_i \cap t_j$ has empty interior whenever $i \neq j$, and
- $\cup_{i=1}^{\infty} t_i = \mathbb{R}^2$.

- A **patch** is a finite subset of T. The **support** of a patch is the union of its tiles.
- If T is a tiling, $x \in \mathbb{R}^2$, $T + x$ is the tiling formed by translating every tile in T by x.
- T is **aperiodic** if $T + x \neq T$ for all $x \in \mathbb{R}^2 \setminus \{0\}$.

Example: Penrose Tiling



Tilings

There are uncountably many Penrose tilings, even up to tranlsation.

However, all Penrose tilings look similar locally.

For any $r > 0$, there are only a finite number of patches of radius r possible in Penrose tilings (finite local complexity).

For any patch P, there is an $R > 0$ such that every ball of radius R contains a copy of P (repetitivity).

The Tiling Space

Given a tiling T, look at the set $T + \mathbb{R}^2 = \{T + x \mid x \in \mathbb{R}^2\}$.

We can put a metric on this set that satisfies the following: T_1 and T_2 are close if

1. $T_1 = T_2 + x$ for some small x.
2. T_1 agrees with T_2 exactly on a large ball around the origin, then disagrees elsewhere.

In most cases, 1 looks like a disc while 2 looks like a Cantor set.

The Tiling Space

Complete $T + \mathbb{R}^2$ in the metric $\longrightarrow \Omega_T$, the **continuous hull** of T.

Ω_T is the set of all tilings T' such that every patch in T' appears somewhere in T.

Finite local complexity $\Longrightarrow \Omega_T$ compact. (Radin-Wolff)

Repetitivity $\Longrightarrow (\Omega_T, \mathbb{R}^2)$ minimal. (Solomyak)

Substitution Rules

We build tilings from a finite set of polygons $\mathcal{P} = \{p_1, p_2, \ldots, p_N\}$, called the set of **prototiles**.

Prototiles may carry labels.

A **substitution** on $\mathcal{P}$ is

- A scaling constant $\lambda > 1$
- A rule ω such that for each $p \in \mathcal{P}$, $\omega(p)$ is a patch with support λp whose tiles are translates of elements of $\mathcal{P}$.

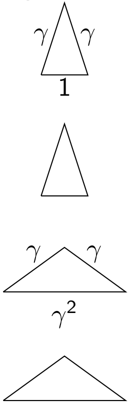
ω can be applied to tilings and patches consisting of translates of $\mathcal{P}$ by applying it to each tile.

ω can be iterated, since $\omega(p)$ is a patch.

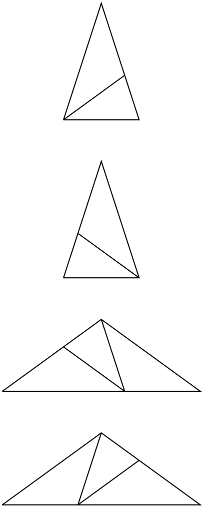
If t is a tile, $\omega^n(t)$ is called an n**th order supertile**.

Example: Penrose Tiling

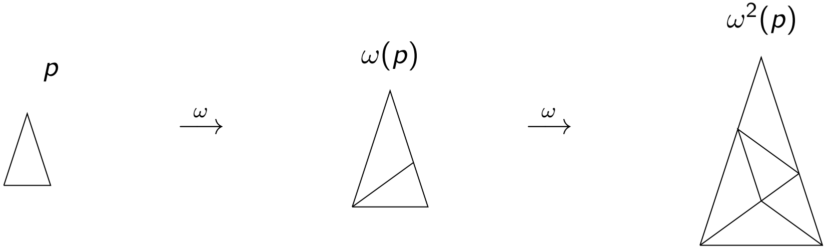
Prototiles
(+ rotates by $\frac{\pi}{5}$)
$\gamma =$ golden ratio



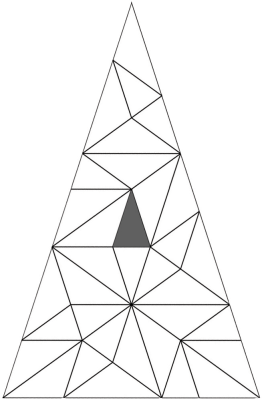
$\stackrel{\omega}{\stackrel{\lambda=\gamma}{\longrightarrow}}$



Example: Penrose Tiling



Producing a Tiling from a Substitution Rule

$\omega^4($  $) =$ 

$p \subset \omega^4(p) \subset \omega^8(p) \ldots$

$\omega^{4n}(p) \subset \omega^{4(n+1)}(p)$
Then

$$T = \bigcup_{n=1}^{\infty} \omega^{4n}(p)$$

is a tiling.

Properties of the Tiling Space

If T is formed as above, every tile in T is a translate of an element of $\mathcal{P}$. If they only meet edge-to-edge in T, T has finite local complexity.

We call ω **primitive** if there exists some n such that such that $\omega^n(p_i)$ contains a copy of p_j for every $p_i, p_j \in \mathcal{P}$.

If T is formed by a primitive substitution rule, then T has repetitivity. Hence $\Omega_T \to \Omega$.

We restrict our attention to tilings that have FLC, that come from a primitive substitution rule, and such that Ω contains no periodic tilings.

In this case $\omega : \Omega \to \Omega$ is a homeomorphism.

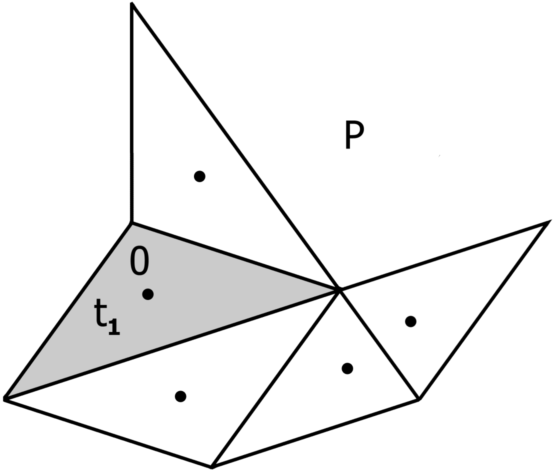
The Punctured Tiling Space

We replace each prototile $p \in \mathcal{P} \longrightarrow (p, x(p))$, where $x(p) \in$ the interior of p. The point $x(p)$ is called the **puncture** of p. If $t \in T$, then $t = p + y$ for some y and so we define $x(t) = x(p) + y$.

Define $\Omega_{punc} \subset \Omega$ as the set of all tilings $T \in \Omega$ such that the origin is on a puncture of a tile in T, ie, $x(t) = 0$ for some $t \in T$. Ω_{punc} is called the **punctured tiling space** or **punctured hull**.

Ω_{punc} is homeomorphic to a Cantor set. Its topology is generated by clopen sets of the following form: if P is a patch and $t \in P$, then let

$$U(P, t) = \{T \in \Omega_{punc} \mid P - x(t) \subset T\}$$



If T looks like this around the origin $0 \in \mathbb{R}^2$, then $T \in U(P, t_1)$.

Groupoids associated to Tilings

Let $\mathcal{R}_{punc} = \{(T, T+x) \mid T, T+x \in \Omega_{punc}\}$. Then $\mathcal{R}_{punc}$ is an equivalence relation.

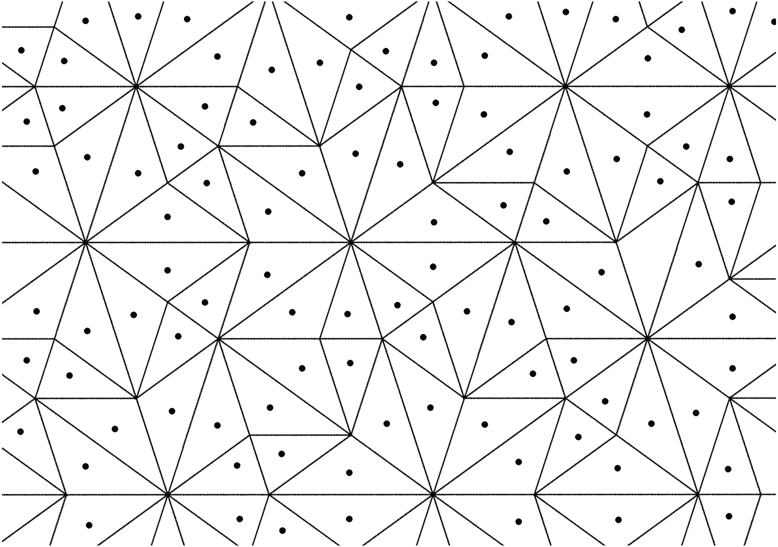
With the topology from $\Omega_{punc} \times \mathbb{R}^2$, it becomes an **étale** groupoid. It is locally compact, σ-compact, the diagonal is open, and the range and source maps are local homeomorphisms. Its unit space is Ω_{punc}.

We build $\mathcal{R}_{AF} \subset \mathcal{R}_{punc}$ from the substitution.

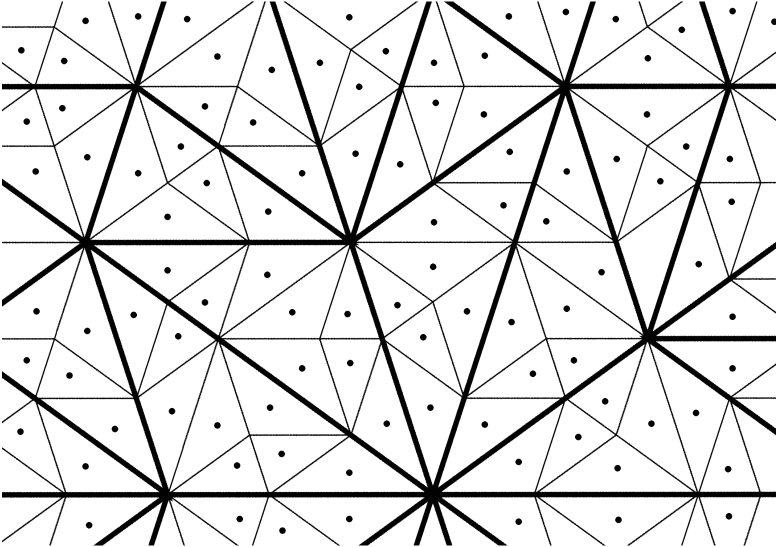
Since $\omega : \Omega \to \Omega$ is invertible, so is ω^n. Hence every tiling in Ω has a unique decomposition into nth order supertiles.

Define $\mathcal{R}_n \subset \mathcal{R}_{punc}$ by saying $(T, T-x) \in \mathcal{R}_n$ if 0 and x are punctures in the same nth-order supertile in T.

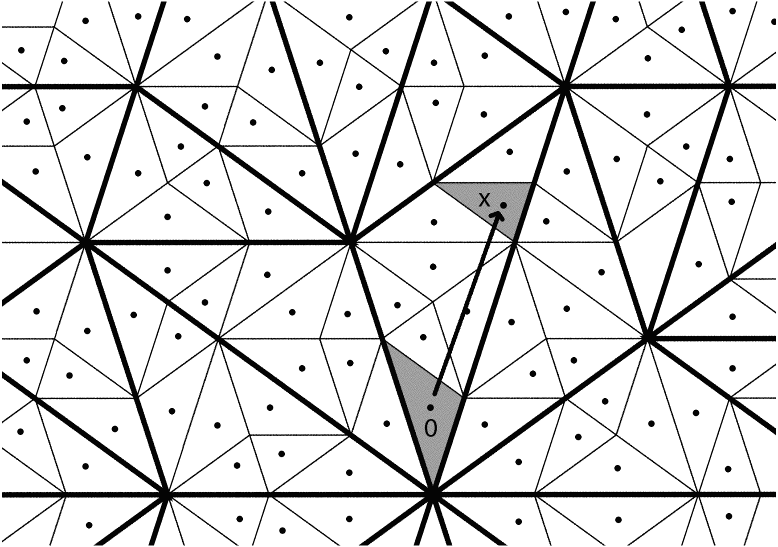
$\mathcal{R}_n$ are nested compact open sub equivalence relations of $\mathcal{R}_{punc} \longrightarrow$ $\mathcal{R}_{AF} = \cup \mathcal{R}_n$ is an AF subgroupoid of $\mathcal{R}_{punc}$.



A punctured tiling T.



T has unique decomposition into 2nd order supertiles.



$(T, T - x) \in \mathcal{R}_2.$

The Tiling Algebra

The C*-algebra of a tiling is

$$A_\omega := C^*(\mathcal{R}_{punc})$$

This is $C_c(\mathcal{R}_{punc})$ with convolution product completed in a suitable norm. This algebra was studied extensively by Kellendonk and Putnam.

$$AF_\omega := C^*(\mathcal{R}_{AF})$$

is an AF-subalgebra of A_ω.

Anderson, Putnam (1996) - $A_\omega \sim_m C(\Omega) \rtimes \mathbb{R}^2$, hence simple. They used this to calculate the K-theory.

The Tiling Algebra

Putnam (1999) - Order on K-theory of A_ω is determined by its unique trace.

Phillips (2002) - Generalized this result to C*-algebras of *almost AF Cantor groupoids* (notably, minimal actions of $\mathbb{Z}^d$ on the Cantor set). Also proved that such algebras have real rank zero and stable rank one.

Conjectured that all such algebras have tracial rank zero. This would imply that tiling algebras would be classified by their K-theory.

- Used presence a "large" AF-subalgebra.

Finite Symmetry Groups

Most of the tilings we are interested in display some finite symmetries.

If T is a Penrose tiling in Ω, then rotating T by $\frac{\pi}{5}$ gives us another element of Ω. Same for flipping over any edge direction.

$\Rightarrow D_{10}$ acts on Ω by homeomorphisms (as do subgroups).

We can choose punctures carefully so that elements of D_{10} act on Ω_{punc} and hence on $\mathcal{R}_{punc}$ and A_ω.

$$\alpha : D_{10} \to Aut(A_\omega)$$

$$\alpha_g(f)(T, T') = f(gT, gT')$$

Thus we can form the crossed product $A_\omega \rtimes G$ for any $G < D_{10}$.

Finite Symmetry Groups

In general, a group G will act on A_ω if G acts on $\mathcal{P}$ and commutes with the substitution.

Since $g\omega^n(t) = \omega^n(gt)$,

$$\alpha_g(\mathcal{R}_n) = \mathcal{R}_n$$

$\Rightarrow AF_\omega \rtimes G$ is an AF algebra.

In the case of the Penrose tiling, D_{10} acts freely on the prototiles, but this is not true in general. However, we can replace $\mathcal{P}$ with $\mathcal{P}'$ that respects the original substitution such that a given symmetry group acts freely.

$\Rightarrow$ Homeomorphic Ω, Mortia equivalent A_ω, but AF_ω need not be isomorphic.

Finite Symmetry Groups

Proposition

If G is a finite group that acts freely on $\mathcal{P}$ and commutes with ω, then $A_\omega \rtimes G$ is the C^*-algebra of an almost AF Cantor groupoid (and hence has real rank zero, stable rank one, and order on K-theory is determined by traces).

The large AF-algebra in this case is $AF_\omega \rtimes G \cong C^*(\mathcal{R}_{AF} \rtimes G)$.

The incidence matrix of $AF_\omega \rtimes G$ is primitive, so it is simple and has a unique trace.

By Phillips, $A_\omega \rtimes G$ also has a unique trace.

Finite Symmetry Groups

Proposition

If G acts freely on the prototiles, then

1. $\alpha : G \to Aut(AF_\omega)$ *has the Rokhlin property*
2. **IF** A_ω *has tracial rank zero, then*
   - $\alpha : G \to Aut(A_\omega)$ *has the tracial Rokhlin property and*
   - $A_\omega \rtimes G$ *also has tracial rank zero.*

Rokhlin property and tracial Rokhlin property are freeness conditions.

K-theory

The crossed product $A_\omega \rtimes G$ is strongly Morita equivalent to $C(\Omega) \rtimes (\mathbb{R}^2 \rtimes G)$.

Chabert, Echterhoff, Nest (2003) – If G is a finite subgroup of $SO(2)$, then

$$K_*(C(\Omega) \rtimes (\mathbb{R}^2 \rtimes G)) = K_*(C(\Omega) \rtimes G).$$

Echterhoff, Emerson (2010) – Compute $K_*(C(X) \rtimes G)$ where G acts properly on some compact X.

They produce an ideal I of $C(X) \rtimes G$ strongly Morita equivalent to $C(X/G)$ and use excision to write down a six-term exact sequence.

K-theory

When $K^1(\Omega/G)$ is free, then

$$K_1(C(\Omega) \rtimes G) \cong K^1(\Omega/G)$$

$$K_0(C(\Omega) \rtimes G) \cong K^0(\Omega/G) \oplus \mathbb{Z}^n$$

Each G-orbit contributes a copy of $\mathbb{Z}$ for each non-trivial character of its stabilizer subgroup.

K-theory

Example: Penrose

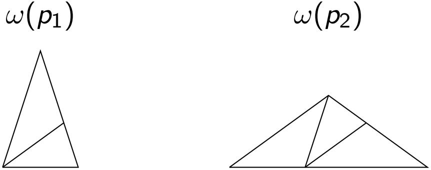
$$K^0(\Omega/\mathbb{Z}_{10}) \cong \mathbb{Z}^3 \qquad K^1(\Omega/\mathbb{Z}_{10}) \cong \mathbb{Z}$$

There are two $\mathbb{Z}_{10}$-orbits with fixed points, each with stabilizer subgroup $\mathbb{Z}_5$.

These contribute 8 copies of $\mathbb{Z}$

$$K_0(C(\Omega) \rtimes \mathbb{Z}_{10}) \cong K_0(A_\omega \rtimes \mathbb{Z}_{10}) \cong \mathbb{Z}^{11}$$
$$K_1(C(\Omega) \rtimes \mathbb{Z}_{10}) \cong K_1(A_\omega \rtimes \mathbb{Z}_{10}) \cong \mathbb{Z}$$

K-theory of the AF algebra



Taking the crossed product by D_{10} has the effect of "modding out" by the group.

$$A_n \rtimes D_{10} \cong \mathbb{M}_{(\#D_{10})(\#\omega^n(p_1))} \oplus \mathbb{M}_{(\#D_{10})(\#\omega^n(p_2))}$$

There is one copy of p_1 and one copy of p_2 in $\omega(p_1)$. There is one copy of p_1 and two copies of p_2 in $\omega(p_2)$.

K-theory of the AF algebra

$$A_n \rtimes D_{10} \hookrightarrow A_{n+1} \rtimes D_{10}$$

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mapsto \left(\begin{array}{cc|ccc} A & 0 & & & \\ 0 & B & & & \\ \hline & & A & 0 & 0 \\ & & 0 & B & 0 \\ & & 0 & 0 & B \end{array}\right)$$

$$\Rightarrow K_0(AF_\omega \rtimes D_{10}) \cong \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}} \mathbb{Z}^2 \longrightarrow \cdots$$

$$\Rightarrow K_0(AF_\omega \rtimes D_{10}) \cong \mathbb{Z} + \gamma\mathbb{Z} \text{ where } \gamma \text{ is the golden ratio.}$$

K-theory

If you are seeing this slide, I ran out of time! Oops!

Penrose:

$$K_0(A_\omega \rtimes \mathbb{Z}_{10}) \cong \mathbb{Z}^{11}$$
$$K_1(A_\omega \rtimes \mathbb{Z}_{10}) \cong \mathbb{Z}$$

$$K_0(AF_\omega \rtimes D_{10}) \cong \mathbb{Z} + \gamma\mathbb{Z}$$